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Fluid dynamics laws using four vectors

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Abstract: Governing equations in classical fluid dynamics are derived by considering flows taking place in three- dimensional space. The present paper considers the fluid flow in the setting of 4-dimensional space-time. Furthermore, the conservation laws associated with fluid flow are presented using scalar product of four vectors. The method is significant because of its successful applications in similar fields like electrodynamics.

Keywords: Four vector algebra, Helicity, Hydrodynamic charge, Conservation equations.
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1. Introduction

Notation in terms of vectors is customarily deployed to express the physical quantities. However, this is limited to 3D space. 19th-century mathematician William Clifford introduced Geometric Algebra combining ideas formulated by Hamilton and Grassman. In his Space-Time Algebra (1966), David Hestenes used geometric algebra to explain classical mechanics concepts[1]. He applied these techniques to electrodynamics, Lie groups, relativity, quantum theory, conformal and projective geometry. Doran, Lasenby and Lasenby [2] as well as Bromborsky have done further work with Geometric Algebra [3].

Considering the language of four vector Algebra, the traditional method of working with the 3D coordinates and time coordinate separately, can be replaced with a 4-component concept. The four components are taken in such a way that the first component corresponds to time and the other three components as the space coordinate x, y & z, respectively. Susan and Vedan (2019) introduced the four vector concept using multivectors in Fluid dynamics [4]. The concept was extended to non-barotropic flows by Susan, Parameswaran and Vedan (2020) [5]. Sulayemann Demur has made a similar approach (2017)[6] where he introduced the concept of 4 vector derivative to discuss the incompressible fluid flow. Susan and M.J. Vedan (2019) also discuss the conservation laws treating time coordinate and the three space coordinates separately [4].

2. Space Time Algebra (STA)

Space Time Algebra (STA) is the geometric algebra of space and time. This algebra is well known as geometric algebra of Minkowski space-time. Within the literature we can find a similarity among the geometric algebra $G(4, 0)$ or G_4 [7] of the Euclidean four-dimensional space R^4 and the geometric algebra G(1, 3) of Minkowski space-time R1,3 [8]. In this paper, we confine ourselves to the geometric algebra within Euclidean space-time.

Let us consider the Euclidean space time $\mathrm{R}^4 \cong T \times \mathrm{R}^3$, where $\mathrm{T} \cong \mathrm{R}$ is the time and R^3 is the Euclidean space. Any general point in this space-time is represented by a set of coordinates (x_0, x_1, x_2, x_3) where the first coordinate represent the time component and the other three, the space components. We consider $v = (v_1, v_2, v_3)$ be a divergence free velocity field in R^3. Euler equation associated with an ideal incompressible flow with constant density is given by,

$$\partial_t v + (v.\nabla)v = \frac{-\nabla p}{\rho} - \nabla\varphi \quad (1)$$

The Navier-Stokes' Equation associated with viscous incompressible fluids are described by ,

$$\partial_t v + (v.\nabla)v = \frac{-\nabla p}{\rho} - \nabla\varphi - \nu\nabla^2 v \quad (2)$$

where ρ represents the mass density, p denotes the pressure, the potential for the volume force field is denoted by φ and ν is the kinematic viscosity (Batchelor, 1967).

3. Four Vector Fields in Fluid Flows

3.1 Four vector derivative

Let the four vector derivative be given by the equation

$$\boldsymbol{\nabla} = e_0\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z} \quad (3)$$

3.2 Four vector Velocity

If $v = (v_1, v_2, v_3)$ be a divergence-free velocity field in R^3 satisfying Euler's or Navier-Stokes' equations as in (1) or (2). The four-vector velocity field of a four dimensional fluid flow V = (*f*, *v*), can be expressed as a vector in G_4 [4] [9] as

$$\mathbf{V} = fe_0 + v_1e_1 + v_2e_2 + v_3e_3 \quad (4)$$

along with the incompressibility condition or the divergence free condition

$$\boldsymbol{\nabla}.\mathbf{V} = \frac{\partial f}{\partial t} + \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0 \quad (5)$$

This V satisfies the Euler's equation [10].

3.3 Four vector Associated with hydrodynamic charge density and current

The four vector associated with the *q*, hydrodynamic charge density, and *J* hydrodynamic current can be denoted with $\mathbf{J}$.

3.4 Four vector Associated with Helicity

The four vector associated with the helicity density h and the helicity or the torsion current vector **H** [11] can be expressed with H, termed as topological torsion tensor[12].

3.5 Flow field

We consider the flow field is of the form $\xi = \partial_t + v_1\partial_{x1} + v_2\partial_{x2} + v_3\partial_{x3}$ or $\xi = \partial_t + v_i\partial x_i$, as a four vector ξ.

4. Conservation equation and Evolution equation

4.1 Conservation equation

Using the four vector derivative, the conservation equation can be represented as $\boldsymbol{\nabla}\cdot\boldsymbol{X} = 0$, where $\boldsymbol{X}$ is any fluid dynamic element.

4.1.1 Conservation equation associated with the hydrodynamic charge

The conservation equation in association with the hydrodynamic charge 4− vector $\mathbf{J}$ can be given by $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$, valid for both ideal flows as well as viscous flows.

This is in agreement with the conservation equation

$$\partial_t q + \nabla \cdot J = 0$$

valid for both ideal and viscous flows, where q and J are the hydrodynamic charge density and the hydrodynamic current. [9]

4.1.2 Conservation equation associated with the Helicity

The conservation equation associated with the helicity four vector can be written as $\boldsymbol{\nabla} \cdot \mathrm{H} = 0$ for perfect fluids, which is the same as the

$$\partial_t h + \nabla \cdot \mathbf{H} = 0$$

the conservation equation of helicity [4].

4.1.3 Conservation of entropy

If the entropy is denoted by S, then the conservation of entropy may be expressed in terms of four vectors as $\xi S = 0$ which is same as the

$$\partial_t S + (v \cdot \nabla) S = 0.$$

4.2 Evolution equation

If we can represent any Fluid dynamical quantity X in the equation of the form $\boldsymbol{\nabla} \cdot \boldsymbol{X} = Y$, where Y any function of some other fluid dynamical quantity.

4.2.1 Evolution equation associated with Helicity

Evolution equation associated with the four-vector Helicity of viscous flows [13] is given by

$$\boldsymbol{\nabla} \cdot -\mathrm{H} = -2v(\nabla \times w).w.$$

Here v is the kinematic viscosity and w is the vorticity vector in three dimensions. The same is similar to the helicity evolution equation given as

$$\partial_t h + \nabla \cdot \mathbf{H} = -2v(\nabla \times w).w$$

for viscous fluids [4].

5. Conclusion

The four-vector differential operator and the four component representation of the fluid elements incorporating the time component together with the three space coordinates will help in the shorthand representation of equations.

6. Further Scope

The same concept can be extended to the other physical systems paving the way to further research in this field. Four vector notation also can be used to derive an equivalent expression for material derivative so that the continuity equations and Euler equation can be modified to the four dimensional Euclidean space-time Manifold.

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