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Lattice of Convex Sets of a Connected Graph

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Abstract: It is known that the set of all convex sets of a finite connected graph together with empty set partially ordered by set inclusion relation forms a lattice. In this paper we studied some of the properties of these lattices for Trees and complete graphs.

Keywords: Convex sets, connected graphs, Trees
MSC subject classification: 06B99, 05C38.

1. Introduction

It is found that the lattice of normal subgroups of a group[5], the lattice of sublattices of a lattice[8], the lattice of convex sublattices of a lattice[9], [10], the lattice of subalgebras of a Boolean algebra[14], and the lattice of convex subgraphs of a directed graph[13]can be applied to study the internal structure of a group, lattice, Boolean algebra and directed graph.

Motivated from the above studies, the set of all convex sets of a finite connected graph together with the empty set is considered in [11] and it is found that this set also forms a lattice with respect to the partial order set inclusion. In this paper we studied properties of these lattices when the graph is a tree. Also the set of all path sets of a connected graph is studied in [12]. The Lattice of Convex Edge sets of a Connected directed graph and its properties are studied in [1],[2], The Lattice of Trail Sets of a Connected Graph and also The Lattice of Trail Sets of a Connected Directed Graph are studied in [3], Covering Graph of a Lattice is studied in [4].

After introducing some basic concepts, notations and stating some fundamental results, in section 2 we have shown that the set of all convex sets of a finite connected graph together with the empty set, forms a lattice with respect to set inclusion with an example. Some of the results proved in [11] are mentioned.

In section 3, length of Con(G) is obtained when G is a Tree. In particular when G is a path, length and cardinality of Con(G) are obtained. Also, length of Con(G) is obtained when G is a complete graph.
For terminologies and notations used in this paper we refer to [6] and [7].

Preliminaries

Let G be a finite connected graph. V(G) be the vertex set of G. A set $C \subseteq V(G)$ is said to be convex in G if for every two vertices $u, v \in C$, the vertex set of every u-v geodesic is contained in C. For a finite connected graph G, let the set of all convex sets in G together with empty set be denoted by Con(G). Define a binary relation $\leq$ on Con(G) by, for $A, B \in Con(G)$, $A \leq B$ if and only if $A \subseteq B$. Then clearly $\leq$ is a partial order on Con(G). Moreover $< Con(G), \subseteq >$ forms a lattice where for $A, B \in Con(G)$, $A \wedge B = A \cup B$ and $A \vee B = < A \cap B >$, where $< A \cap B >$ is the convex set generated by $A \cap B$ or equivalently the smallest convex set containing $A \cap B$.

For example, the lattice given in Fig.2 represents the lattice $< Con(G), \subseteq >$ of the connected graph G given in Fig.1.

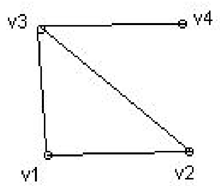

Figure 1

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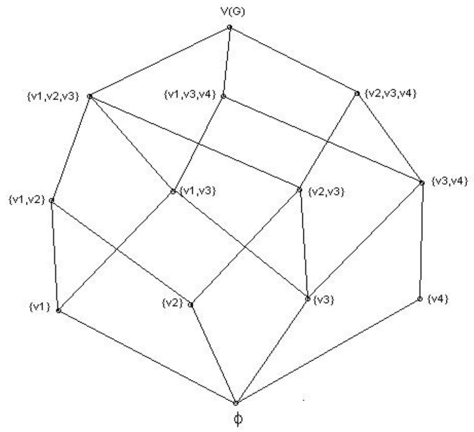

Figure 2

Throughout this paper we use Con(G) to represent the lattice < Con(G), ⊆ >

Con(G) is atomic where atoms are convex sets containing only one vertex.
An element A ∈ Con(G) is doubly irreducible if and only if A = {v} where v is an end point of G (i.e. v is a pendant vertex of G).

Following statements are equivalent in a connected graph G
1. G is complete
2. Con(G) is distributive
3. Con(G) is modular
4. Con(G) is semimodular
5. Con(G) satisfies lower covering condition.

If a graph G is complete or a cycle, then Con(G) is complemented. But converse need not be true.
For any connected graph G, Con(G) is planar if and only if the graph is a path.
The above results are proved in [11].

On The Lattice < CON(G), ⊆ >

Theorem 3.1: If G is a tree with n vertices, then l(Con(G)) = n.
Proof: Let G be a tree with n vertices. Let $v_1, v_2, v_3, \ldots\ldots, v_n$ be the vertices of G. Then $\emptyset \prec \{v_1\} \prec \{v_1, v_2\} \prec \cdots \prec \{v_1, v_2, \ldots v_n\}$ is the maximum chain.
Hence $l\big(Con(G)\big) = n$

Remark 3.2: If G is a Path with n vertices, then l(Con(G)) = n.

Theorem 3.3: If G is a Path with n vertices, then $|Con(G)| = \frac{n(n+1)}{2} + 1$

Proof: There are n convex sets with single vertex, $n-1$ convex sets with two vertices, $n-2$ convex sets with three vertices and so on, finally one convex set with n vertices. Including empty set,

$$|Con(G)| = n + (n-1) + (n-2) + \cdots + 1 + 1 = \frac{n(n+1)}{2} + 1$$

Theorem 3.4: If G is complete graph with n vertices, then l(Con(G)) = n.
Proof: Let G be a complete graph with n vertices. Let $v_1, v_2, v_3, \ldots\ldots, v_n$ be the vertices of G. Then $\emptyset \prec \{v_1\} \prec \{v_1, v_2\} \prec \cdots \prec \{v_1, v_2, \ldots v_n\}$ is the maximum chain.
Hence $l\big(Con(G)\big) = n$

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