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Novelty of Implicative Filters in Lattice Pseudo Wajsberg Algebras

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Abstract
The NIm.Fi (Normal Implicative Filter) and NFIm.Fi (Normal Fuzzy Implicative Filter) of LPWA (Lattice pseudo-Wajsberg Algebra) are introduced in this work, and we use its illustrations to explore some of their related properties.

Keywords: FIm.Fi , LPWA, NIm.Fi , FNIm.Fi, Cartesian product.

Mathematical Subject classification: 30G10

1. Introduction

PWA's were introduced by Rodica Ceterchi[1]. Wajsberg algebras give rise to PWA's(Pseudo-Wajsberg algebras). In this article, we define NIm.Fi's of LPWA and discuss their characteristics. We also look into the fuzzification of their Cartesian and normal product.

2. NIm.Fi OF LPWA

The references [1, 2, 3], [4, 5, 6, 7, 8], and [9, 10] are used for all of the fundamental definitions.

A NIm.Fi is a non-empty subset of LPWA $\mathcal{A}_1$ defined by $\mathcal{F}_1$.Then it satisfies $\acute{x} \longmapsto \acute{y}$ iff $\acute{x} \rightsquigarrow \acute{y} \in \mathcal{F}_1 \forall\, \acute{x}, \acute{y} \in \mathcal{A}_1$.

2.1. Example: The quasi complements and binary operations of the poset $\mathcal{A}_1 = \{0, p_1, q_1, 1\}$ with $0 \leq p_1 \leq q_1 \leq 1$. (Refer table in [5])

The table [5] shows that, $\mathcal{F}_{1_1} = \{0,1\}$ is a NIm.Fi of $\mathcal{A}_1$.

But $\mathcal{F}_{1_2} = \{p_1, q_1\}$ is not a NIm.Fi of $\mathcal{A}_1$.

Since, $(p_1 \longmapsto q_1) = 1 \notin \mathcal{F}_{1_2}$& $(p_1 \rightsquigarrow q_1) = 1 \notin \mathcal{F}_{1_2}$.

2.2. Proposition: Let $\mathcal{F}_1$ be a non-empty subset of $\mathcal{A}_1$ and $\mathcal{A}_1$ be a LPWA. Then $\mathcal{F}_1$ is a NIm.Fi of $\mathcal{A}_1$ iff the following circumstances are true $\forall\, \acute{x}, \acute{y}, \acute{z} \in \mathcal{A}_1$.

(i) $1 \in \mathcal{F}_1$

(ii) $\acute{y} \in \mathcal{F}_1$& $\acute{x} \longmapsto (\acute{y} \rightsquigarrow \acute{z}) \in \mathcal{F}_1 \Rightarrow \acute{x} \rightsquigarrow \acute{z} \in \mathcal{F}_1$

(iii) $\acute{y} \in \mathcal{F}_1$& $\acute{x} \rightsquigarrow (\acute{y} \longmapsto \acute{z}) \in \mathcal{F}_1 \Rightarrow \acute{x} \longmapsto \acute{z} \in \mathcal{F}_1$.

***Proof for* (i):** Fix, $\acute{x} \longmapsto (\acute{y} \rightsquigarrow \acute{z}) \in \mathcal{F}_1$ and $\acute{y} \in \mathcal{F}_1$

We have $\acute{y} \rightsquigarrow (\acute{x} \longmapsto \acute{z}) = \acute{x} \longmapsto (\acute{y} \rightsquigarrow \acute{z}) \in \mathcal{F}_1$ (By reference [1])

Thus $\acute{y} \rightsquigarrow (\acute{x} \longmapsto \acute{z}) \in \mathcal{F}_1 \Rightarrow \acute{x} \rightsquigarrow \acute{z} \in \mathcal{F}_1$ (By reference [3])

In similar manner,$(\acute{x} \longmapsto \acute{z}) \in \mathcal{F}_1$.

Conversely, suppose that $\mathcal{F}_1$ satisfies (i),(ii) and (iii)

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To prove $\mathcal{F}_1$ is a NIm.Fi of LPWA $\mathcal{A}_1$.
Let $\acute{x} \in \mathcal{F}_1, \acute{x} \mapsto \acute{y} \in \mathcal{F}_1$ then both $\acute{x}$, $1 \rightsquigarrow (\acute{x} \mapsto \acute{y}) \in \mathcal{F}_1$ (By reference [1])
From (ii), we have $1 \mapsto \acute{y} \in \mathcal{F}_1$
So $\acute{y} \in \mathcal{F}_1$. (By reference [3])
Let $\acute{x} \mapsto \acute{y} \in \mathcal{F}_1$, to prove $\acute{x} \rightsquigarrow \acute{y} \in \mathcal{F}_1$
Let $\acute{x} \leq (\acute{x} \mapsto \acute{y}) \rightsquigarrow \acute{y}$ and by reference [1], we have $\acute{x} \mapsto \big((\acute{x} \mapsto \acute{y}) \rightsquigarrow \acute{y}\big) = 1 \in \mathcal{F}_1$
From (ii), we have $\acute{x} \rightsquigarrow \acute{y} \in \mathcal{F}_1$
Let $\acute{x} \rightsquigarrow \acute{y} \in \mathcal{F}_1$; $\acute{x} \leq (\acute{x} \rightsquigarrow \acute{y}) \mapsto \acute{y}$ then by reference [1], $\acute{x} \rightsquigarrow \big((\acute{x} \rightsquigarrow \acute{y}) \mapsto \acute{y}\big) = 1 \in \mathcal{F}_1$
From (iii), $\acute{x} \mapsto \acute{y} \in \mathcal{F}_1$
Thus $\mathcal{F}_1$ is a NIm.Fi of LPWA.

2.3. Proposition: If an Im.Fi $\mathcal{F}_1$ of $\mathcal{A}_1$ is a NIm.Fi, then the following conditions hold $\forall\, \acute{x}, \acute{y} \in \mathcal{A}_1$
(i) $\acute{x} \in \mathcal{F}_1 \,\&\, (\acute{x} \mapsto \acute{y}) \mapsto \acute{y} \in \mathcal{F}_1$
(ii) $\acute{x} \in \mathcal{F}_1 \,\&\, (\acute{x} \rightsquigarrow \acute{y}) \rightsquigarrow \acute{y} \in \mathcal{F}_1$.
***Proof* for (i):** If $\mathcal{F}_1$ is a NIm.Fi of LPWA $\mathcal{A}_1, \acute{x} \in \mathcal{F}_1$ and $\acute{x} \leq (\acute{x} \rightsquigarrow \acute{y}) \rightsquigarrow \acute{y}$ then by reference [1], we have $(\acute{x} \rightsquigarrow \acute{y}) \rightsquigarrow \acute{y} \in \mathcal{F}_1$
From the definition of NIm.Fi , we have $(\acute{x} \mapsto \acute{y}) \mapsto \acute{y} \in \mathcal{F}_1$
In similar manner, we can prove that $(\acute{x} \rightsquigarrow \acute{y}) \rightsquigarrow \acute{y} \in \mathcal{F}_1$.

3. NFIm.Fi of LPWA
An Inconstant FIm.Fi ϕ_1 of LPWA $\mathcal{A}_1$ is known as a normal fuzzy, if $\phi_1(\acute{x}) = \phi_1(1)$ $\forall$ $x_1 \in \mathcal{A}_1$.

3.1. Example: Take the collection $\mathcal{A}_1 = \{0, s_1, t_1, u_1, 1\}$. Create a partial ordering "≤" on $\mathcal{A}_1$ that includes the binary operations " $\mapsto$ " , " $\rightsquigarrow$ " and quasi complements " $^{-}$ " , " $^{\sim}$ " such that $0 \leq s_1 \leq t_1,\ u_1 \leq 1$.
(i) Take into consideration a fuzzy ϕ_1 on $\mathcal{A}_1$ as, $\phi_1(\acute{x}) = \begin{cases} 1 & \text{if } \acute{x} = 1 \\ 0.4 & \text{Otherwise} \end{cases}$ $\forall\ \acute{x} \in \mathcal{A}_1$.
Then, ϕ_1 is a NIm.Fi of LPWA $\mathcal{A}_1$. Refer table in [5]
(ii) If $\phi_1(\acute{x}) = \begin{cases} 1 & \text{if } \acute{x} = 1 \\ 0.2 & \text{Otherwise} \end{cases}$ $\forall\, \acute{x} \in \mathcal{A}_1$. Then, ϕ_1 is not a NFIm.Fi of LPWA $\mathcal{A}_1$.
Refer table in [5]

3.2. Proposition: Define a fuzzy set $\phi_1{}^{\star}$ in $\mathcal{A}_1$ as $\phi_1{}^{\star}(\acute{x}) = \phi_1(\acute{x}) + 1 - \phi_1(1)$ $\forall\, \acute{x} \in \mathcal{A}_1$. Then $\phi_1{}^{\star}$ is NFIm.Fi ϕ_1 of $\mathcal{A}_1$ such that $\phi_1 \subseteq \phi_1{}^{\star}$ with ϕ_1 is a NFIm.Fi of LPWA.
***Proof*.** To demonstrate that, $\phi_1{}^{\star}$ is NFIm.Fi of LPWA $\mathcal{A}_1$.
(i) Let $\phi_1{}^{\star}(1) = \phi_1(1) + 1 - \phi_1(1) = 1 \geq \phi_1{}^{\star}(\acute{x})$
$$\phi_1{}^{\star}(1) \geq \phi_1{}^{\star}(\acute{x})$$
(ii) To prove $\phi_1{}^{\star}(\acute{y}) = \phi_1(\acute{y}) \geq \min\{\phi_1{}^{\star}(\acute{x} \mapsto \acute{y}),\ \phi_1{}^{\star}(\acute{x})\}$
Now $\phi_1{}^{\star}(\acute{y}) = \phi_1(\acute{y}) + 1 - \phi_1(1)$
$$\geq \min\big\{\big(\phi_1(\acute{x} \mapsto \acute{y}), \phi_1(\acute{x})\big)\big\} + 1 - \phi_1(1)$$ (By reference [4])
$$= \min\{\phi_1(\acute{x}) + 1 - \phi_1(1), \phi_1(\acute{x} \mapsto \acute{y}) + 1 - \phi_1(1)\}$$
Thus $\phi_1{}^{\star}(\acute{y}) = \min\{\phi_1{}^{\star}(x_1 \mapsto y_1), \phi_1{}^{\star}(\acute{x})\}$
(iii) To prove that $\phi_1{}^{\star}(\acute{y}) = \phi_1(\acute{y}) \geq \min\{\phi_1{}^{\star}(\acute{x} \rightsquigarrow \acute{y}), \phi_1{}^{\star}(\acute{x})\}$
Now $\phi_1{}^{\star}(\acute{y}) = \phi_1(\acute{y}) + 1 - \phi_1(1)$
$$\geq \min\big\{\big(\phi_1(\acute{x} \rightsquigarrow \acute{y}), \phi_1(\acute{x})\big)\big\} + 1 - \phi_1(1)$$ (By reference [4])
$$= \min\{\phi_1(\acute{x}) + 1 - \phi_1(1), \phi_1(\acute{x} \rightsquigarrow \acute{y}) + 1 - \phi_1(1)\}$$
$\phi_1{}^{\star}(\acute{y}) = \min\{\phi_1{}^{\star}(\acute{x} \rightsquigarrow \acute{y}), \phi_1{}^{\star}(\acute{x})\}$
Thus $\phi_1{}^{\star}$ is a FIm.Fi of LPWA $\mathcal{A}_1$.
Clearly $\phi_1{}^{\star}(\acute{x}) = \phi_1{}^{\star}(1)$ $\forall \acute{x} \in \mathcal{A}_1$.
$\phi_1{}^{\star}$ is NFIm.Fi of LPWA $\mathcal{A}_1$.
Thus, it is obvious that $\phi_1(\acute{x}) \subseteq \phi_1{}^{\star}(\acute{x})$ $\forall\, \acute{x} \in \mathcal{A}_1$.

3.3. Remark: Let $(\mathcal{A}_1, \longmapsto, \rightsquigarrow, ^{-}, ^{\sim}, 1)$ be a LPWA and $\hbar_1: X_1 \longmapsto Y_1$ be an onto homomorphism for any FIm.Fi of ϕ_1 in Y, define a mapping ${\phi_1}^{\hbar_1}: X_1 \longmapsto [0,1]$ such that ${\phi_1}^{\hbar_1}(\acute{x}) = \phi_1(\hbar_1(\acute{x}))\ \forall\ \acute{x} \in \mathcal{A}_1$.

3.4. Proposition: Let $\mathcal{A}_1$ be a LPWA and $\hbar_1: X_1 \longmapsto Y_1$be an onto homomorphism for any FIm.Fi of ϕ_1 in Y_1, define a mapping ${\phi_1}^{\hbar_1}: X_1 \longmapsto [0,1]$ such that ${\phi_1}^{\hbar_1}(\acute{x}) = \phi_1(\hbar_1(\acute{x}))\quad \forall\ \acute{x} \in \mathcal{A}_1$. Then $\phi_1$is a NFIm.Fi of LPWA $\mathcal{A}_1$ iff ${\phi_1}^{\hbar_1}$ is a NFIm.Fi of LPWA $\mathcal{A}_1$.

Proof.

(i) ${\phi_1}^{\hbar_1}(1) = \phi_1(\hbar_1(1)) = \phi_1(1) \geq \phi_1(\hbar_1(\acute{x})) = {\phi_1}^{\hbar_1}(\acute{x})\ \forall\ \acute{x} \in \mathcal{A}_1$

(ii)${\phi_1}^{\hbar_1}(\acute{y}) = \phi_1(\hbar_1(\acute{y})) \geq \min\{\phi_1(\hbar_1(\acute{x}) \longmapsto \hbar_1(\acute{y})), \phi_1(\hbar_1(\acute{x}))\}$

$= \min\{\phi_1(\hbar_1(\acute{x} \longmapsto \acute{y})), \phi_1(\hbar_1(\acute{x}))\}$

${\phi_1}^{\hbar_1}(\acute{y}) = \min\{{\phi_1}^{\hbar_1}(\acute{x} \longmapsto \acute{y}), {\phi_1}^{\hbar_1}(\acute{x})\}\ \forall\ \acute{x}, \acute{y} \in \mathcal{A}_1$

(iii)${\phi_1}^{\hbar_1}(\acute{y}) = \phi_1(\hbar_1(\acute{y})) \geq \min\{\phi_1(\hbar_1(\acute{x}) \rightsquigarrow \hbar_1(\acute{y})), \phi_1(\hbar_1(\acute{x}))\}$

$= \min\{\phi_1(\hbar_1(\acute{x} \rightsquigarrow \acute{y})), \phi_1(\hbar_1(\acute{x}))\}$

${\phi_1}^{\hbar_1}(\acute{y}) = \min\{{\phi_1}^{\hbar_1}(\acute{x} \rightsquigarrow \acute{y}), {\phi_1}^{\hbar_1}(\acute{x})\}\ \forall\ \acute{x}, \acute{y} \in \mathcal{A}_1$.

Hence ${\phi_1}^{\hbar_1}$ is a FIm.Fi of LPWA $\mathcal{A}_1$

To prove ${\phi_1}^{\hbar_1}$ is a NFIm.Fi of LPWA $\mathcal{A}_1$, We have ${\phi_1}^{\hbar_1}(\acute{y}) = \phi_1(\hbar_1(1)) = {\phi_1}^{\hbar_1}(1) = 1$

Thus ${\phi_1}^{\hbar_1}$is NFIm.Fi of LPWA $\mathcal{A}_1$

Conversely, suppose ${\phi_1}^{\hbar_1}$ is a NFIm.Fi of LPWA $\mathcal{A}_1$

To prove $\phi_1$is a NFIm.Fi of LPWA $\mathcal{A}_1$

(i) If $\hbar_1$ is onto, then there exist $\acute{x} \in \mathcal{A}_1$ such that $\hbar_1(\acute{y}) = \acute{x}$.

We have

$$\phi_1(1) = \phi_1(\hbar_1(1)) = {\phi_1}^{\hbar_1}(1) \geq {\phi_1}^{\hbar_1}(\acute{y}) = {\phi_1}^{\hbar_1}(\hbar_1(\acute{y})) = \phi_1(\acute{x}) \forall\ \acute{x}, \acute{y} \in \mathcal{A}_1$$

(ii) If $\hbar_1$ is onto, then there exist $\acute{x}, \acute{y} \in \mathcal{A}_1$such that $\hbar_1(a) = \acute{x}$ and $\hbar_1(a) = \acute{y}$

We have $\phi_1(\acute{y}) = \phi_1(\hbar_1(b)) = {\phi_1}^{\hbar_1}(b) \geq \min\{{\phi_1}^{\hbar_1}(a \longmapsto b), {\phi_1}^{\hbar_1}(a)\}$

$= \min\{\phi_1(\hbar_1(a) \longmapsto \hbar_1(b)), \phi_1(\hbar_1(a))\}$

$\phi_1(\acute{y}) = \min\{\phi_1(\acute{x} \longmapsto \acute{y}), \phi_1(\acute{x})\}\ \forall\ \acute{x}, \acute{y} \in \mathcal{A}_1$.

(iii) If $\hbar_1$is onto, then there exist $\acute{x}, \acute{y} \in \mathcal{A}_1$ such that $\hbar_1(a) = \acute{x}$ and $\hbar_1(a) = \acute{y}$

We have $\phi_1(\acute{y}) = \phi_1(\hbar_1(b)) = {\phi_1}^{\hbar_1}(b) \geq \min\{{\phi_1}^{\hbar_1}(a \rightsquigarrow b), {\phi_1}^{\hbar_1}(a)\}$

$= \min\{\phi_1(\hbar_1(a) \rightsquigarrow \hbar_1(b)), \phi_1(\hbar_1(a))\}$

$\phi_1(\acute{y}) = \min\{\phi_1(\acute{x} \rightsquigarrow \acute{y}), \phi_1(\acute{x})\}\ \forall\ \acute{x}, \acute{y} \in \mathcal{A}_1$.

Hence $\phi_1$is FIm.Fi of LPWA $\mathcal{A}_1$.

We have $\phi_1(1^1) = \phi_1((1)) = {\phi_1}^{\hbar_1}(1) = 1$ (since ${\phi_1}^{\hbar_1}$ is normal)

Thus ϕ_1 is NFIm.Fi of LPWA $\mathcal{A}_1$.

4. Cartesian product of FIm.Fi

4.1. Proposition: Let $\phi_1$and ψ_1 be two FIm.Fi 's of a LPWA $\mathcal{A}_1$. Then $\phi_1 \times \psi_1$ is a FIm.Fi in $\mathcal{A}_1 \times \mathcal{A}_1$.

Proof.

Let $(\acute{x}, \acute{y}) \in \mathcal{A}_1 \times \mathcal{A}_1$. Since $\phi_1$and ψ_1 be two FIm.Fi 's in $\mathcal{A}_1$.

We have $(\phi_1 \times \psi_1)(1,1) = \min\{\phi_1(1), \psi_1(1)\} \geq \min\{\phi_1(\acute{x}), \psi_1(\acute{y})\}\ \forall\ \acute{x}, \acute{y} \in \mathcal{A}_1$

$(\phi_1 \times \psi_1)(1,1) = (\phi_1 \times \psi_1)(\acute{x}, \acute{y})$

Let$(\acute{x}, \acute{x}^\star), (\acute{y}, \acute{y}^\star) \in \mathcal{A}_1 \times \mathcal{A}_1$

Clearly$(\acute{x} \longmapsto \acute{y}, \acute{x}^\star \longmapsto \acute{y}^\star) = (\acute{x}, \acute{x}^\star) \longmapsto (\acute{y}, \acute{y}^\star); (\acute{x}, \acute{x}^\star) \rightsquigarrow (\acute{y}, \acute{y}^\star)$

$(\phi_1 \times \psi_1)(\acute{x}, \acute{y}) = \min\{\phi_1(\acute{x}), \psi_1(\acute{y})\}$

$(\phi_1 \times \psi_1)(\acute{y}, \acute{y}^\star) = \min\{\phi_1(\acute{y}), \psi_1(\acute{y}^\star)\}$

$= \min\{\min\{\phi_1(\acute{x}), \phi_1(\acute{x} \longmapsto \acute{y})\}, \min\{\psi_1(\acute{x}^\star), \psi_1(\acute{x}^\star \longmapsto \acute{y}^\star)\}\}$

$= \min\{\min\{\phi_1(\acute{x}), \psi_1(\acute{y}^\star)\}, \min\{\phi_1(\acute{x} \longmapsto \acute{y}), \psi_1(\acute{x}^\star \longmapsto \acute{y}^\star)\}\}$

$= \min\{(\phi_1 \times \psi_1)(\acute{x}, \acute{x}^\star), (\phi_1 \times \psi_1)(\acute{x} \longmapsto \acute{y}, \acute{x}^\star \longmapsto \acute{y}^\star)\}$

$(\phi_1 \times \psi_1)(\acute{y}, \acute{y}^\star) = \min\left\{(\phi_1 \times \psi_1)(\acute{x}, \acute{x}^\star), (\phi_1 \times \psi_1)\left((\acute{x}, \acute{x}^\star) \longmapsto \left((\acute{y}, \acute{y}^\star)\right)\right)\right\}$

Similarly,

$$(\phi_1 \times \psi_1)(\acute{y}, \acute{y}^\star) = \min\{(\phi_1 \times \psi_1)(\acute{x}, \acute{x}^\star), (\phi_1 \times \psi_1)((\acute{x}, \acute{x}^\star) \rightsquigarrow (\acute{y}, \acute{y}^\star))\}$$

Thus $\phi_1 \times \psi_1$ is a FIm.Fi of LPWA $\mathcal{A}_1$.

4.2. Proposition: Let $\phi_1$be a FIm.Fi of a LPWA $\mathcal{A}_1$ and $\phi_{1_{\psi_1}}$ be the strongest fuzzy relation on $\mathcal{A}_1$. Then $\psi_1$is a FIm.Fi of a LPWA $\mathcal{A}_1$ iff $\phi_{1_{\psi_1}}$ is a FIm.Fi of a LPWA of $\mathcal{A}_1 \times \mathcal{A}_1$.

Proof.

(i) $\phi_{1_{\psi_1}}(\acute{x}, \acute{y}) = \min\{\psi_1(\acute{x}), \psi_1(\acute{y})\} \leq \min\{\psi_1(1), \psi_1(1)\}$

$\phi_{1_{\psi_1}}(\acute{x}, \acute{y}) \leq \phi_{1_{\psi_1}}(1,1)$

(ii) Let $(\acute{x}, \acute{x}^\star), (\acute{y}, \acute{y}^\star) \in \mathcal{A}_1 \times \mathcal{A}_1$

$\phi_{1_{\psi_1}}(\acute{y}, \acute{y}^\star) = \min\{\psi_1(\acute{y}), \psi_1(\acute{y}^\star)\} \geq \min\{\min\{\psi_1(\acute{x}), \psi_1(\acute{x} \longmapsto \acute{y})\}, \min\{\psi_1(\acute{x}^\star), \psi_1(\acute{x}^\star \longmapsto \acute{y}^\star)\}\}$

$= \min\{\min\{\psi_1(\acute{x}), \psi_1(\acute{x}^\star)\}, \min\{\psi_1(\acute{x} \longmapsto \acute{y}), \psi_1(\acute{x}^\star \longmapsto \acute{y}^\star)\}\}$

$= \min\left\{\phi_{1_{\psi_1}}\left((\acute{x}, \acute{x}^\star)\right), \phi_{1_{\psi_1}}(\acute{x} \longmapsto \acute{y}, \acute{x}^\star \longmapsto \acute{y}^\star)\right\}$

$= \min\left\{\phi_{1_{\psi_1}}\left((\acute{x}, \acute{x}^\star)\right), \phi_{1_{\psi_1}}(\acute{x} \longmapsto \acute{y}; \acute{x}^\star \longmapsto \acute{y}^\star)\right\}$

$\phi_{1_{\psi_1}}(\acute{y}, \acute{y}^\star) = \min\left\{\phi_{1_{\psi_1}}(\acute{x}, \acute{x}^\star), \phi_{1_{\psi_1}}\left((\acute{x}, \acute{x}^\star) \longmapsto (\acute{y}, \acute{y}^\star)\right)\right\}$

Similarly, $\phi_{1_{\psi_1}}(\acute{y}, \acute{y}^\star) = \min\left\{\phi_{1_{\psi_1}}(\acute{x}, \acute{x}^\star), \phi_{1_{\psi_1}}\left((\acute{x}, \acute{x}^\star) \rightsquigarrow (\acute{y}, \acute{y}^\star)\right)\right\}$

Therefore $\phi_{1_{\psi_1}}$ is a FIm.Fi of LPWA of $\mathcal{A}_1 \times \mathcal{A}_1$.

Conversely, suppose $\phi_{1_{\psi_1}}$ is a FIm.Fi of LPWA of $\mathcal{A}_1 \times \mathcal{A}_1$.

Then

(i) $\psi_1(1) \leq \min\{\psi_1(1), \psi_1(1)\}$

$\phi_{1_{\psi_1}}(1,1) \geq \phi_{1_{\psi_1}}(\acute{x}, \acute{x}) = \min\{\psi_1(\acute{x}), \psi_1(\acute{y})\} = \psi_1(\acute{x})$

$\psi_1(1) \geq \psi_1(\acute{x})\ \forall \acute{x} \in \mathcal{A}_1$.

(ii) $\psi_1(y_1) \leq \min\{\psi_1(\acute{y}), \psi_1(1)\} = \phi_{1_{\psi_1}}(y_1, 1)$

$\geq \min\left\{\phi_{1_{\psi_1}}(\acute{x}, 1), \phi_{1_{\psi_1}}\left((\acute{x}, 1) \longmapsto (\acute{y}, 1)\right)\right\}$

$= \min\left\{\phi_{1_{\psi_1}}(\acute{x}, 1), \phi_{1_{\psi_1}}\left((\acute{x}, \acute{y}) \longmapsto (1,1)\right)\right\} = \min\{\min\{\psi_1(\acute{x}), \psi_1(1)\}, \min\{\psi_1(\acute{x} \longmapsto \acute{y}), \psi_1(1)\}\}$

$\psi_1(\acute{y}) = \min\{\psi_1(\acute{x}), \psi_1(\acute{x} \longmapsto \acute{y})\}$

(iii) $\psi_1(\acute{y}) \leq \min\{\psi_1(\acute{y}), \psi_1(1)\} = \phi_{1_{\psi_1}}(\acute{y}, 1)$

$\geq \min\left\{\phi_{1_{\psi_1}}(\acute{x}, 1), \phi_{1_{\psi_1}}\left((\acute{x}, 1) \rightsquigarrow (\acute{y}, 1)\right)\right\}$

$= \min\left\{\phi_{1_{\psi_1}}(\acute{x}, 1), \phi_{1_{\psi_1}}\left((\acute{x}, \acute{y}) \rightsquigarrow (1,1)\right)\right\}$

$= \min\{\min\{\psi_1(\acute{x}), \psi_1(1)\}, \min\{\psi_1(\acute{x} \rightsquigarrow \acute{y}), \psi_1(1)\}\}$

$\psi_1(\acute{y}) = \min\{\psi_1(\acute{x}), \psi_1(\acute{x} \rightsquigarrow \acute{y})\}$

Hence ψ_1 is a FIm.Fi of LPWA $\mathcal{A}_1$.

4.3. Proposition: Let ψ_1 be a FIm.Fi of a LPWA $\mathcal{A}_1$ and $\phi_{1_{\psi_1}}$ the strongest fuzzy relation on $\mathcal{A}_1$. If ψ_1 is NFIm.Fi of a LPWA $\mathcal{A}_1$, then $\phi_{1_{\psi_1}}$is NFIm.Fi of a LPWA $\mathcal{A}_1 \times \mathcal{A}_1$.

Proof.

$\phi_{1_{\psi_1}}(1,1) = \min\{\psi_1(1), \psi_1(1)\} \geq \min\{\psi_1(\acute{x}), \psi_1(\acute{y})\} = \phi_{1_{\psi_1}}(\acute{x}, \acute{y})$

$\phi_{1_{\psi_1}}(1,1) = \phi_{1_{\psi_1}}(\acute{x}, \acute{y})$

Let $(\acute{x}, \acute{y}) \in \mathcal{A}_1 \times \mathcal{A}_1$

$\phi_{1_{\psi_1}}(z_1, w) = \min\{\psi_1(\acute{z}), \psi_1(w)\} \geq \min\{\min\{\psi_1(x_1), \psi_1(\acute{x} \longmapsto \acute{z})\}, \min\{\psi_1(\acute{y}), \psi_1(\acute{y} \longmapsto w)\}\}$

$$= \min\{\min\{\psi_1(\acute{x}), \psi_1(\acute{y})\}, \min\{\psi_1(\acute{x} \longmapsto \acute{z}), \psi_1(\acute{y} \longmapsto w)\}\}$$

$$= \min\left\{\phi_{1_{\psi_1}}(\acute{x}, \acute{y}), \phi_{1_{\psi_1}}(\acute{x} \longmapsto \acute{z}, \acute{y} \longmapsto w)\right\}$$

$$\phi_{1_{\psi_1}}(\acute{z}, w) = \min\left\{\phi_{1_{\psi_1}}(\acute{x}, \acute{y}), \phi_{1_{\psi_1}}\left((\acute{x}, \acute{y}) \longmapsto (\acute{z}, w)\right)\right\}$$

Let $(\acute{x}, \acute{y}) \in \mathcal{A}_1 \times \mathcal{A}_1$

$$\phi_{1_{\psi_1}}(\acute{z}, w) = \min\{\psi_1(\acute{z}), \psi_1(w)\} \geq \min\{\min\{\psi_1(\acute{x}), \psi_1(\acute{x} \rightsquigarrow \acute{z})\}, \min\{\psi_1(\acute{y}), \psi_1(\acute{y} \rightsquigarrow w)\}\}$$

$$= \min\{\min\{\psi_1(\acute{x}), \psi_1(\acute{y})\}, \min\{\psi_1(\acute{x} \rightsquigarrow \acute{z}), \psi_1(\acute{y} \rightsquigarrow w)\}\}$$

$$= \min\left\{\phi_{1_{\psi_1}}(\acute{x}, \acute{y}), \phi_{1_{\psi_1}}(\acute{x} \rightsquigarrow \acute{z}, \acute{y} \rightsquigarrow w)\right\}$$

$$\phi_{1_{\psi_1}}(\acute{z}, w) = \min\left\{\phi_{1_{\psi_1}}(\acute{x}, \acute{y}), \phi_{1_{\psi_1}}\left((\acute{x}, \acute{y}) \rightsquigarrow (\acute{z}, w)\right)\right\}$$

Therefore, $\phi_{1_{\psi_1}}$ is FIm.Fi of LPWA $\mathcal{A}_1 \times \mathcal{A}_1$.

Also $\phi_{1_{\psi_1}}(1,1) = \min\{\psi_1(1), \psi_1(1)\} = \min\{1,1\} = 1$.

Hence $\phi_{1_{\psi_1}}$ is a NFIm.Fi of LPWA $\mathcal{A}_1 \times \mathcal{A}_1$.

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