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Sombor Index of Fuzzy Graphs and its Applications

Hanumantha Reddy D T [1,2], M V Chakradhara Rao [1] and S. M. Hosamani [3]

[1]*Department of Mathematics*
Presidency University, Itgalpur, Rajanakunte, Yelahanka,
Bengaluru-560064, Karnataka, India.
[2]*Department of Mathematics*
Government College for Women, Chintamani-563125
Karnataka, India.
[3]*Department of Mathematics*
Rani Channamma University, Belagavi-591156
Karnataka, India.

Abstract

The Sombor index has proved its importance in mathematical chemistry due to its high predicting power in physico-chemical properties of chemical compounds. In this paper, we have explore the properties of the Sombor index of fuzzy graphs. The study presented the relationship between the Sombor index and the first Zagreb index of fuzzy graphs. As an application of the Sombor index, we have analyzed the internet network of Bharati Airtel Ltd India for enhancing the efficiency and effectiveness of internet systems.

Keywords: *First Zagreb index; Second Zagreb index; Sombor index; .*

Subject Classification: 05C09; 05C72.

1 Introduction

Let $G=(V,\omega,\rho)$ be a simple fuzzy graph of order n and size m. The membership values of the vertices $\{v_1,v_2,v_3,\cdots,v_n\}$ and edges $\{e_1,e_2,e_3,\cdots,e_m\}$ of a fuzzy graph G are $\{\omega(v_1),\omega(v_2),\omega(v_3),\cdots,\omega(v_n)\}$ and $\{\rho(e_1),\rho(e_2),\rho(e_3),\cdots,\rho(e_m)\}$ respectively. The degree of a vertex $v\in V(G)$ is denoted by $d_G(v)$ and is defined as the sum of the membership values of the edges incident to a vertex $v\in V(G)$. Let P_n,C_n,K_n and $K_{1,n-1}$ denotes the fuzzy path, fuzzy cycle, fuzzy complete graph and fuzzy star respectively. For any undefined terminology in this paper may be found in [12].

In the field of molecular chemistry, topological indices(TIs) are molecular descriptors which are calculated on the molecular graph of a chemical compound. These TIs are numerical quantities of a graph which describes its topology. Zagreb index $M_1(G)$ is one such TIs which is degree-based TI and introduced by Gutman and Trinajstic in 1972 [6] and this TI is used to calculate π-electron energy of a conjugate system. One of the most useful topological indices are the Zagreb indices which are defined as:

$$M_1(G) = \sum_{i=1}^{n} d_G(v_i)^2 \tag{1}$$

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v) \tag{2}$$

where M_1 and M_2 are the first and second Zagreb indices respectively.

Let $G = (V, \omega, \rho)$ be a fuzzy graph then the first Zagreb index of fuzzy graphs[9] is defined as follows:

$$M_1(G) = \sum_{i=1}^{n} [\omega(v_i)d_G(v_i)]^2$$

Gutman[7] defined a new vertex degree-based topological index, named the Sombori index, and defined for a graph G as follows:

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$$

1.1 Motivation and Research Questions

Islam and Pal [14] have studied the the first Zagreb index of fuzzy graphs and shown its application in modeling MCDM problem. Recently, Jana and Ghorai [10] have studied the first entire Zagreb index of fuzzy graph and demonstrated its applications in finding the requirement of development of internet system in the different states of India. Likewise, a novel class of degree-based topological molecular descriptors was proposed, the so-called Sombor indices[7]. It was found that this degree-based index exert modest discriminative potential, when tested on a large group of isomers. Motivated by the works mentioned above here we have studied the Sombor index of fuzzy graphs and shown its applications in strengthening the internet system in the different states of India. We considered the following research questions:

1. What is the Sombor index of standard class of fuzzy graphs?
2. What is the relationship between the first Zagreb index and Sombor index of fuzzy graphs?
3. Does the Sombor index values for vertex and edge critical graphs remains same?
4. Bounds for Sombor index of fuzzy graphs in terms of order and size of G.
5. What are the applications of this index?

1.2 Objectives of the work

Numerous topological indices have been studied for crisp graphs and obtained many applications. But in many practical applications it is seen that many situations cannot be modeled using crisp graphs. We need to define a fuzzy graph to answer this question. In this paper the Sombor index for fuzzy graphs is defined and some results are related to vertex and edge critical fuzzy graphs. Also, obtained relationship between the first Zagreb index and Sombor index of fuzzy graphs. At the end of this paper, we applied the first entire Sombor index in internet network systems.

1.3 Structure of the Study

The structure of the article is as follows: in Section 2, we studied the Sombor index of a fuzzy graph and provided some results on sub graphs, paths, cycle, complete and stars of fuzzy graphs. Also obtained the Sombor index of vertex and edge critical graphs. In Section 3, an application of the Sombor index in development in internet networking system is discussed.

2 Results

Let $G = (V, \omega, \rho)$ be a fuzzy graph then the Sombor index of fuzzy graphs is defined as follows:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{(\omega(u)d_G(u))^2 + (\omega(v)d_G(v))^2}$$

Example 1. Let G be a fuzzy graph with $V(G) = \{p, q, r, s\}$ such that $\omega(p) = 0.8$, $\omega(q) = 0.7$, $\omega(r) = 0.6$, $\omega(s) = 0.5$, $\rho(pq) = 0.4$, $\rho(pr) = 0.5$, $\rho(ps) = 0.5$, $\rho(rs) = 0.4$, $\rho(qs) = 0.3$ as shown in figure 1. Then $d_G(p) = 1.4$, $d_G(q) = 0.7$, $d_G(r) = 0.9$ and $d_G(s) = 1.2$. Now

$$SO(G) = \sum_{uv \in E(G)} \sqrt{(\omega(u)d_G(u))^2 + (\omega(v)d_G(v))^2}$$

$$= \sqrt{(\omega(p)d_G(p))^2 + (\omega(r)d_G(r))^2} + \sqrt{(\omega(p)d_G(p))^2 + (\omega(s)d_G(s))^2}$$

$$+\sqrt{(\omega(p)d_G(p))^2 + (\omega(q)d_G(q))^2} + \sqrt{(\omega(r)d_G(r))^2 + (\omega(s)d_G(s))^2}$$

$$+\sqrt{(\omega(s)d_G(s))^2 + (\omega(q)d_G(q))^2}$$

$$= \sqrt{[(0.8)(1.4)]^2 + [(0.6)(0.9)]^2} + \sqrt{[(0.8)(1.4)]^2 + [(0.5)(0.2)]^2}$$

$$+\sqrt{[(0.8)(1.4)]^2 + [(0.7)(0.7)]^2} + \sqrt{[(0.6)(0.9)]^2 + [(0.5)(1.2)]^2}$$

$$+\sqrt{[(0.5)(1.2)]^2 + [(0.7)(0.7)]^2}$$

$$= 5.3223.$$

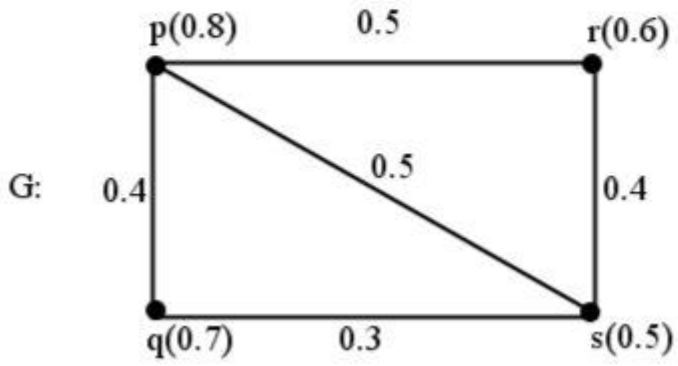

Figure 1. A fuzzy graph on 4-vertices.

Theorem 1 *Let* P_n, C_n *and* K_n *denotes fuzzy path, cycle, complete graphs respectively. Then*

1. $SO(P_n) \leq 2(n-3)\sqrt{2} + 2\sqrt{5}$

2. $SO(C_n) \leq 2\sqrt{2}n$

3. $SO(K_n) \leq \frac{\sqrt{2}}{2}n(n-1)^2$

Proof. We consider the following cases:

Case 1. Let $G = P_n$ be a fuzzy path with $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(P_n) = \{e_1, e_2, e_3, \cdots, e_{n-1}\}$. Let $\omega_1, \omega_2, \omega_3, \cdots, \omega_n$ and $\rho_1, \rho_2, \rho_3, \cdots, \rho_{n-1}$ be the membership values of vertices and edges of P_n respectively. Then, clearly $d_G(v_1) = \rho(e_1)$, $d_G(v_n) = \rho(e_{n-1})$ and $d_G(v_i) = \rho(e_i) + \rho(e_{i+1})$ for $2 \leq i \leq n-2$. Therefore,

$$SO(P_n) = \sum_{uv\in E(G)} \sqrt{(\omega(u)d_G(u))^2 + (\omega(v)d_G(v))^2}$$

$$= \sqrt{(\omega(v_1)d_G(v_1))^2 + \omega(v_2)d_G(v_2)^2}$$

$$+ \sqrt{(\omega(v_{n-1})d_G(v_{n-1}))^2 + \omega(v_n)d_G(v_n)^2}$$

$$+ \sum_{v_iv_j\in E(G)-\{e_1,e_{n-1}\}} \sqrt{(\omega(v_i)d_G(v_i))^2 + \omega(v_j)d_G(v_j)^2}$$

$$= \sqrt{(\omega(v_1)\rho(e_1))^2 + (\omega(v_2)(\rho(e_1) + \rho(e_2)))^2}$$

$$+ \sqrt{(\omega(v_n)\rho(e_{n-1}))^2 + (\omega(v_{n-1})(\rho(e_{n-1}) + \rho(e_{n-2})))^2}$$

$$+ \sum_{v_iv_j\in E(G)-\{e_1,e_{n-1}\}} + \sqrt{(\omega(v_j)(\rho(e_j) + \rho(e_{j+1})))^2}$$

$$= \sqrt{\omega(v_1)^2\rho(e_1)^2 + (\omega(v_2)^2(\rho(e_1)^2 + \rho(e_2)^2 + 2\rho(e_1)\rho(e_2)))}$$

$$+ \sqrt{\omega(v_n)^2\rho(e_{n-1})^2 + (\omega(v_{n-1})^2(\rho(e_{n-1})^2 + \rho(e_{n-2})^2 + 2\rho(e_{n-1})\rho(e_{n-2})))}$$

$$+\sum_{v_iv_j\in E(G)-\{e_1,e_{n-1}\}}\sqrt{\begin{array}{c}(\omega(v_i)^2(\rho(e_i)^2+\rho(e_{i+1})^2+2\rho(e_i)\rho(e_{i+1})))+(\omega(v_j)^2(\rho(e_j)^2+\rho(e_{j+1})^2\\+2\rho(e_j)\rho(e_{j+1})))\end{array}}$$

Since, $o\le\omega(v)\le 1$ and $0\le\rho(e)\le 1$. Therefore,

$$SO(P_n)\le\sqrt{1^2\cdot 1^2+1^2\cdot 1^2+2(1^2)\cdot 1\cdot 1}+\sqrt{1^2\cdot 1^2+1^2\cdot 1^2+1^21^2+2(1^2)\cdot 1\cdot 1}$$

$$+(n-3)\sqrt{1^2\cdot 1^2+1^2\cdot 1^2+2(1^2)\cdot 1\cdot 1+1^2\cdot 1^2+1^2\cdot 1^2+2(1^2)\cdot 1\cdot 1}$$

$$=2\sqrt{2}(n-3)+2\sqrt{5}.$$

Case 2. Let $G=C_n$ be a fuzzy cycle with $V(C_n)=\{v_1,v_2,v_3,\cdots,v_n\}$ and $E(P_n)=\{e_1,e_2,e_3,\cdots,e_n\}$. Let $\omega_1,\omega_2,\omega_3,\cdots,\omega_n$ and $\rho_1,\rho_2,\rho_3,\cdots,\rho_n$ be the membership values of vertices and edges of C_n respectively. Then, clearly $d_G(v_i)=\rho(e_i)+\rho(e_{i+1})$. Therefore,

$$SO(C_n)=\sum_{uv\in E(G)}\sqrt{(\omega(u)d_G(u))^2+\omega(v)d_G(v))^2}$$

$$=\sum_{v_iv_j\in E(G)}\sqrt{(\omega(v_i)(\rho(e_i)+\rho(e_{i+1})))^2}+\sqrt{(\omega(v_j)(\rho(e_j)+\rho(e_{j+1})))^2}$$

$$+\sum_{v_iv_j\in E(G)}\sqrt{(\omega(v_i)^2(\rho(e_i)^2+\rho(e_{i+1})^2+2\rho(e_i)\rho(e_{i+1})))+(\omega(v_j)^2(\rho(e_j)^2+\rho(e_{j+1})^2+2\rho(e_j)\rho(e_{j+1})}$$

Since, $o\le\omega(v)\le 1$ and $0\le\rho(e)\le 1$. Therefore,

$$SO(C_n)\le n\sqrt{1^2\cdot 1^2+1^2\cdot 1^2+2(1^2)\cdot 1\cdot 1+1^2\cdot 1^2+1^2\cdot 1^2+2(1^2)\cdot 1\cdot 1}$$

$$=2\sqrt{2}n.$$

Case 3. Let $G=K_n$ be a fuzzy complete graph with $V(K_n)=\{v_1,v_2,v_3,\cdots,v_n\}$ and $E(K_n)=\{e_1,e_2,e_3,\cdots,e_n\}$. Let $\omega_1,\omega_2,\omega_3,\cdots,\omega_n$ and $\rho_1,\rho_2,\rho_3,\cdots,\rho_n$ be the membership values of vertices and edges of K_n respectively. Then, clearly $d_G(v_i)=\sum_{v_i\sim e_j}\rho(e_j)$. Therefore,

$$SO(K_n)=\sum_{uv\in E(G)}\sqrt{(\omega(u)d_G(u))^2+\omega(v)d_G(v))^2}$$

$$=\sum_{v_iv_j\in E(G)}\sqrt{(\omega(v_i)^2(\sum_{v_i\sim e_j}\rho(e_j)^2))+(\omega(v_j)^2(\sum_{v_j\sim e_i}\rho(e_i)^2))}$$

$$\le\sum_{v_iv_j\in E(G)}\sqrt{\omega(v_i)^2(n-1)\rho(e_j)^2+\omega(v_j)^2(n-1)\rho(e_i)^2}$$

Since, $o\le\omega(v)\le 1$ and $0\le\rho(e)\le 1$. Therefore,

$$SO(K_n) \leq \sum_{v_iv_j \in E(G)} \sqrt{(n-1)+(n-1)}$$
$$= \frac{n(n-1)}{2}\sqrt{2n-2}.$$

Lemma 1 *Let* $G = K_{1,n-1}$ *be a fuzzy star and satisfies the condition* $\omega(o) \leq \omega(v)$ *where* o *is the center of the fuzzy star, then*

$$SO(K_{1,n-1}) \leq (n-1)\sqrt{n^2-2n+2}$$

Proof. Let $G = K_{1,n-1}$ be a fuzzy star graph with $V(K_{1,n-1}) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(K_{1,n-1}) = \{e_1, e_2, e_3, \cdots, e_{n-1}\}$. Let $\omega_1, \omega_2, \omega_3, \cdots, \omega_n$ and $\rho_1, \rho_2, \rho_3, \cdots, \rho_n$ be the membership values of vertices and edges of K_n respectively. Let $v_1 = o$ be the center of the star. It is given that $\omega(o) \leq \omega(v)$. Then, clearly $d_G(v_i) = \omega(o)$ and $d_G(0) = (n-1)\omega(o)$. Therefore,

$$SO(K_{1,n-1}) = \sum_{uv \in E(G)} \sqrt{(\omega(u)d_G(u))^2 + \omega(v)d_G(v))^2}$$
$$= \sum_{v_iv_j \in E(G)} \sqrt{(\omega(v_i)^2(d_G(o))^2) + (\omega(v_j)^2(d_G(v_j))^2)}$$
$$= \sum_{v_iv_j \in E(G)} \sqrt{(\omega(v_i)^2((n-1)\omega(o))^2) + (\omega(v_j)^2(\omega(o))^2}$$

Since, $o \leq \omega(v) \leq 1$ and $0 \leq \rho(e) \leq 1$. Therefore,

$$SO(K_{1,n-1}) \leq (n-1)[\sqrt{(1^2 \cdot ((n-1)^2 \cdot 1^2)) + (1^2 \cdot 1^2)}]$$
$$= (n-1)\sqrt{n^2-2n+2}.$$

Theorem 2 *Let* $G = (V, \omega, \rho)$ *be a fuzzy graph and* $ZF_1(G)$ *denote the first Zagreb index for fuzzy graphs. Then* $SO(G) \leq ZF_1(G)$*.*

Proof. Let $G = (V, \omega, \rho)$ be a fuzzy graph. By definition of Sombor index for fuzzy graphs, we have

$$SO(G) = \sum_{uv \in E(G)} \sqrt{(\omega(u)d_G(u))^2 + \omega(v)d_G(v))^2}$$
$$= \sum_{uv \in E(G)} \sqrt{(\omega(u)d_G(u) + \omega(v)d_G(v))^2 - 2\omega(u)\omega(v)d_G(u)d_G(v)}$$
$$\leq \sum_{uv \in E(G)} \sqrt{(\omega(u)d_G(u) + \omega(v)d_G(v))^2}$$
$$= \sum_{uv \in E(G)} (\omega(u)d_G(u) + \omega(v)d_G(v))$$
$$= ZF_1(G)$$

Example 2. Let G be a fuzzy graph with $V(G) = \{p, q, r, s\}$ such that $\omega(p) = 0.8$, $\omega(q) =$

0.7, $\omega(r) = 0.5$, $\omega(s) = 0.6$, $\rho(pq) = 0.4$, $\rho(qr) = 0.4$, $\rho(qs) = 0.3$, $\rho(rs) = 0.3$ as shown in figure 2. Then $d_G(p) = 0.4$, $d_G(q) = 1.1$, $d_G(r) = 0.7$, and $d_G(s) = 0.6$. Now

$$SO(G) = \sum_{uv\in E(G)} \sqrt{(\omega(u)d_G(u))^2 + (\omega(v)d_G(v))^2}$$

$$= \sqrt{(\omega(p)d_G(p))^2 + (\omega(q)d_G(q))^2} + \sqrt{(\omega(q)d_G(q))^2 + (\omega(r)d_G(r))^2}$$

$$+\sqrt{(\omega(q)d_G(q))^2 + (\omega(s)d_G(s))^2} + \sqrt{(\omega(r)d_G(r))^2 + (\omega(s)d_G(s))^2}$$

$$= \sqrt{[(0.8)(0.4)]^2 + [(0.7)(1.1)]^2} + \sqrt{[(0.7)(1.1)]^2 + [(0.5)(0.7)]^2}$$

$$+\sqrt{[(0.7)(1.1)]^2 + [(0.6)(0.6)]^2} + \sqrt{[(0.5)(0.7)]^2 + [(0.6)(0.6)]^2}$$

$$= 3.0316.$$

Now consider the first Zagreb index of fuzzy graphs:

$$ZF_1(G) = \sum_{uv\in E(G)} [\omega(u)d_G(u) + \omega(v)d_G(v)]$$

$$= (\omega(p)d_G(p) + \omega(q)d_G(q)) + (\omega(q)d_G(q) + \omega(r)d_G(r))$$

$$+(\omega(q)d_G(q) + \omega(s)d_G(s)) + (\omega(r)d_G(r) + \omega(s)d_G(s))$$

$$= [(0.8)(0.4)] + [(0.7)(1.1)] + [(0.7)(1.1)] + [(0.5)(0.7)]$$

$$+[(0.7)(1.1)] + [(0.6)(0.6)] + [(0.5)(0.7)] + [(0.6)(0.6)]$$

$$= 4.09.$$

Thus, clearly $SO(G) < ZF_1(G)$.

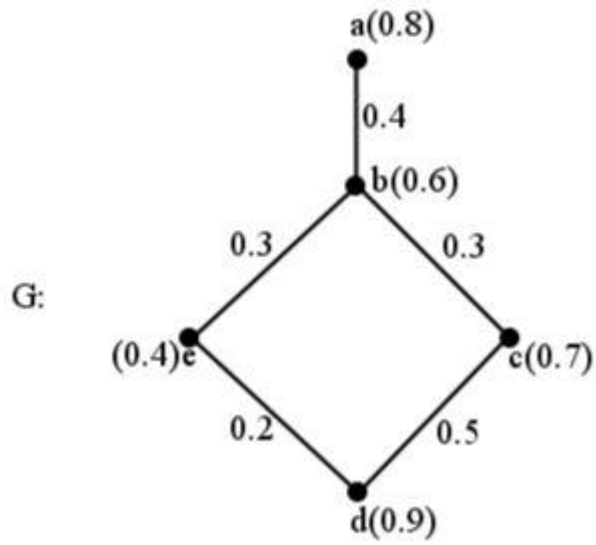

Figure 2. A fuzzy graph G.

Corollary 3 *Let* $G = (V, \omega, \rho)$ *be a n-vertex fuzzy graph with m-edges. Then*

1. $SO(G) \leq n^2T_v^2$
2. $SO(G) \leq 4n^2m^2$

where T_v is a total vertex degree of G.

Proof. The proof follows from Theorem 2 and Theorem 3.1 in [9].

Theorem 4 *Let* $G = (V, \omega, \rho)$ *be a n-vertex fuzzy graph with m-edges. Then* $SO(G) \geq SO(G - e)$, *where* $e \in E(G)$.

Proof. Let $G = (V, \omega, \rho)$ be a fuzzy graph and $H = G - e$ is a graph obtained by removing an edge $e \in E(G)$. The membership values in G and H are given by the relationship: $\omega_G(v) \geq \omega_H(v)$ and $\rho_G(e) \geq \rho_H(e)$. This shows that $d_G(v) \geq d_H(v)$ and $d_G(e) \geq d_H(v)$. Now,

$$SO(G) = \sum_{uv \in E(G)} \sqrt{(\omega(u)d_G(u))^2 + \omega(v)d_G(v))^2}$$

$$\geq \sum_{uv \in E(H)} \sqrt{(\omega(u)d_H(u))^2 + \omega(v)d_H(v))^2}$$

$$= SO(H)$$

$$= SO(G - e).$$

Example 2. Let G be a fuzzy graph with $V(G) = \{a, b, c, d, e\}$ such that $\omega(a) = 0.8$, $\omega(b) = 0.6$, $\omega(c) = 0.7$, $\omega(d) = 0.9$, $\omega(e) = 0.4$, $\rho(ab) = 0.4$, $\rho(bc) = 0.3$, $\rho(be) = 0.3$, $\rho(cd) = 0.5$, $\rho(de) = 0.2$ as shown in figure 3. Then $d_G(a) = 0.4$, $d_G(b) = 1.0$, $d_G(c) = 0.8$, $d_G(d) = 0.7$ and $d_G(e) = 0.5$. Now

$$SO(G) = \sum_{uv \in E(G)} \sqrt{(\omega(u)d_G(u))^2 + (\omega(v)d_G(v))^2}$$

$$= \sqrt{(\omega(a)d_G(a))^2 + (\omega(b)d_G(b))^2} + \sqrt{(\omega(b)d_G(b))^2 + (\omega(c)d_G(c))^2}$$

$$+ \sqrt{(\omega(b)d_G(b))^2 + (\omega(e)d_G(e))^2} + \sqrt{(\omega(c)d_G(c))^2 + (\omega(d)d_G(d))^2}$$

$$+ \sqrt{(\omega(d)d_G(d))^2 + (\omega(e)d_G(e))^2}$$

$$= \sqrt{[(0.8)(0.4)]^2 + [(0.6)(1.0)]^2} + \sqrt{[(0.6)(1.0)]^2 + [(0.7)(0.8)]^2}$$

$$+ \sqrt{[(0.6)(1.0)]^2 + [(0.4)(0.5)]^2} + \sqrt{[(0.7)(0.8)]^2 + [(0.9)(0.7)]^2}$$

$$+ \sqrt{[(0.4)(0.5)]^2 + [(0.9)(0.7)]^2}$$

$$= 3.5917.$$

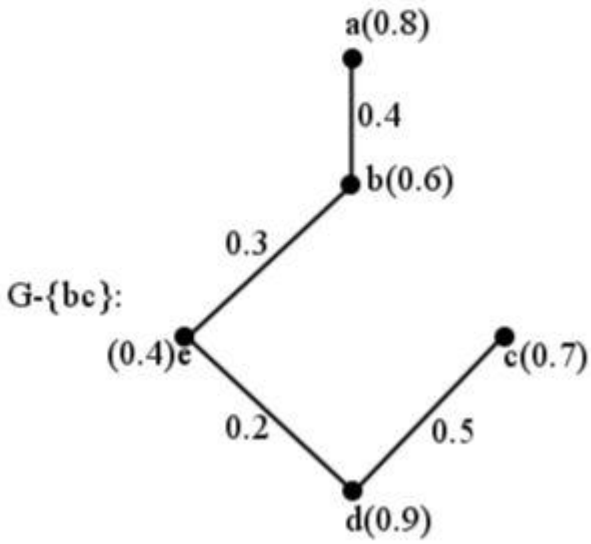

Figure 3. A fuzzy graph on 5-vertices.

Now, let $H = G - \{bc\}$ be a graph obtained by removing an edge $bc \in E(G)$. The membership values of the vertices of H will remain same as in G but there is a change in degrees of the vertices b and c in H. Then $d_G(b) = 0.7$ and Then $d_G(c) = 0.5$. Now

$$SO(H) = \sum_{uv \in E(H)} \sqrt{(\omega(u)d_H(u))^2 + (\omega(v)d_H(v))^2}$$

$$= \sqrt{(\omega(a)d_G(a))^2 + (\omega(b)d_G(b))^2} + \sqrt{(\omega(b)d_G(b))^2 + (\omega(e)d_G(e))^2}$$

$$+\sqrt{(\omega(c)d_G(c))^2 + (\omega(d)d_G(d))^2} + \sqrt{(\omega(d)d_G(d))^2 + (\omega(e)d_G(e))^2}$$

$$= \sqrt{[(0.8)(0.4)]^2 + [(0.6)(0.7)]^2} + \sqrt{[(0.6)(0.7)]^2 + [(0.4)(0.5)]^2}$$

$$+\sqrt{[(0.7)(0.5)]^2 + [(0.9)(0.7)]^2} + \sqrt{[(0.9)(0.7)]^2 + [(0.4)(0.5)]^2}$$

$$= 2.349$$

Thus, clearly $SO(G) > SO(G) - e$.

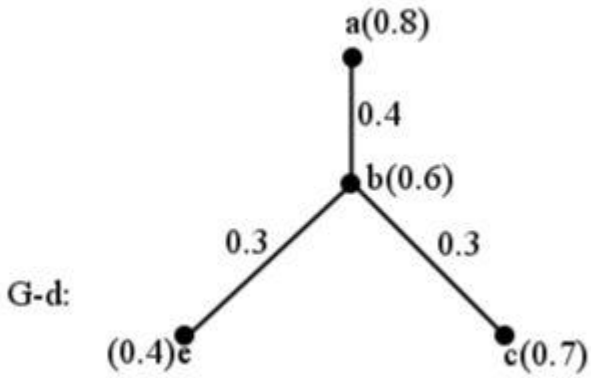

Figure 4. A fuzzy graph G-{bc}.

Theorem 5 *Let* $G = (V, \omega, \rho)$ *be a n-vertex fuzzy graph. Then* $SO(G) \geq SO(G - v)$, *where* $v \in V(G)$.

Proof. Let $G = (V, \omega, \rho)$ be a fuzzy graph and $H = G - v$ is a graph obtained by removing a vertex $v \in V(G)$. The membership values in G and H are given by the relationship: $\omega_G(v) \geq \omega_H(v)$ and $\rho_G(e) \geq \rho_H(e)$. This shows that $d_G(v) \geq d_H(v)$ and $d_G(e) \geq d_H(v)$. Now,

$$SO(G) = \sum_{uv \in E(G)} \sqrt{(\omega(u)d_G(u))^2 + \omega(v)d_G(v))^2}$$

$$\geq \sum_{uv \in E(H)} \sqrt{(\omega(u)d_H(u))^2 + \omega(v)d_H(v))^2}$$

$$= SO(H)$$

$$= SO(G - v).$$

Example 3. Let $H = G - \{d\}$ be a fuzzy graph obtained by removing a vertex d from figure 2. Clearly, the membership values of H remains same for the vertices and there is a change in degrees of vertices e and c. Then $d_H(c) = 0.3$ and $d_H(e) = 0.3$. Now

$$SO(H) = \sum_{uv \in E(H)} \sqrt{(\omega(u)d_H(u))^2 + (\omega(v)d_H(v))^2}$$

$$= \sqrt{(\omega(a)d_G(a))^2 + (\omega(b)d_G(b))^2} + \sqrt{(\omega(b)d_G(b))^2 + (\omega(c)d_G(c))^2}$$

$$+ \sqrt{(\omega(b)d_G(b))^2 + (\omega(e)d_G(e))^2}$$

$$= \sqrt{[(0.8)(0.4)]^2 + [(0.6)(1.0)]^2} + \sqrt{[(0.6)(1.0)]^2 + [(0.7)(0.3)]^2}$$

$$+ \sqrt{[(0.6)(1.0)]^2 + [(0.4)(0.3)]^2}$$

$$= 1.9036.$$

Thus, clearly $SO(G) > SO(G) - v$.

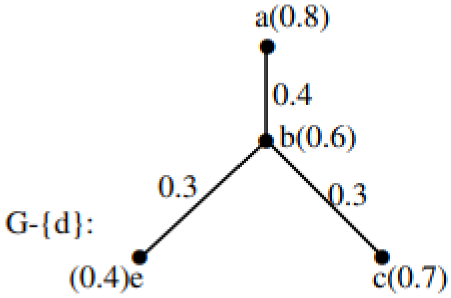

Figure 5. A fuzzy graph $G - \{d\}$.

3 Applications of Sombor Index of Fuzzy Graphs

In the modern day, the internet is the most important part of our regular life. Here in this paper, we analyzed the Bharati Airtel Ltd internet system in India. The data of Reliance Bharati Airtel Ltd Ltd internet users are given in Table 1. These data were taken from $https://www.trai.gov.in/sites/default/files/PR_No.58of2023_0.pdf$, accessed on 5 th July 2023 . Then, we constructed a Bharati Airtel Ltd internet system graph (see Figure 6). Here, the whole graph is similar to a fuzzy star where Bharati Airtel Ltd(C) is the center of the star and each state is a pendent vertex of the star.

The membership values of each vertex($\omega(v)$) and an edge($\rho(e)$) of Bharati Airtel internet network is obtained by the following methods:

$$\omega(v) = min\{1, \frac{\text{TotalBharatiAirtelinternetusersinthestate}}{\text{Totalpopulationinthestate}}\}$$

$$\rho(e) = min\{1, \frac{\text{TotalBharatiAirtelinternetusersinthestate}}{\text{Totalpopulationinthestate}} + \frac{\text{Populationpersentageinthisstate}}{100}\}$$

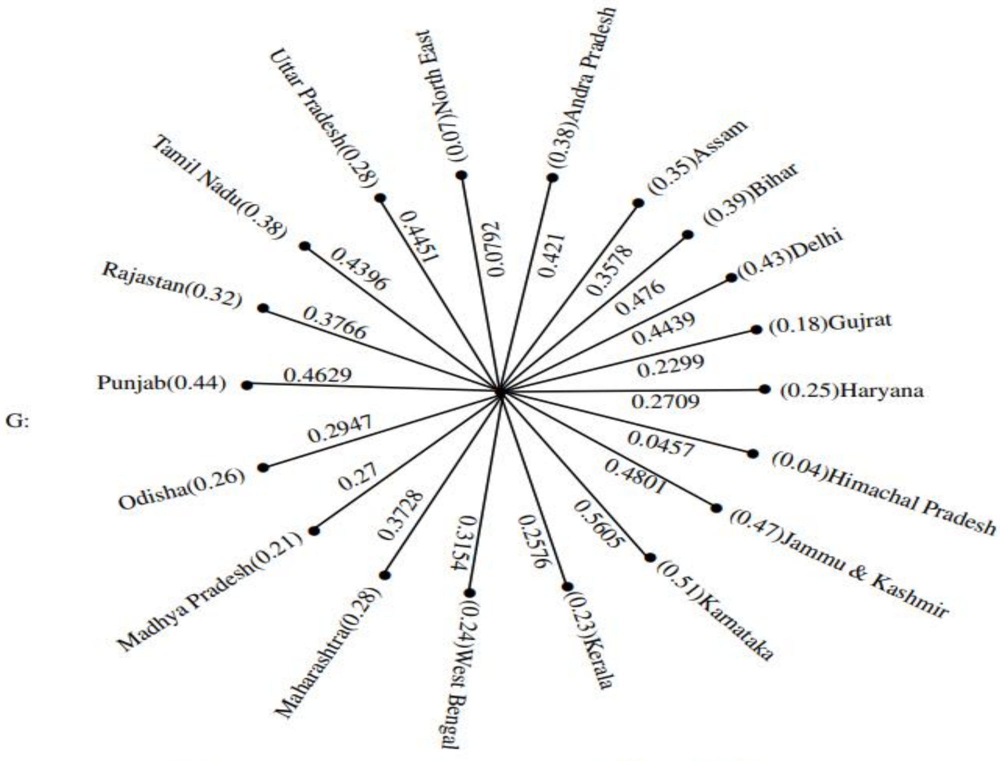

Figure 6. Fuzzy graph of internet network of Bharati Airtel.

Table 1. Data of internet users for all states.

State	Internet Users	Total population	Population percentage
	(in Millions)	(in Millions)	of the state
Andra Pradesh(AP)	32.4	84.66	4.1
Assam(AS)	11.09	31.16	2.58
Bihar(BI)	40.03	100.8	8.6
Delhi(DE)	17.10	38.89	1.39
Gujarat(GJ)	11.3	60.38	4.99
Haryana(HR)	6.5	25.35	2.09
Himachal Pradesh(HP)	3.4	68.56	0.57
Jammu & Kashmir (JK)	5.8	12.26	1.01
Karnataka(KA)	31.2	61.13	5.05
Kerala(KL)	8.0	33.35	2.76
West Bengal(WB)	22.54	91.34	7.54
Madhya Pradesh(MP)	15.36	72.59	6
Maharashtra(MH)	30.98	110.23	9.28
North East(NE)	5.9	80.4	0.92
Odisha(OD)	11.27	41.94	3.47
Punjab(PB)	12.26	27.7	2.29
Rajastan(RJ)	22.51	68.62	5.66
Tamil Nadu(TN)	27.84	72.13	5.96
Uttar	55.10	190.95	16.51

Pradesh(UP)			

Let

$$X = \frac{\text{Totalinternetusers}}{\text{Totalpopulation}}$$

$$Y = \frac{\text{BharatiAirtelInternetUsers}}{\text{Totalpopulation}}$$

and

$$Z = \frac{\text{BharatiAirtelInternetUsers}}{\text{Totalpopulation}} + \frac{\text{Populationpercentage}}{100}$$

. **Table 2. Some values with respect to internet users.**

State	**Bharati Airtel**	**X**	**Y**	**Z**
	(Internet Users)	**(in Millions)**	**(in Millions)**	**(in Millions)**
Andra Pradesh(AP)	32.4	1.09	0.38	0.421
Assam(AS)	11.09	0.52	0.35	0.3758
Bihar(BI)	40.03	0.73	0.39	0.476
Delhi(DE)	17.10	1.36	0.43	0.4439
Gujarat(GJ)	11.3	0.65	0.18	0.2299
Haryana(HR)	6.5	0.59	0.25	0.2709
Himachal Pradesh(HP)	3.4	0.27	0.04	0.0457
Jammu & Kashmir (JK)	5.8	0.92	0.47	0.4801
Karnataka(KA)	31.2	0.89	0.51	0.5605
Kerala(KL)	8.0	0.02	0.73	0.2576
West Bengal(WB)	22.54	0.58	0.24	0.3154
Madhya Pradesh(MP)	15.36	0.62	0.21	0.27

Maharashtra(MH)	30.98	0.54	0.28	0.3728
North East(NE)	5.9	0.26	0.07	0.0792
Odisha(OD)	11.27	0.51	0.26	0.2947
Punjab(PB)	12.26	0.99	0.44	0.4629
Rajastan(RJ)	22.51	0.76	0.32	0.3766
Tamil Nadu(TN)	27.84	0.85	0.38	0.4396
Uttar Pradesh(UP)	55.10	0.57	0.28	0.4451

The weight of the central vertex of fuzzy graph of internet network of Bharati Airtel is given by

$$\omega(c) = \frac{\text{TotalnumberofBharatiAirtelUsers}}{\text{Totalpopulation}}$$

$$= \frac{37{,}09{,}87{,}201}{114{,}31{,}31{,}456}$$

$$= 0.3245.$$

The degree of the vertex c is given by:

$$d_G(c) = \sum_{e\in E(G)} \rho(e)$$

$$= 6.6177.$$

The Sombor index of a graph G is given by:

$$SO(G) = \sum_{uv\in E(G)} \sqrt{(\omega(u)d_G(u))^2 + (\omega(v)d_G(v))^2}$$

$$= \sqrt{(\omega(c)d_G(c))^2 + (\omega(AP)d_G(AP))^2} + \sqrt{(\omega(c)d_G(c))^2 + (\omega(AS)d_G(AS))^2}$$

$$+\sqrt{(\omega(c)d_G(c))^2 + (\omega(BI)d_G(BI))^2} + \sqrt{(\omega(c)d_G(c))^2 + (\omega(DE)d_G(DE))^2}$$

$$+\sqrt{(\omega(c)d_G(c))^2 + (\omega(GJ)d_G(GJ))^2} + \sqrt{(\omega(c)d_G(c))^2 + (\omega(HR)d_G(HR))^2}$$

$$+\sqrt{(\omega(c)d_G(c))^2 + (\omega(HP)d_G(HP))^2} + \sqrt{(\omega(c)d_G(c))^2 + (\omega(JK)d_G(JK))^2}$$

$$+\sqrt{(\omega(c)d_G(c))^2 + (\omega(KA)d_G(KA))^2} +$$

$\sqrt{(\omega(c)d_G(c))^2+(\omega(KL)d_G(KL))^2}$

$+\sqrt{(\omega(c)d_G(c))^2+(\omega(WB)d_G(WB))^2}+\sqrt{(\omega(c)d_G(c))^2+(\omega(MP)d_G(MP))^2}$

$+\sqrt{(\omega(c)d_G(c))^2+(\omega(MH)d_G(MH))^2}+\sqrt{(\omega(c)d_G(c))^2+(\omega(NE)d_G(NE))^2}$

$+\sqrt{(\omega(c)d_G(c))^2+(\omega(OD)d_G(OD))^2}+\sqrt{(\omega(c)d_G(c))^2+(\omega(PB)d_G(PB))^2}$

$+\sqrt{(\omega(c)d_G(c))^2+(\omega(RJ)d_G(RJ))^2}+\sqrt{(\omega(c)d_G(c))^2+(\omega(TN)d_G(TN))^2}$

$+\sqrt{(\omega(c)d_G(c))^2+(\omega(UP)d_G(UP))^2}$

$=\sqrt{4.4845+0.02559}+\sqrt{4.4845+0.0173}+\sqrt{4.4845+0.0344}+\sqrt{4.4845+0.0364}$

$+\sqrt{4.4845+0.0074}+\sqrt{4.4845+0.0045}+\sqrt{4.4845+0.00003}+\sqrt{4.4845+0.0509}$

$+\sqrt{4.4845+0.09171}+\sqrt{4.4845+0.0035}+\sqrt{4.4845+0.0057}+\sqrt{4.4845+0.0032}$

$+\sqrt{4.4845+0.00108}+\sqrt{4.4845+0.00003}+\sqrt{4.4845+0.0058}+\sqrt{4.4845+0.0412}$

$+\sqrt{4.4845+0.0145}+\sqrt{4.4845+0.0279}+\sqrt{4.4845+0.0155}$

$= 40.3454.$

The $SO(G)$ of states (vertex) are given in the Table 3, and they are calculated by the formula:

$$SO(State) = SO(G) - SO(G - State)$$

3.1 Decision Making

From the Table 4, we have SO(NE)< SO(HP)< SO(KL)< SO(HR)< SO(MP)< SO(GJ)< SO(OD) < SO(WB) < SO(AP) < SO(AS) < SO(RJ) < SO(MH) < SO(DE) < SO(PB) < SO(JK)< SO(TN)< SO(UP)< SO(BI)< SO(KA).

Now, the least sombor index of a vertex indicates that the vertex is most crucial for the development of internet network systems. Then, the state having the largest population percentage that does not use the internet becomes the first to develop an internet network system. Then we can order the states as follows, and needing more development:

North East, Himachal Pradesh, Kerala, Haryana, Madhya Pradesh, Gujrat, Odhisa, West Bengal, Andra Pradesh, Assam, Rajastan, Maharastra, Delhi, Punjab, Jammu & Kashmir, Tamil Nadu, Uttar Pradesh, Bihar and Karnataka.

Table 3. Membership values and degrees of the fuzzy graph of Figure 6.

State	MV of the State (Vertex)	Degree of the state vertex	Degree of an edge between state and center of the star
AP	0.38	0.421	6.1967
AS	0.35	0.3758	6.2419
BI	0.39	0.476	6.1417
DE	0.43	0.4439	6.1738
GJ	0.18	0.2299	6.3878
HR	0.25	0.2709	6.3468
HP	0.04	0.0457	6.572
JK	0.47	0.4801	6.1376
KA	0.51	0.5605	6.0572
KL	0.23	0.2576	6.3601
MP	0.21	0.27	6.3477
MH	0.28	0.3228	6.2449
OD	0.26	0.2947	6.323
PB	0.44	0.4629	6.1548
RJ	0.32	0.3766	6.2411
TN	0.38	0.4396	6.1781
UP	0.28	0.4451	6.1726
WB	0.24	0.3154	6.3023
NE	0.07	0.0792	6.5385

Table 4. Value of Sombor index for all states.

State	SO(G)-State	SO(State)
AP	40.1854	0.1600
AS	40.1775	0.1679
BI	40.0525	0.2929
DE	40.1127	0.2323
GJ	40.2259	0.1195
HR	40.2483	0.0971
HP	40.3244	0.0210
JK	40.1094	0.2360
KA	39.9897	0.3557
KL	40.2524	0.0930
WB	40.1885	0.1569
MP	40.2271	0.1183
MH	40.1439	0.2015
NE	40.3393	0.0061
OD	40.2244	0.121
PB	40.1109	0.2347
RJ	40.1528	0.1926
TN	40.1032	0.2422
UP	40.0669	0.27895

Conclusion: In this article, the sombor index was introduced as a graph parameter to quantify the structural characteristics of a graph. This paper provided bounds for some standard class of fuzzy graphs and applied these results to real-life problems in the field of internet system development. To analyze the Bharati Airtel Ltd. internet system in India, this paper constructed an internet system graph. In this graph, the least Sombor index of a vertex indicates that the vertex is most crucial for the development of the internet network system. According to this paper, the state with the highest proportion of people who do not use the internet is the first to develop an internet network system.

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