Existence of Cyclic Kite Configuration and Its Application to the Newtonian Four-Body Problem for Mean Motion.

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Abstract

This paper deals with the existence of cyclic kite configuration in the Newtonian four body problem. By considering the center of mass of the system as the origin the coordinates of the four- point masses have been expressed in terms of radius R of the common circular orbit in three theorems. We have also proved that the masses lying on both sides of the axis of symmetry of the kite configuration are equal to the non-dimensional mass parameter μ then explicit and unique representations of other two masses in terms of the mass parameter μ and the total mass 'M' became possible. Further by taking angular velocity $\vec{\omega} = n\vec{k}$ and using first two theorems the equations of motion of each of the four point masses relative to the other three have been derived in synodic frame and hence the mean motion ' n' of the synodic frame was expressed as a function of mass parameter μ .Later on an attempt has been made for fixing the domain of μ .

Keywords: Cyclic kite configuration, mass parameter, axis of symmetry, mean motion.

Cyclic kite configuration: Kite configuration is a quadrilateral with two pairs of equal adjacent sides whereas in a cyclic kite configuration, the vertices of the kite lie on the circumference of a common circle.

1. Introduction

In the field of four body configuration MacMillon etal (1932) proved two theorems in detail for the existence of quadrilateral configuration. Brumberg(1957) studied permanent configuration of the Four-body problem. Albouy (1996) discussed the symmetric central configuration of equal masses. Long etal (2002) analyzed the four-body central configuration with the masses $(\beta,\beta,\alpha,\alpha)$ & $(\beta,\alpha,\beta,\beta)$ in which they proved that a convex non-collinear planar four body central configuration with three equal masses must be a kite.

Chavela etal (2007) invented that if in a convex four body central configuration two equal masses are located at opposite vertices of a quadrilateral and at most the mass of one of the remaining particles is larger than the equal masses then the configuration must be a kite shaped quadrilateral. Albouy etal (2008) established the relationship between the masses and some geometric properties of the quadrilateral configuration. Further they proved that the planar four body problem is a convex central configuration and is symmetric with respect to one diagonal if and only if masses of two particles on the other diagonal are equal

. During discussions of central configuration for the planar Newtonian four body problem, Pina etal (2009) developed an algorithm to construct general four-body configuration on dividing the directed areas by the corresponding scalar areas. These algorithms became important tools for finding the new properties of symmetric and non-symmetric central configuration. Pina (2010) developed the new coordinates of the four particles forming kite configuration in terms of principal moments of inertia and Eulerian angles independently. Cors etal (2012) studied cyclic central configuration of point masses in the Newtonian four body problem by using six mutual distances of the four-point masses as their coordinates and showed that the four-point masses form two dimensional plane surface. They extensively investigated two symmetric families, the kite configuration and the isosceles trapezoidal configuration. Further they have shown that if any two masses of a four-body cyclic central configuration are equal, then the configuration has a line of symmetry. Balint Erdi etal (2016) extended the work of Cors etal (2012) in three cases (one in convex case and two in concave cases) by expressing the masses of the central configuration in terms of angle coordinates Further they claimed that formulas derived represents the exact analytical solutions of the four-body problem.

Deng etal (2017) used mutual distances as the coordinates and proved that the diagonals of a cyclic central quadrilateral cannot be perpendicular except that the configuration is a kite. Corbera etal(2017) proved that the diagonals of a four body central configuration are perpendicular then the configuration must be a kite. Further they verified the same theorem in the four-vortex convex central configuration. Hassan et al (2017) studied planar Newtonian four-body problem and applied it to the different central configurations to develop some tools for further research.

Benhammouda etal (2020) identified the global regions in the mass parameter space by using action minimizing orbits of the central kite configuration in the four-body problem. Also, they presented the dynamical aspects of periodic solutions in the problem under consideration and showed that the minimizers of the classical action functional restricted to the hodographic solutions are nothing rather they are Keplerian elliptical solutions. Finally, they provided numerical explorations of the periodic and quazi-periodic solutions by Poin-Care section in broad sense.

From the discussions of the authors like Chavela (2007),Albouy (2008), Deng (2017) and Corbera (2017), it is found that a quadrilateral is a kite if and only if the diagonals are perpendicular to each other and at least one diagonal is the line of symmetry. In present work I have proved three existence theorems to establish unique expression of four masses forming kite configuration and these results were applied to rotating frame for finding mean motion**.**

From the definition; the cyclic kite configuration can be classified into three classes as follows.

2 (i) The kite formed by the combination of one equilateral triangle and one isosceles triangle with common base;

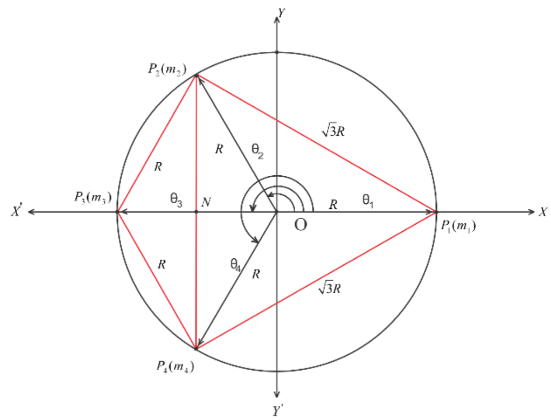
2(ii)The kite formed by the combination of two congruent right-angled isosceles triangles with common base and

2(iii)The kite formed by the combination of two non-congruent arbitrary isosceles triangles with common base.

2.Existence of kite configuration

Theorem 2.1: When the cyclic kite configuration is formed by the combination of one equilateral triangle and one isosceles triangle with the common base P_2NP_4.

Statement:-The necessary and sufficient conditions for the existence of cyclic kite configuration with four positive point masses $m_k\ (k=1,2,3,4)$ at the respective vertices P_k of the kite are that there exists a non-dimensional mass parameter $\mu \in \left(0, \frac{M}{3}\right)$ such that $m_2 = m_4 = \mu$,then , $m_1 = \frac{M-\mu}{2}, m_3 = \frac{M-3\mu}{2}$, where $m_1 + m_2 + m_3 + m_4 = M$.



Kite configuration of the system 2(*i*)
Figure−1

Proof. Let $P_1P_2P_3P_4$ be a cyclic kite formed by the combination of one equilateral triangle $P_1P_2P_4$ and one isosceles triangle

$P_2P_3P_4$ with the common base P_2NP_4 and axis of symmetry $P_1OP_3 = 2R$. Let P_k be the positions of the four positive point masses m_k moving on a common circular orbit with center at O, radius R and diameter $P_1OP_3 = 2R$ as the axis of symmetry. Considering the center of the circle as the origin, the axis of symmetry as the x-axis and a line YOY' parallel to P_2NP_4 and perpendicular to P_1OP_3 as the y- axis. Using the properties of cyclic quadrilateral $P_1P_2\ P_3P_4$ in the form of cyclic kite configuration and the cyclic equilateral triangle $P_1P_2P_4$, the coordinates of the four point masses can be written as $P_k(R\cos\theta_k, R\sin\theta_k)$, where $\theta_1 = 0^0, \theta_2 = 120^0$, $\theta_3 = 180^0, \theta_4 = 240^0$.

Thus, the position vectors of the four-point masses are given by $\vec{r}_k = R\cos\theta_k\hat{i} + R\sin\theta_k\hat{j} = \overrightarrow{OP_k}$

$$\Rightarrow \vec{r}_1 = R\hat{i}, \vec{r}_2 = -\frac{R}{2}\hat{i} + \frac{\sqrt{3}}{2}R\hat{j}, \vec{r}_3 = -R\hat{i}, \vec{r}_4 = -\frac{R}{2}\hat{i} - \frac{\sqrt{3}}{2}R\hat{j}.$$

The position vector of the center of mass of the system of four-point masses is given by

$$\vec{c} = M^{-1}\sum_{k=1}^{4} m_k\vec{r}_k.$$

If we consider the center' of mass as the origin, then $\vec{c} = \hat{0} \Rightarrow m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4 = \hat{0}$

$$\Rightarrow m_1R\hat{i} + m_2(-\frac{R}{2}\hat{i} + \frac{\sqrt{3}}{2}R\hat{j}) + m_3(-R\hat{i}) + m_4(-\frac{R}{2}\hat{i} - \frac{\sqrt{3}}{2}R\hat{j}) = 0$$

$$\Rightarrow R\{2(m_1 - m_3) - (m_2 + m_4)\}\hat{i} + \sqrt{3}R\{m_2 - m_4\}\hat{j} = 0$$

Taking scalar product of $\hat{i}$ and $\hat{j}$ we get

$$\left.\begin{aligned} 2(m_1 - m_3) - (m_2 + m_4) = 0 \\ m_2 = m_4 \end{aligned}\right\} \qquad (1)$$

Also, we know that

$$m_1 + m_2 + m_3 + m_4 = M \qquad (2)$$

The equations (1) and (2) yield

$$m_1 - m_2 - m_3 = 0 \qquad (3)$$

$$m_1 + 2m_2 + m_3 = M \qquad (4)$$

Let us introduce a non- dimensional mass parameter $\mu \in \left(0, \frac{M}{3}\right)$ such that $m_2 = m_4 = \mu$ then from (3) & (4) we get

$$m_1 = \frac{M - \mu}{2}, m_3 = \frac{M - 3\mu}{2}.$$

Thus, the necessary conditions for four positive point masses m_k situated at the vertices of a cyclic kite configuration formed by an equilateral triangle $P_1P_2P_4$ and an isosceles triangle $P_2P_3P_4$ are

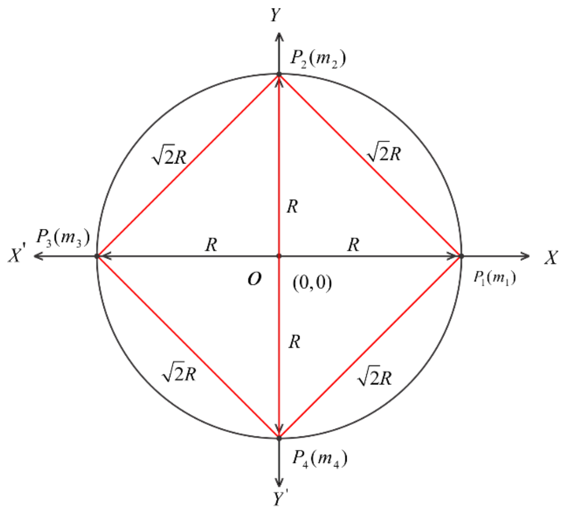
$$\left.\begin{aligned} &(a) m_2 = m_4 = \mu,\ m_1 = \frac{M-\mu}{2},\ m_3 = \frac{M-3\mu}{2} \\ &(b) m_1 = m_3 + 2\mu \\ &(c) P_1P_2 = P_2P_4 = P_1P_4 = \sqrt{3}R\ i.e.; r_{12} = r_{24} = r_{14} = \sqrt{3}R \\ &(d)\ P_2P_3 = P_3P_4 = R\ i.e.; r_{23} = r_{34} = R \\ &(e) P_1P_3 = 2R\ i.e.; r_{13} = 2R \end{aligned}\right\} \qquad (5)$$

Conversely the results of 5(a) , $m_2 = m_4 = \mu$ represents that the diagonal P_1OP_3 is the line of symmetry, 5(b) shows that $m_1 > m_3$ and from the results 5(c,d,e) $P_1P_2^{\,2} + NP_2^{\,2} = P_1N^2$ **i**.e.; the diagonals are perpendicular to each other. Thus the results of (5) represent the sufficient conditions for the existence of cyclic kite configuration**.**

Theorem 2.2: When the kite is formed by two congruent isosceles right-angled triangles**:**

Statement:- The necessary and sufficient conditions for the existence of a cyclic kite configuration with four positive point masses $m_k\ (k=1,2,3,4)$ at the respective vertices P_k of the kite are that there exists a non-dimensional mass parameter $\mu \in (0, M/2)$ such that $m_2 = m_4 = \mu$, then $m_1 = (M/2) - \mu = m_3$, where

$$m_1 + m_2 + m_3 + m_4 = M.$$



Kite configuration of the system 2(*ii*)
Figure−2

Proof: Let $P_1P_2P_3P_4$ be the cyclic kite formed by the combination two congruent isosceles right- angled triangles $P_1P_2P_4$ & $P_2P_3P_4$ with the common bas $P_2OP_4 = 2R$ and axis of symmetry $P_1OP_3 = 2R$ **.** Let P_k be the positions of the four positive point masses m_k on the cyclic kite ss .Taking the center O of the circle as the origin and two perpendicular diameters P_1OP_3 & P_2OP_4 as the x- axis and y-axis respectively, then the position vectors of the four point masses are $\overrightarrow{OP_1} = R\hat{i} = \vec{r}_1$, $\overrightarrow{OP_2} = R\hat{j} = \vec{r}_2$, $\overrightarrow{OP_3} = -R\hat{i} = \vec{r}_3$, $\overrightarrow{OP_4} = -R\hat{j} = \vec{r}_4$ respectively where $\hat{i}$ and $\hat{j}$ are the unit vectors along the x-axis and y –axis respectively .If we take the center of mass as the origin then by using theorem2.1 we have, $m_1R\hat{i} + m_2R\hat{j} - m_3R\hat{i} - m_4R\hat{j} = \hat{0}$,

Here either of P_1OP_3 *or* P_2OP_4 may be taken as the axis of symmetry.

Taking scalar product of $\hat{i}$ and $\hat{j}$ we get

$$m_1 = m_3\ \&\ m_2 = m_4 \qquad (6)$$

Introduction of (6) in (2) gives

$$m_1 + m_2 = M/2 \qquad (7)$$

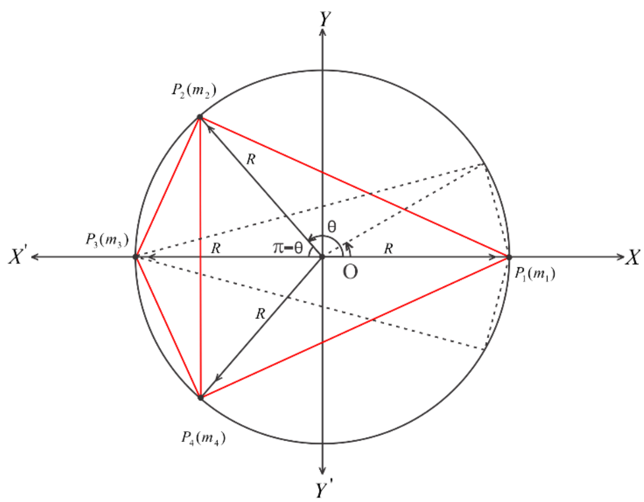
Let us introduce a non-dimensional mass parameter $\mu \in (0, M/2)$ such that $m_2 = m_4 = \mu$, then from (2) & (7) $m_1 = m_3 = (M/2) - \mu$. Both the masses of each pair are equal so both the diagonals are the axes of symmetry of the cyclic kite configuration .Thus the required necessary conditions for the existence of cyclic kite configuration are

$$\left.\begin{aligned} &(a) m_2 = m_4 = \mu,\ m_1 = m_3 = (M/2) - \mu, \\ &(b) m_1 = m_3 \geq m_4 = m_2, \\ &(c) P_1P_2 = P_2P_3 = P_3P_4 = P_1P_4 = \sqrt{2}R \text{ i.e.; } r_{12} = r_{23} = r_{34} = r_{14} = \sqrt{2}R \\ &(d)\ P_1P_3 = P_2P_4 = 2R \text{ i.e.; } r_{13} = r_{24} = 2R \end{aligned}\right\} \quad (8)$$

Conversely the results8(a) (similar to Long et al (2002)) and 8(b) justify that both the diagonals are the axes of symmetry and 8(c),(d) are the sufficient conditions for the perpendicularity of the diagonals.

Theorem 2.3: When the kite is formed by the combination of two non-congruent arbitrary isosceles triangles with 2(iii)

Statement: The necessary and sufficient conditions for the existence of cyclic kite configuration are that there exists a parameter $\mu \in \left(0, \frac{M}{2(1-\cos\theta)}\right)$ where $\theta \in (0,\pi)$ such that $m_2 = m_4 = \mu,$ then $m_1 = \frac{M}{2} - \mu(1+\cos\theta)$, $m_3 = \frac{M}{2} - \mu(1-\cos\theta)$, *where* $m_1 + m_2 + m_3 + m_4 = M.$



Kite configuration of the system 2(iii)
Figure−3

Proof Let $P_1P_2P_3P_4$ be a cyclic kite configuration formed by the combination of two non-congruent arbitrary isosceles triangles $P_1P_2P_4$ & $P_2P_3P_4$ with the common base P_2NP_4 . Let the four point masses m_1, m_2, m_3, m_4 be moving on a common circular orbit of radius R and center O. Considering $P_1P_2 = P_1P_4$ & $P_2P_3 = P_3P_4$ as the pair of adjacent equal sides.Let $< P_1OP_2 =< P_1OP_4 = \theta$ then $< P_2OP_3 =< P_4OP_3 = \pi - \theta$ .Taking the axis of symmetry P_1OP_3 as the x-axis, O as the origin (the bary–center) and the line YOY' as the y-axis then the position vectors of the four point masses are given by $\overrightarrow{OP_1} = R\hat{i} = \vec{r}_1, \overrightarrow{OP_2} = R\cos\theta\hat{i} + R\sin\theta\hat{j} = \vec{r}_2,$
$\overrightarrow{OP_3} = -R\hat{i} = \vec{r}_3, \overrightarrow{OP_4} = R\cos\theta\hat{i} - R\sin\theta\hat{j} = \vec{r}_4,$

As in previous theorems, $m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4 = \hat{0}$

$$\Rightarrow m_1R\hat{i} + m_2(R\cos\theta\hat{i} + R\sin\theta\hat{j}) + m_3(-R\hat{i}) + m_4(R\cos\theta\hat{i} - R\sin\theta\hat{j}) = \hat{0}$$

$$\Rightarrow R(m_1 + m_2\cos\theta - m_3 + m_4\cos\theta)\hat{i} + R(m_2 - m_4)\sin\theta\hat{j} = \hat{0}$$

By taking scalar products of $\hat{i}$ & $\hat{j}$ we get

$$m_1 + (m_2 + m_4)\cos\theta - m_3 = 0 \tag{9}$$

$$m_2 = m_4 \tag{10}$$

Collaboration of the equations (9) and (10) gives

$$m_1 + 2m_2\cos\theta - m_3 = 0 \tag{11}$$

Introduction of (10) in (2) gives

$$m_1 + 2m_2 + m_3 = M \tag{12}$$

Addition of (11) and (12) gives

$$m_1 + m_2(1+\cos\theta) = \frac{M}{2} \tag{13}$$

Let us introduce a parameter $\mu \in \left(0, \frac{M}{2(1-\cos\theta)}\right)$ such that $m_2 = m_4 = \mu$ where $\theta \in (0,\pi)$ then from(12) & (13) we get $m_1 = (M/2) - \mu(1+\cos\theta)$ & $m_3 = (M/2) - \mu(1-\cos\theta)$.

Thus, the necessary conditions for the existence of cyclic kite configuration are

$$\left.\begin{array}{ll}
(a) & m_2 = m_4 = \mu, \\
(b) & m_1 \geq m_2 = m_4 \geq m_3, \\
(c) & m_3 = m_1 + 2\mu\cos\theta, \\
(d) & m_1 = (M/2) - \mu(1+\cos\theta), \qquad m_3 = (M/2) - \mu(1-\cos\theta) \\
(e) & \left(\begin{array}{l} P_1P_2 = P_1P_4 = 2R\sin\frac{\theta}{2}, P_2P_3 = P_3P_4 = 2R\cos\frac{\theta}{2}, \\ P_2P_4 = 2R\sin\theta, P_1P_3 = 2R \end{array}\right)
\end{array}\right\} \tag{14}$$

Conversely the results 14(a) and 14(b) will justify that the line P_1OP_3 is a line of symmetry and the results of 14(e) will justify the perpendicularity of the diagonals of the kite.

2.4 Collapse of the kite configuration:

If the equal point masses lying on opposite sides of the axis of symmetry of the kite configuration are very closed to either of the point masses of the axis of symmetry, then collision may happen and hence the kite configuration may collapse. So there should some restricted domain of θ for existence of kite configuration. The necessary condition may be taken as $2P_2P_3 \geq R$

$i.e.; 4R\cos\frac{\theta}{2} \geq R \Rightarrow \cos\frac{\theta}{2} \geq \frac{1}{4} \Rightarrow \cos\frac{\theta}{2} \geq 0.25 \Rightarrow (14.5)^{\circ} \leq \frac{\theta}{2} \leq (75.5)^{\circ} \Rightarrow 29^{\circ} \leq \theta \leq 151^{\circ}$ Thus the kite configuration will exist for $\theta \in [\pi/6, 5\pi/6]$ and will collapse if $\theta \leq 29^{0}$ *and* $\theta \geq 151^{0}$ approximately.

3. Justification of the results of three theorems with the results of previous authors:

The above results of three theorems are the representations of the sufficient conditions for at least one line of symmetry and
for perpendicularity of the diagonals of the kite indeed we have verified some results of the previous authors like Long & Sun (2002), Chavela & Santoprete (2007), Albouy (2008), Pina & Lonngi (2009), Pina (2010), Roberts (2011) Roberts et al (2012) and Roberts (2017) etc. The results (a), (b), (c) of (5) justify the theories of Chavela and Santoprete (2007), Albouy (2008), Pina & Lonngi (2009) and Roberts (2011,2012) etc. The results of system 5(c,d,e) satisfy the conditions of Long &Sun (2009) given by:

$$(r_{24}^3 - r_{12}^3)(r_{23}^3 - r_{34}^3)(r_{13}^3 - r_{14}^3) = (r_{12}^3 - r_{14}^3)(r_{13}^3 - r_{34}^3)(r_{24}^3 - r_{23}^3). \quad (15)$$

This is the necessary and sufficient condition of four body planar central configuration that the six mutual distances r_{ij} determine a geometrically realizable planar configuration. The results of (5)vanish the Caley –Manger determinant.

$$i.e.; V(r) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix} = 3R^6 \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 2 \\ 1 & -2 & -1 & 1 \end{vmatrix} = -3R^6 \begin{vmatrix} -2 & 0 & 1 \\ 0 & -2 & -1 \\ 3 & -1 & -2 \end{vmatrix} = 0. \quad (16)$$

Thus the square of six mutual distances r_{ij} satisfy $V(r) = 0$ that is the volume of the tetrahedron formed the four point masses is zero i.e., cyclic kite configuration is planar.
Again

$$P(r) = r_{12}r_{34} + r_{14}r_{23} - r_{24}r_{13} = 0.$$

Thus, the four-point masses forming cyclic kite configuration and numbered sequentially lie on a common circle with diagonals, r_{13} & r_{24} where $r_{24} \leq r_{13}$.

The twice of the directed areas of the four parts of kite configuration $P_1P_2P_3P_4$ are given by

$$\left.\begin{aligned} \vec{S}_1 = \vec{r}_2 \wedge \vec{r}_3 + \vec{r}_3 \wedge \vec{r}_4 + \vec{r}_4 \wedge \vec{r}_2, \vec{S}_2 = \vec{r}_1 \wedge \vec{r}_4 + \vec{r}_4 \wedge \vec{r}_3 + \vec{r}_3 \wedge \vec{r}_1 \\ \vec{S}_3 = \vec{r}_1 \wedge \vec{r}_2 + \vec{r}_2 \wedge \vec{r}_4 + \vec{r}_4 \wedge \vec{r}_1, \vec{S}_4 = \vec{r}_1 \wedge \vec{r}_3 + \vec{r}_3 \wedge \vec{r}_2 + \vec{r}_2 \wedge \vec{r}_1 \end{aligned}\right\}$$

$$\Rightarrow \vec{S_1} = (-\frac{R}{2}\hat{i} + \frac{\sqrt{3}}{2}R\hat{j}) \wedge (-R\hat{i}) + (-R\hat{i}) \wedge (-\frac{R}{2}\hat{i} - \frac{\sqrt{3}}{2}R\hat{j}) + (-\frac{R}{2}\hat{i} - \frac{\sqrt{3}}{2}R\hat{j}) \wedge (-\frac{R}{2}\hat{i} + \frac{\sqrt{3}}{2}R\hat{j}) = \frac{\sqrt{3}}{2}R^2,$$

$$\Rightarrow \vec{S}_1 = \frac{\sqrt{3}}{2}R^2\hat{k}, \quad \vec{S}_2 = -\sqrt{3}R^2\hat{k}, \quad \vec{S}_3 = 3\frac{\sqrt{3}}{2}R^2\hat{k}, \quad \vec{S}_4 = -\sqrt{3}R^2\hat{k}. \quad (17)$$

Thus, the directed areas $\vec{S_1}$ & $\vec{S_3}$ are normals to the plane of motion of the point masses whereas $\vec{S_2}$ & $\vec{S_4}$ are the anti-normals to the plane of motion that is all directed areas are parallel to each other**.**

Also $$\sum_{k=1}^{4}\overrightarrow{S_k}=\overrightarrow{S_!}+\overrightarrow{S_2}+\overrightarrow{S}_3+\overrightarrow{S_4}=0. \qquad (18)$$

As $P_k(x_k,y_k)\Rightarrow P_1\equiv(R,0),P_2\equiv(-\frac{R}{2},\frac{\sqrt{3}}{2}R),P_3\equiv(-R,0),P_4\equiv(-\frac{R}{2},-\frac{\sqrt{3}}{2}R)$ then

$$\sum_{k=1}^{4}S_k x_k=\sum_{k=1}^{4}S_k y_k=0 \text{ and consequently } \sum_{k=1}^{4}S_k\vec{r}_k=0,$$

$$\sum_{m=1}^{4}\sum_{n=1}^{4}r_{mn}^2S_mS_n=\sum_{m=1}^{4}\sum_{n=1}^{4}\left|\vec{r}_m-\vec{r}_n\right|^2S_mS_n=\sum_{m=1}^{4}\sum_{n=1}^{4}(\vec{r}_m^2+\vec{r}_n^2-2\vec{r}_m.\vec{r}_n)S_mS_n$$
$$=\sum_{m=1}^{4}S_m\vec{r}_m^{\,2}\sum_{n=1}^{4}S_n+\sum_{m=1}^{4}S_m\sum_{n=1}^{4}S_n\vec{r}_n^{\,2}-2\sum_{m=1}^{4}S_m\vec{r}_m.\sum_{n=1}^{4}S_n\vec{r}_n$$
$$=\left(\sum_{m=1}^{4}S_m\vec{r}_m^{\,2}\right)0+0\left(\sum_{n=1}^{4}S_n\vec{r}_n^{\,2}\right)-2\left(\sum_{m=1}^{4}S_m\vec{r}_m\right).\left(\sum_{n=1}^{4}S_n\vec{r}_n\right)=0. \qquad (19)$$

where $$\vec{r}_m=x_m\hat{i}+y_m\hat{j}.$$

Using 5(c),(d),8(c) ,(d)and 14(e)

$$\left.\begin{aligned}\left|\vec{S}_1\right|&=\frac{r_{23}r_{34}r_{24}}{4R}=\frac{3\sqrt{3}}{2}R^2=\Delta_1,\left|\vec{S}_2\right|=\frac{r_{13}r_{34}r_{14}}{4R}=\sqrt{3}R^2=\Delta_2\\ \left|\vec{S}_3\right|&=\frac{r_{12}r_{24}r_{14}}{4R}=\frac{\sqrt{3}}{2}R^2=\Delta_3,\left|\vec{S}_4\right|=\frac{r_{12}r_{23}r_{13}}{4R}=\sqrt{3}R^2=\Delta_4.\end{aligned}\right\} \qquad (20)$$

$$\Rightarrow\Delta_1\geq\Delta_2=\Delta_4\geq\Delta_3. \qquad (21)$$

Now using the results of 5(c, d , e) and a matrix A we have

$$\left.\begin{aligned}\left[r_{23}^2r_{34}^2r_{24}^2\right]A\left[r_{23}^2r_{34}^2r_{24}^2\right]^T&=\left[4\Delta_1^2\right]\\ \left[r_{13}^2r_{34}^2r_{14}^2\right]A\left[r_{13}^2r_{34}^2r_{14}^2\right]^T&=\left[4\Delta_2^2\right]\\ \left[r_{12}^2r_{24}^2r_{14}^2\right]A\left[r_{12}^2r_{24}^2r_{14}^2\right]^T&=\left[4\Delta_3^2\right]\\ \left[r_{12}^2r_{23}^2r_{13}^2\right]A\left[r_{12}^2r_{23}^2r_{13}^2\right]^T&=\left[4\Delta_4^2\right]\end{aligned}\right\} where\ \ A=\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix} \qquad (22)$$

Thus, the results of 5(c, d e) justified the results of previous authors. In similar way all the results from (15) to (23) can be verified by the results of (8) and (14) also**.**

The partial derivatives of the Caley-Manger determinant with respect to the variables $r_{ij}^2(i\neq j=1,2,3,4)$ are given by

$$\frac{\partial V(r)}{\partial r_{ij}^2}=\left(-1\right)^{i+j-1}32\Delta_i\Delta_j. \qquad (23)$$

4.Mean motion

To develop the works of synodic kite configuration we need mean motion of the rotating frame .Let us find it by using theorems 2.1 and 2.2 separately .

4.1 **The mean motion using theorem 2.1**

Let the plane of motion of the four point masses rotate with the angular velocity $\vec{\omega} = n\hat{k}$ about the z-axis OZ ; $\hat{r}_{ij}$ be the unit vector along the line P_iP_j joining the i-th and j-th point masses ,then the equation of motion of the i-th point mass relative to the other three point masses is given by

$$-m_i n^2 \vec{r}_i = \sum_{j=1}^{4} \frac{\partial V_{ij}}{\partial r_{ij}} \hat{r}_{ij} \,, \qquad (\hat{i} \neq \hat{j}) \qquad (24)$$

where the potential V_{ij} between the i-th & j-th point masses and gravitational constant G are given by the relation

$$-V_{ij} = G \frac{m_i m_j}{|\vec{r}_{ij}|}, \qquad (25)$$

and the unit vector is given by $\hat{r}_{ij} = \frac{\vec{r}_{ij}}{|\vec{r}_{ij}|} = \frac{\overrightarrow{OP}_j - \overrightarrow{OP}_i}{|\overrightarrow{OP}_j - \overrightarrow{OP}_i|} = \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|}.$ $\qquad (26)$

The equations (25), (26) & (27) yield $n^2 \vec{r}_i + G \sum_{j=1}^{4} \frac{m_j \vec{r}_{ij}}{|\vec{r}_{ij}|^3} = 0, (i \neq j)$

$$\Rightarrow n^2 \vec{r}_i + G \sum_{j=1}^{4} \frac{m_j (\vec{r}_j - \vec{r}_i)}{|\vec{r}_{ij}|^3} = 0. \qquad (27)$$

Putting i=1 and expanding the summation on j we get

$$n^2 \vec{r}_1 + G \left[\frac{m_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_{12}|^3} + \frac{m_3 (\vec{r}_3 - \vec{r}_1)}{|\vec{r}_{13}|^3} + \frac{m_4 (\vec{r}_4 - \vec{r}_1)}{|\vec{r}_{14}|^3} \right] = 0 \qquad (28) .$$

Thus, the equations of motion of the system of four-point masses of the cyclic kite configuration are

$$\left.\begin{aligned}
&R^3 n^2 \vec{r}_1 + G\left[\frac{\mu(\vec{r}_2 - \vec{r}_1)}{3\sqrt{3}} + \frac{(M-3\mu)(\vec{r}_3 - \vec{r}_1)}{16} + \frac{\mu(\vec{r}_4 - \vec{r}_1)}{3\sqrt{3}} \right] = 0, \\
&R^3 n^2 \vec{r}_2 + G\left[\frac{(M-\mu)(\vec{r}_1 - \vec{r}_2)}{6\sqrt{3}} + \frac{(M-3\mu)(\vec{r}_3 - \vec{r}_2)}{2} + \frac{\mu(\vec{r}_4 - \vec{r}_2)}{3\sqrt{3}} \right] = 0, \\
&R^3 n^2 \vec{r}_3 + G\left[\frac{(M-\mu)(\vec{r}_1 - \vec{r}_3)}{16} + \mu(\vec{r}_2 - \vec{r}_3) + \mu(\vec{r}_4 - \vec{r}_3) \right] = 0, \\
&R^3 n^2 \vec{r}_4 + G\left[\frac{(M-\mu)(\vec{r}_1 - \vec{r}_4)}{6\sqrt{3}} + \frac{\mu(\vec{r}_2 - \vec{r}_4)}{3\sqrt{3}} + \frac{(M-3\mu)(\vec{r}_3 - \vec{r}_4)}{2} \right] = 0.
\end{aligned}\right\} \qquad (29)$$

Choosing units of mass, units of force, units of distances in such a way that

$M = m_1 + m_2 + m_3 + m_4 = 1, G = 1,$ maximum distance between the two-point masses as unity then $R = \frac{1}{2}, m_1 = \frac{1-\mu}{2}, m_3 = \frac{1-3\mu}{2}, m_2 = m_4 = \mu, \quad \mu \in (0, \frac{1}{3}).$

Arranging the terms of the above four equations as the linear combinations of position vectors we get

$$\left(\frac{n^2}{8} - \frac{2\mu}{3\sqrt{3}} - \frac{1-3\mu}{16}\right)\vec{r}_1 + \frac{\mu}{3\sqrt{3}}\vec{r}_2 + \frac{1-3\mu}{16}\vec{r}_3 + \frac{\mu}{3\sqrt{3}}\vec{r}_4 = 0, \quad (30)$$

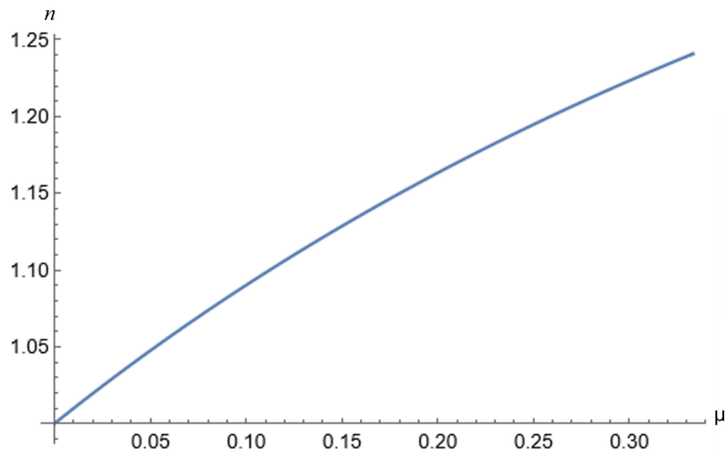
$$\left(\frac{1-\mu}{6\sqrt{3}}\right)\vec{r}_1 + \left(\frac{n^2}{8} - \frac{1+\mu}{6\sqrt{3}} - \frac{1-3\mu}{2}\right)\vec{r}_2 + \frac{1-3\mu}{2}\vec{r}_3 + \frac{\mu}{3\sqrt{3}}\vec{r}_4 = 0, \quad (31)$$

$$\left(\frac{1-\mu}{16}\right)\vec{r}_1 + \mu\vec{r}_2 + \left(\frac{n^2}{8} - \frac{1-\mu}{16} - 2\mu\right)\vec{r}_3 + \mu\vec{r}_4 = 0, \quad (32)$$

$$\left(\frac{1-\mu}{6\sqrt{3}}\right)\vec{r}_1 + \frac{\mu}{3\sqrt{3}}\vec{r}_2 + \frac{1-3\mu}{2}\vec{r}_3 + \left(\frac{n^2}{8} - \frac{1+\mu}{6\sqrt{3}} - \frac{1-3\mu}{2}\right)\vec{r}_4 = 0. \quad (33)$$

Here the above system of equations (30,31,32,33) represents the equations of motion of the four-point masses forming cyclic kite configuration in synodic frame. Eliminating $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$ from the above four equations we have

$$\begin{vmatrix} \frac{n^2}{8} - \frac{2\mu}{3\sqrt{3}} - \frac{1-3\mu}{16} & \frac{\mu}{3\sqrt{3}} & \frac{1-3\mu}{16} & \frac{\mu}{3\sqrt{3}} \\ \frac{1-\mu}{6\sqrt{3}} & \frac{n^2}{8} - \frac{1+\mu}{6\sqrt{3}} - \frac{1-3\mu}{2} & \frac{1-3\mu}{2} & \frac{\mu}{3\sqrt{3}} \\ \frac{1-\mu}{16} & \mu & \frac{n^2}{8} - \frac{1-\mu}{16} - 2\mu & \mu \\ \frac{1-\mu}{6\sqrt{3}} & \frac{\mu}{3\sqrt{3}} & \frac{1-3\mu}{2} & \frac{n^2}{8} - \frac{1+\mu}{6\sqrt{3}} - \frac{1-3\mu}{2} \end{vmatrix} = 0 \quad (34)$$



Graph between mass parameter versus mean motion

Figure -4 *system* $2(i)$

The equation (34) gives the mean motion 'n' of the synodic frame as a function of mass parameter μ. Using Mathematica 11in (34) the mean motion is given by $n = f_1(\mu)$. The graph in figure- 4 shows that the mean motion $'n'$ is a continuous monotonic increasing function of the mass parameter μ. The mean motion increases with the increase of the mass parameter μ.

4.2. The mean motion using theorem2.2

Similar to theorem 2.1, the equations of motion of four-point masses of the kite configuration of the theorem 2.2 are

$$8R^3n^2\vec{r}_1 + G\left[2\sqrt{2}\mu(\vec{r}_2-\vec{r}_1) + (\frac{M}{2}-\mu)(\vec{r}_3-\vec{r}_1) + 2\sqrt{2}\mu(\vec{r}_4-\vec{r}_1)\right] = 0,$$

$$8R^3n^2\vec{r}_2 + G\left[2\sqrt{2}(\frac{M}{2}-\mu)(\vec{r}_1-\vec{r}_2) + 2\sqrt{2}(\frac{M}{2}-\mu)(\vec{r}_3-\vec{r}_2) + \mu(\vec{r}_4-\vec{r}_2)\right] = 0,$$

$$8R^3n^2\vec{r}_3 + G\left[(\frac{M}{2}-\mu)(\vec{r}_1-\vec{r}_3) + 2\sqrt{2}\mu(\vec{r}_2-\vec{r}_3) + 2\sqrt{2}\mu(\vec{r}_4-\vec{r}_3)\right] = 0,$$

$$8R^3n^2\vec{r}_4 + G\left[2\sqrt{2}(\frac{M}{2}-\mu)(\vec{r}_1-\vec{r}_4) + \mu(\vec{r}_2-\vec{r}_4) + 2\sqrt{2}(\frac{M}{2}-\mu)(\vec{r}_3-\vec{r}_4)\right] = 0.$$

Choosing units as in case (i) we get $M=1, G=1, R=\frac{1}{2}$ and $P_1(\frac{1}{2},0), P_2(0,\frac{1}{2}), P_3(-\frac{1}{2},0), P_4(0,-\frac{1}{2})$

Rearrangement of above four equations yields

$$\left[n^2-(4\sqrt{2}-1)\mu-\frac{1}{2}\right]\vec{r}_1+2\sqrt{2}\mu\vec{r}_2+(\frac{1}{2}-\mu)\vec{r}_3+2\sqrt{2}\mu\vec{r}_4=0,$$

$$\sqrt{2}(1-2\mu)\vec{r}_1+\left[n^2+(4\sqrt{2}-1)\mu-2\sqrt{2}\right]\vec{r}_2+\sqrt{2}(1-2\mu)\vec{r}_3+\mu\vec{r}_4=0,$$

$$(\frac{1}{2}-\mu)\vec{r}_1+2\sqrt{2}\mu\vec{r}_2+\left[n^2-(4\sqrt{2}-1)\mu-\frac{1}{2}\right]\vec{r}_3+2\sqrt{2}\mu\vec{r}_4=0,$$

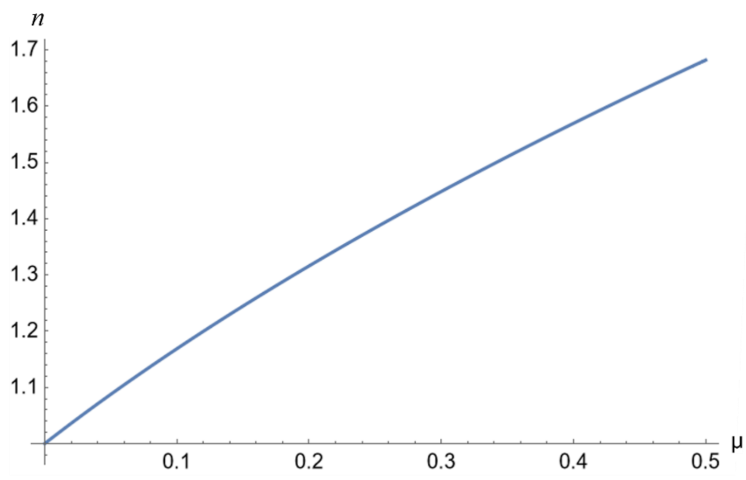
$$\sqrt{2}(1-2\mu)\vec{r}_1+\mu\vec{r}_2+\sqrt{2}(1-2\mu)\vec{r}_3+\left[n^2+(4\sqrt{2}-1)\mu-2\sqrt{2}\right]\vec{r}_4=0.$$

Eliminating $\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4$ from the above equations, we get

$$\begin{vmatrix} a & 2\sqrt{2}\mu & 1/2-\mu & 2\sqrt{2}\mu \\ \sqrt{2}(1-2\mu) & b & \sqrt{2}(1-2\mu) & \mu \\ 1/2-\mu & 2\sqrt{2}\mu & a & 2\sqrt{2}\mu \\ \sqrt{2}(1-2\mu) & \mu & \sqrt{2}(1-2\mu) & b \end{vmatrix}=0 \qquad (35)$$

where $a=n^2-(4\sqrt{2}-1)\mu-1/2 \quad b=n^2+(4\sqrt{2}-1)\mu-2\sqrt{2}.$

This equation gives different values of the mean motion 'n' as a function of the mass parameter μ. Those values of μ for which n is positive are valid.



Graph between mass parameter versus mean motion

Figure – 5 system 2(ii)

The equation (35) gives the mean motion 'n' of the synodic frame as a function of mass parameter μ. Using Mathematica 11in (35) the mean motion is given by $n = f_2(\mu)$. The graph in figure- 5 shows that the mean motion $'n'$is a continuous monotonic increasing function of the mass parameter μ but the rate of increase of 'n'is low in comparison of the figure-**4.**

For $\mu \in (0,3^{-1})$ the rate of increase of' ' n' is higher than that of for $\mu \in (0,2^{-1})$.

4.3. Exact domain of the mass parameter μ using theorem2.1:

If $\mu = 0$, then $m_2 = m_4 = 0 \,\&\, m_1 = m_3 = 1/2$ then the four-body cyclic kite configuration will be reduced to two body straight line configuration lying on the axis of symmetry P_1OP_3 i.e.; $\mu = 0$ is a contradiction for the existence of cyclic kite configuration .
Thus from (34) when $\mu = 0$ we get

$$\begin{vmatrix} \frac{n^2}{8}-\frac{1}{16} & 0 & \frac{1}{16} & 0 \\ \frac{1}{6\sqrt{3}} & \frac{n^2}{8}-\frac{1}{6\sqrt{3}}-\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{16} & 0 & \frac{n^2}{8}-\frac{1}{16} & 0 \\ \frac{1}{6\sqrt{3}} & 0 & \frac{1}{2} & \frac{n^2}{8}-\frac{1}{6\sqrt{3}}-\frac{1}{2} \end{vmatrix} = 0 \Rightarrow n = 0,1,2.354, \qquad (36)$$

Here $n = 0, 2.354 \notin (f_1(0), f_1(3^{-1}))$ (ref. graph of figure -4, so the cyclic straight-line configuration is rotating with mean motion n=1.
If we put $\mu = 1/3$ then $m_1 = m_2 = m_4 = 1/3 \,\&\, m_3 = 0$,then the cyclic kite configuration converges to the cyclic equilateral triangular configuration and hence a contradiction for the existence of a cyclic kite configuration.
From (34) when $\mu = 1/3$ we have

$$\begin{vmatrix} \frac{n^2}{8}-\frac{2}{9\sqrt{3}} & \frac{1}{9\sqrt{3}} & 0 & \frac{1}{9\sqrt{3}} \\ \frac{1}{9\sqrt{3}} & \frac{n^2}{8}-\frac{2}{9\sqrt{3}} & 0 & \frac{1}{9\sqrt{3}} \\ \frac{1}{24} & \frac{1}{3} & \frac{n^2}{8}-\frac{17}{24} & \frac{1}{3} \\ \frac{1}{9\sqrt{3}} & \frac{1}{9\sqrt{3}} & 0 & \frac{n^2}{8}-\frac{2}{9\sqrt{3}} \end{vmatrix} = 0 \Rightarrow n = 0,1.241,2.38, \qquad (37)$$

Here $n = 0 \,\&\, n = 2.38$ are invalid so the cyclic equilateral triangular configuration is rotating with mean motion $n = 1.241$. Thus for $\mu = 0$ & $\mu = 1/3$ the kite configuration does not exist; consequently the exact domain of μ in the system 2(i) is the open interval $(0,3^{-1})$

4.4. Exact domain of the mass parameter μ using theorem 2.2:

If we put $\mu = 0$, then $m_2 = m_4 = 0 \,\&\, m_1 = m_3 = 1/2$ and the four-body cyclic kite configuration will be reduced to two body straight line configuration lying on the axis of symmetry P_1OP_3 i.e.; a contradiction for the existence of kite configuration .

From (35) when $\mu = 0$

$$\begin{vmatrix} n^2-1/2 & 0 & 1/2 & 0 \\ \sqrt{2} & n^2-2\sqrt{2} & \sqrt{2} & 0 \\ 1/2 & 0 & n^2-1/2 & 0 \\ \sqrt{2} & 0 & \sqrt{2} & n^2-2\sqrt{2} \end{vmatrix} = 0 \Rightarrow n = 0,1,1.682. \quad (38)$$

Here $n = 0,1.682 \notin(f_1(0), f_1(2^{-1}))$ so the cyclic straight- line configuration lying on axis of symmetry is rotating with the mean motion $n = 1$.

If we put $\mu = 1/2$ then $m_2 = m_4 = 1/2 \,\&\, m_1 = m_3 = 0$ then the cyclic kite configuration again will be reduced to the cyclic straight-line configuration lying on the y-axis i.e.; a contradiction for the existence of cyclic kite configuration.

From Equation (35) when $\mu = 1/2$ we get

$$\begin{vmatrix} n^2-2\sqrt{2} & \sqrt{2} & 0 & \sqrt{2} \\ 0 & n^2-1/2 & 0 & 1/2 \\ 0 & \sqrt{2} & n^2-2\sqrt{2} & \sqrt{2} \\ 0 & 1/2 & 0 & n^2-1/2 \end{vmatrix} = 0 \Rightarrow n = 0\ ,1,1.674 \quad (39)$$

Here $n = 0,1.674$ are invalid **(**similar to (38)**)** so the cyclic straight- line configuration lying on the y- axis is rotating with the mean motion $n = 1$.Thus for $\mu = 1\backslash 2$, the kite configuration of the system 2(ii) does not exist ,so the exact domain of μ is the open interval $\left(0, 2^{-1}\right)$.

5.Conclusion: For the first time in section-2; I have established the necessary and sufficient conditions for the existence of cyclic kite configuration by using the definition of central configuration (i.e.; center of mass of the system be taken as the origin) in three theorems. The four masses forming kite configuration were uniquely expressed in three theorems separately. Among the four masses of the kite two are of equal masses and situated at equal distances from the axis of symmetry but on opposite sides of it. Other two masses lying on the axis of symmetry are of different masses in the first theorem. The range of μ in the first theorem is the open interval (0,1/3) whereas in the second theorem it is (0,1/2) so the pair of opposite particles are of equal masses. In section 2.4 the values of θ give the range of existence and collapsing range of the cyclic kite configuration. For $\theta \in [\pi/6, 5\pi/6]$ the kite exists and $0 \le \theta \le 29^0$ *and* $151^0 \le \theta \le 180^0$ approximately are the collapsing range of the kite. In section -3 many results of previous authors have been well verified by the results of our three theorems. The righthand side of equation (23) differs radically with the results of previous authors. To study the motion of a satellite in the gravitational field of a kite configuration, the mean motion of the rotating frame has been established by using first two theorems separately in section -4. During the justification of exact domain of the mass parameter μ ,it was found that only two body systems can rotate with unit mean motion but not the higher order systems. The higher order systems can rotate with the mean motion greater than unity. According to G.E. Roberts (2011), the satellite motion in the gravitational field of a cyclic kite can be studied and further the existence and stability of Lagrangian points also in future. For this the masses $\frac{1-\mu}{2}, \mu, \frac{1-3\mu}{2}, \mu$ *and* n are very important tools **.**

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