Beta Distribution in Fuzzy Game theory

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Abstract

In this paper we deal with fuzzy game problem in which the payoff are represented as triangular and trapezoidal fuzzy numbers. We use beta distribution to convert the fuzzy numbers in to crisp numbers. By applying matrix method the value of the game is obtained.

Keywords: *Fuzzy sets, Triangular fuzzy numbers, Trapezoidal fuzzy numbers, Fuzzy game problem, Matrix method.*

1 Introduction

A game is a mathematical model of a real conflict situation, the analysis of which is carried out according to certain rules. It facilitates the players to determine the optimal strategy in the game.

The concept of fuzzy sets was first introduced in 1965 by Zadeh [1]. He expanded the notion of membership beyond the “zero-one” logic and utilized the dynamic infinite space between these values. Japanese industrialists used this worthwhile concept to develop a scheme of fully automated subway controls, which demonstrated its real-world application encouraged a whole new wave of researchers to study its theoretical and practical potentials.

The foundation of conventional mathematics is based on real numbers and the process of defuzzifying and ranking fuzzy quantities - such as color or quality of goods - plays a significant role in data analysis, economics and industrial systems, so an extensive amount of research has been dedicated to this specific subject. Some of the previous works in the literature as follows. In 1980, Yager [2] proposed a method for ranking based on their corresponding centroid-index. In 1998 Cheng [3] proposed a centroid-index ranking method for calculating the centroid point of a fuzzy number. In 2000 Yao and Wu [4] used the decomposition principle and the crisp ranking on R to construct a ranking system for fuzzy numbers. In 2003, S.J. Chen and S.M. Chen [5] introduced a procedure for ordering trapezoidal fuzzy numbers based on the center of gravity. In 2006, Asady and Zendehnam [6] presented a method for ranking fuzzy numbers by distance minimization. The fuzzy number ranking method proposed in 2008 by Chen and Wang [7] used α-cuts for this purpose. A method for defuzzying and ranking fuzzy numbers by using the mean value of beta distribution was proposed by Rahmani et al. [8] in 2016.

2 Preliminaries

2.1 Fuzzy Numbers

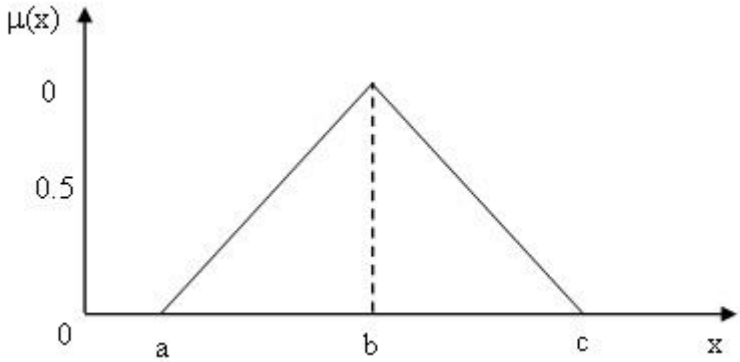
A fuzzy number $\tilde{A}$ is a fuzzy set on the real line R must satisfy the following conditions

(i) There exist at least one $x \in X$ with $\mu_A(x) = 1$
(ii) $\mu_A(x)$ is piecewise continuous

2.2 Triangular Fuzzy Numbers

A fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ is said to be triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}(x)} = \begin{cases} 0 & for\ x < a_1 \\ \dfrac{(x - a_1)}{(a_2 - a_1)} & for\ a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{(a_3 - a_2)} & for\ a_2 \le x \le a_3 \\ 0 & for\ x > a_3 \end{cases}$$



where $a_1 \le a_2 \le a_3$ are real numbers.

Figure 1: Graphical representation of triangular fuzzy numbers

2.3 Trapezoidal Fuzzy Numbers

A fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$ is said to be trapezoidal fuzzy number if its membership function is given by, where $a_1 \le a_2 \le a_3 \le a_4$ are real numbers

$$\mu_{\tilde{A}(x)} = \begin{cases} 0 & for\ x < a_1 \\ \dfrac{(x - a_1)}{(a_2 - a_1\ )} & for\ a_1 \le x \le a_2 \\ 1 & for\ a_2 \le x \le a_3 \\ \dfrac{a_3 - x}{(a_4 - a_3)} & for\ a_3 \le x \le a_4 \\ 0 & for\ x > a_4 \end{cases}$$

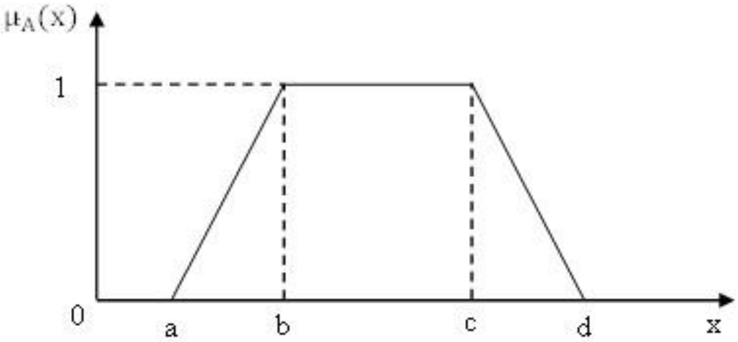


Figure 2: Graphical representation of trapezoidal fuzzy numbers

Statistical Preliminaries : . Let S be the sample space of a random trial with a given probability value and let X be the random variable defined as a real-valued function on S. If X is a discrete random variable, the function $f(x)$ = P (X = x) for any specific value of x within range of X is called probability distribution. When X is a continuous random variable, the function $f(x)$ is known as probability density function of X. Probability distributions and probability densities come in different types, including uniform density and Beta distribution just to name two.

The random variable X is said to have Beta distribution if and only if its probability density is as follows:

$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha).\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0, & Otherwise \end{cases} \qquad (1)$$

where $\alpha > 0$ and $\beta > 0$ are Beta distribution parameters. The mean of Beta distribution is obtained as follows:

$$\mu = \frac{\alpha}{\alpha+\beta} \qquad (2)$$

If $\alpha \geq 1$ and $\beta \geq 1$, then the curve of Beta function will be unimodal. When $\alpha > \beta$, the curve is said to have negative skewness and if $\alpha < \beta$, then the skewness is positive. For $\alpha = \beta$ the curve of Beta function is called symmetric. To better understand the described concept, see Figure 3, where $1 < \alpha < \beta < 2$. Given that the curve of Beta distribution is a unimodal one, $f'(x) = 0$ should have a unique solution. Solving the equation $f'(x) = 0$ gives the following relation between Beta distribution parameters:

$$\beta = (\alpha - 1)\left(\frac{1-x_{mod}}{x_{mod}}\right) + 1 \qquad (3)$$

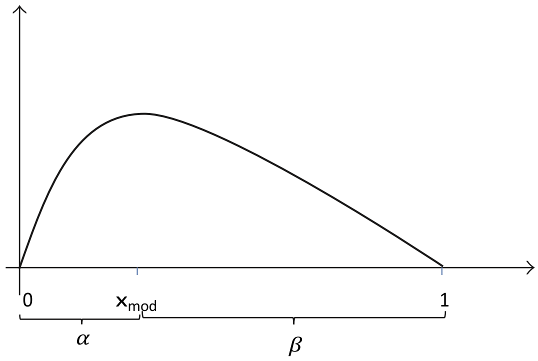


Figure 3: $\alpha < \beta$ and so the curve has positive skewness. where $x_{mod} \in (0, 1)$ is the point at which $f(x)$ has the maximum value. Hence given both values of α and x_{mod} the β parameter can be obtained from (3) and then the mean value of Beta distribution can be calculated by (2).

Uniform density is a special case of Beta distribution. The random variable X is said to have uniform density if and only if its probability density is as follows:

$$f\,(x) = \begin{cases} \frac{1}{p-q} & p < x < q \\ 0 & \end{cases} \qquad (4)$$

The mean value of uniform density can be obtained by the following equation

$$\mu = \frac{p+q}{2} \qquad (5)$$

Using the above-mentioned statistical preliminaries, a crisp number belonging to the interval (0, 1) and corresponding to the triangular fuzzy number can be obtained by (2) and (3) and a crisp number in the interval (0, 1) and corresponding to the trapezoidal fuzzy number can be obtained by (2), (3), and (5)

Defuzzification of Triangular Fuzzy Numbers.

Consider the triangular fuzzy number $a = (l, m, \tilde{u})$. To obtain the crisp real number corresponding to the triangular fuzzy number $a = (l, m, \tilde{u})$. we first project $\tilde{a}$ on the interval (0, 1), which will be in the form of $\tilde{a} = (l - l) / (u - l), (m - l) / (u - l), (u - l) / (u - l)) = (0, (m-l)/(u-l), 1)$ Then we define the parameter α $corre$sponding to the Beta distribution as follows:

$$\alpha = \frac{m-1}{u-1} + 1. \qquad (6)$$

With the above definition, firstly, it is clear that $\alpha \geq 1$ and if $m \neq l$, then the Beta distribution curve will be unimodal. Secondly, if $-l = u - m$, which denotes a symmetric triangular fuzzy number, then $\alpha = \beta = 3/2$ and the Beta distribution curve will be symmetric. Here, the left skewness of the Beta distribution curve is the left side spread of triangular fuzzy number divided by its support set.

In the Beta distribution corresponding to the projection of fuzzy number $\tilde{a}(l, m, \tilde{u})$, we have $x_{mod} = (m - l) / (u - l)$ and by using (6) and substituting it into (3), we get

$$\beta = \frac{u-m}{u-l} + 1. \qquad (7)$$

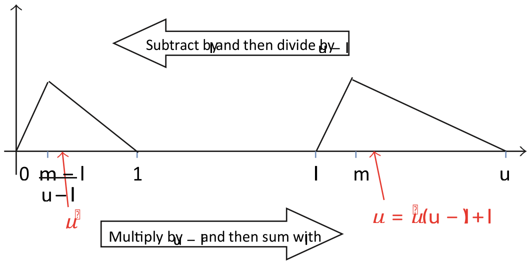


Figure 4: Transfer of the triangular fuzzy number to the interval (0,1) and vice versa.

We use (2), (3), and (7) as shown below to calculate the mean value of Beta distribution corresponding to the fuzzy number:

$$\mu' = \frac{\alpha}{\alpha+\beta}$$

$$= \frac{\frac{m-1}{u-l}+1}{\frac{m-l}{u-l}+\frac{u-m}{u-l}+1}$$

$$= \frac{m+u-2l}{3(u-l)}$$

The real number $\mu_{\tilde{a}}$ which is obtained (as shown below) by transferring μ' from the interval (0, 1) to the interval (l, u), is considered as the real number corresponding to the fuzzy number $\tilde{a} = (l, m, u)$:

$$\mu_{\tilde{a}} = \mu'\ (u - l) + l. \qquad (8)$$

Figure 4 shows the manner of projecting the triangular fuzzy number on the interval (0, 1), and Figure 5 shows the manner of defining the Beta function corresponding to the projection of fuzzy number in the interval (0,1).

The crisp real number $\mu_{\tilde{a}}$ corresponding to the triangular fuzzy number $\tilde{a} = (l, m, u)$ is obtained from the following relation:

$$\mu_{\tilde{a}} = \frac{l+m+u}{3} \qquad (9)$$

Defuzzification of Trapezoidal Fuzzy Numbers

The trapezoidal fuzzy numbers $\tilde{A}$ = (a, b, c, d) are defuzzified by using the following ranking formula

$$\mathrm{R}\,(\tilde{A}) = \frac{2a+7b+7c+2d}{18} \qquad (10)$$

2.7 Description of the paper

In this paper the triangular fuzzy numbers and trapezoidal fuzzy numbers are defuzzied by using β-distribution. Applying matrix method to the defuzzied numbers the value of the game is obtained.

Matrix Method

If the game matrix is in the form of a square matrix, then the optimal strategy as value of the game be obtained by the matrix method. The solution of a two-person zero sum game with mixed strategies with a square payoff matrix may be obtained by using the following formulae:

$$Player\ A's\ \ optimal\ strategy\ = \frac{[1 \quad 1]\, P_{adj}}{[1 \quad 1]\, P_{adj} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$Player\ B's\ \ optimal\ strategy\ = \frac{[1 \quad 1]\, P_{cof}}{[1 \quad 1]\, P_{adj} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$Value\ of\ the\ game\ = (player\ A's\ optimal\ strategies) \times (pay\ off\ matrix\ P_{ij})$$
$$\times (player\ B's\ optimal\ strategies)$$

where P_{adj} = adjoint matrix, P_{cof} = cofact matrix.

Player A's optimal strategies are in the form of a row vector and player B's optimal strategies are in the form of a column vector.

This method can be used to find a solution of a game with size of more than 2×2

3 .Numerical Example

3.1 Fuzzy game problem with payoffs as Triangular Fuzzy Numbers

Consider the following payoff matrix with triangular fuzzy numbers.

$$\begin{array}{cc} & Player\ A \\ Player\ B & \begin{bmatrix} 12,15,18 & 5,7,9 \\ 7,8,9 & 7,9,11 \end{bmatrix} \end{array}$$

The following table gives the crisp values of triangular fuzzy numbers

Strategy	**Triangular Fuzzy Numbers**	$\boldsymbol{\mu_{\tilde{a}} = \frac{\ell + m + u}{3}}$
$\widetilde{a_{11}}$	(12, 15, 18)	15
$\widetilde{a_{12}}$	(5, 7, 9)	7
$\widetilde{a_{21}}$	(7, 8, 9)	8
$\widetilde{a_{22}}$	(7, 9, 11)	9

$$The\ payoff\ matrix = \begin{pmatrix} 15 & 7 \\ 8 & 9 \end{pmatrix}$$

The given fuzzy game is reduced into crisp game problem using beta distribution and solved by maximin-minimax criterian as follows

To find saddle point

Minimum of 1st row = 7

Minimum of 2nd row = 8

Maxi(min) = 8

Maximum of 1st column = 15

Maximum of 2nd column = 9

Mini(max) = 9

∴ saddle point does not exist.

By applying matrix method to find the value of the game.

$$P_{adj} = \begin{pmatrix} 9 & -7 \\ -8 & 15 \end{pmatrix}$$

$$P_{cof} = \begin{pmatrix} 9 & -8 \\ -7 & 15 \end{pmatrix}$$

$$Player\ A's\ \ optimal\ strategy\ = \frac{[1\quad 1]\begin{pmatrix} 9 & -7 \\ -8 & 15 \end{pmatrix}}{[1\quad 1]\begin{pmatrix} 9 & -7 \\ -8 & 15 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \qquad (12)$$

$$= \frac{[1\quad 8]}{9}$$

This solution can be broken down into the optimal strategy for player A as P_1 and P_2 where P_1 and P_2 represent the probabilities of player A's using his strategies A_1 and A_2

$$P_1 = \frac{1}{9}$$

$$P_2 = \frac{8}{9}$$

Similarly, the optimal strategy for player B is obtained as,

$$Player\ B's\ \ optimal\ strategy\ = \frac{[1\quad 1]\begin{pmatrix} 9 & -8 \\ -7 & 15 \end{pmatrix}}{[1\quad 1]\begin{pmatrix} 9 & -8 \\ -7 & 15 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \qquad (13)$$

$$= \frac{[2\quad 7]}{9}$$

$$q_1 = \frac{2}{9}$$

$$q_2 = \frac{7}{9}$$

Using the values obtained in (12) & (13)

$$Value\ of\ the\ game\ V\ = \begin{bmatrix} \frac{1}{9} & \frac{8}{9} \end{bmatrix} \begin{pmatrix} 15 & 7 \\ 8 & 9 \end{pmatrix} \begin{bmatrix} \frac{2}{9} \\ \frac{7}{9} \end{bmatrix}$$

$$= [0.11 \quad 0.88] \begin{pmatrix} 15 & 7 \\ 8 & 9 \end{pmatrix} \begin{bmatrix} 0.22 \\ 0.77 \end{bmatrix}$$

$$V = 8.51$$

3.2 Fuzzy game problem with payoffs as Trapezoidal Fuzzy Numbers

Consider the following payoff matrix with trapezoidal fuzzy numbers

$$\begin{array}{cc} & Player\ A \\ Player\ B & \begin{bmatrix} 5,10,12,17 & 1,2,4,5 \\ 1,4,5,6 & 4,5,9,14 \end{bmatrix} \end{array}$$

The following table gives the crisp values of Trapezoidal fuzzy numbers

Strategy	**Trapezoidal Fuzzy Numbers**	$\mu_{\tilde{a}} = \frac{\mathbf{2a+7b+7c+2d}}{\mathbf{18}}$
$\widetilde{a_{11}}$	(5, 10, 12, 17)	11
$\widetilde{a_{12}}$	(1, 2, 4, 5)	3
$\widetilde{a_{21}}$	(1, 4, 5, 6)	4.27
$\widetilde{a_{22}}$	(4, 5, 9, 11)	7.4

$$The\ payoff\ matrix = \begin{pmatrix} 11 & 3 \\ 4.27 & 7.4 \end{pmatrix}$$

The given fuzzy game is reduced into crisp game problem using ranking formula and solved by minimax-maximin criterian as follows

To find saddle point

Minimum of 1st row = 3

Minimum of 2nd row = 4.27

Maxi(min) = 4.27

Maximum of 1st column = 15

Maximum of 2nd column = 7.4

Mini(max) = 7.4

∴ saddle point does not exist.

By applying matrix method to find the value of the game.

$$P_{adj} = \begin{pmatrix} 7.4 & -3 \\ -4.27 & 11 \end{pmatrix}$$

$$P_{cof} = \begin{pmatrix} 7.4 & -4.27 \\ -3 & 11 \end{pmatrix}$$

$$Player\ A's\ optimal\ strategy = \frac{[1 \quad 1]\begin{pmatrix} 7.4 & -3 \\ -4.27 & 11 \end{pmatrix}}{[1 \quad 1]\begin{pmatrix} 7.4 & -3 \\ -4.27 & 11 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \qquad (14)$$

$$= \frac{[3.14 \quad 8]}{11.13}$$

$$P_1 = \frac{3.13}{11.13}$$

$$P_2 = \frac{8}{11.13}$$

$$Player\ B's\ optimal\ strategy = \frac{[1 \quad 1]\begin{pmatrix} 7.4 & -4.27 \\ -3 & 11 \end{pmatrix}}{[1 \quad 1]\begin{pmatrix} 7.4 & -4.27 \\ -3 & 11 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \quad (15)$$

$$= \frac{[4.4 \quad 6.73]}{11.13}$$

$$q_1 = \frac{4.4}{11.13}$$

$$q_2 = \frac{6.73}{11.13}$$

Using the values from equations (14) & (15)

$$Value\ of\ the\ game\ V = \begin{bmatrix} \frac{3.13}{11.13} & \frac{8}{11.13} \end{bmatrix} \begin{pmatrix} 11 & 3 \\ 4.27 & 7.4 \end{pmatrix} \begin{bmatrix} \frac{4.4}{11.13} \\ \frac{6.73}{11.13} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3.13}{11.13} & \frac{8}{11.13} \end{bmatrix} \begin{pmatrix} 11 & 3 \\ 4.27 & 7.4 \end{pmatrix} \begin{bmatrix} 0.39 \\ 0.60 \end{bmatrix}$$

$$V = 5.9$$

4. Conclusion

In this paper, the triangular and trapezoidal fuzzy numbers are converted in to crisp numbers by using β-distribution. Applying matrix method, the value of the game is obtained. This method is easy for execution.

5.References

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