A Study on the Lattice of Convex Sublattices of $S(B_n)$

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Abstract: The concept of 0-Supermodular lattices was introduced in the thesis of J.Arivukkarasu which appeared in the paper[1]. He proved there that the lattices $S(B_n)$ and $S(D)$ are 0-Supermodular. In this paper, we prove that the lattice of convex sublattices of $S(B_n)$ with respect to the new ordering defined by S.Lavanya and S.ParameshwaraBhatta[7], is 0-Supermodular.

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1. Introduction

Let L be a lattice and $CS(L)$ be the set of all nonempty convex sublattices of L.The work on $\langle CS(L)\cup\{\varphi\},\subseteq\rangle$ has been done by many authors. For example, one can refer to Koh[5,6],chen[2],Marmazeev[8], A.Vethamanickam and R.Subbarayan[12],Sheeba Merlin[13].

In 1996,S.Lavanya and S.ParameshwaraBhatta[7] have introduced a new partial ordering on $CS(L)$ and established many results on $CS(L)$ with respect to that ordering. For example , they have proved that L and $CS(L)$ are in the same equational class. Also from this paper,we know that $[\{0\},L]\cong I(L)$ and $[L,\{1\}]\cong D(L)$,where $I(L)$ is the lattice of all ideals of L and $D(L)$ is the lattice of all dual ideals(Filters) of L.Further works on $CS(L)$ with respect to this ordering can be seen in the papers of P.V.Ramanamurty[9] and R.M.H.Raman[10].

The 0-version of the concept of supermodular lattice namely,a 0-supermodular lattice was introduced in the paper by A.Vethamanickam and J.Arivukkarasu[1].There they have proved that the Eulerian lattice $S(B_n)$is 0-supermodular.0-Supermodularity condition is not an identity. So, the class of all 0-supermodular lattices is not an equationalclass[1]. Though, 0-supermodularity condition is not equational, we attempt in the lines of S.Lavanya and S.ParameshwaraBhatta to establish that $CS[S(B_n)]$ to be 0-supermodular and we have succeeded in this attempt. For general 0-supermodular lattices L the question of deciding the 0-supermodularity of $CS(L)$ remains an open problem.

2. Preliminaries

In this section,we give some basic definitions needed for the development of the paper.

2.1 Poset

A partially ordered set<A, ρ >consists of a non empty set A and a binary relation ρ on A such that ρ satisfies the following properties:

(i) ρ is reflexive.that is, a ρ a for every a $\in$ A.

(ii) ρ is antisymmetric.that is ,whenever a ρ b and b ρ a then a =b.

(iii) ρ is transitive.that is ,whenever a ρ b and b ρ c thena ρ c for every a,b,c $\in$ A

ρ is called a partial ordering relation on A and is usually denoted by the notation $\leq$.

If aposet<A, $\leq$> also satisfies the linearitycondition, that is,either a $\leq$ b or b $\leq$ a, for every a,b $\in$ A, then it is called a **chain.**

If<A, $\leq$> is a poset, a,b $\in$ A, then a and b are **comparable** if either a $\leq$ b or b $\leq$ a.

Otherwise, they are called incomparable elements . In notation, we write a || b.

2.2Lattice

A poset<L; $\leq$> is said to be a lattice if inf{a,b} and sup{a,b} exist for all a,b $\in$ L,

we denote by inf{a,b} = a$\wedge$band sup{a,b}=a$\vee$b

2.3Sublattice.

Let L be any lattice .A non empty subset K of a lattice L which is closed under $\wedge$ and $\vee$ is called a sublattice of L.

2.4Convex sublattice.

A sublattice K of a lattice L is saidto be convex iffa,b $\in$ K, c $\in$ L and $a \leq c \leq b$ imply that c $\in$ K. For example,if a,b $\in$ L, $a \leq b$,the interval $[a, b] = \{x \in L / a \leq x \leq b\}$ is an example of a convex sublattice of L.The collection of allnon empty convex sublattices of a lattice L is denoted by CS(L)

2.5A new partial orderingon CS(L)

We define a binary relation $\leq$ on CS(L) by the following rule: for A,B $\in$ CS(L), A $\leq$ B if and only if "for every a $\in$ A there exists a b $\in$ B such that a $\leq$ b and for every b $\in$ B there exists an a $\in$ A such that b $\geq$ a", clearly '$\leq$' is a partial order on CS(L). Moreover, $\langle CS(L), \leq \rangle$ is a lattice.

2.6 Notations

Let L be a lattice. For a subset A of L, we denote (A],[A) and $\langle A \rangle$ respectively the ideal, the filter and the convex sublattice of L generated by A .Let I(L) be the lattice of ideals of L (ordered by $\subseteq$) and D(L) be the lattice of filters of L (ordered by $\supseteq$)

2.7 Remark:

For A,B $\in$ CS(L), under the new partial ordering $\leq$, we have

$$A \wedge B = \langle \{a \wedge b / a \in A, b \in B\} \rangle = \{x \in L / a \wedge b \leq x \leq a_1 \wedge b_1 \ for\ some\ a, a_1 \in A, b, b_1 \in B\}$$

$$A \vee B = \langle \{a \vee b / a \in A, b \in B\} \rangle = \{x \in L / a \vee b \leq x \leq a_1 \vee b_1\ for\ some\ a, a_1 \in A, b, b_1 \in B\}$$

2.8Supermodular lattice

A lattice L is said to be supermodular if it satisfies the following identity $(a\vee b)\wedge(a\vee c)\wedge(a\vee d)=a\vee[b\wedge c\wedge(a\vee d)]\vee[c\wedge d\wedge(a\vee b)]\vee[b\wedge d\wedge(a\vee c)]$
for all a,b,c,d $\in$ L

2.90-Supermodular lattice

A Lattice L is called 0-Supermodular,ifwhenever b, c, d $\in$ L satisfy $b\wedge c=c\wedge d=b\wedge d=0, then\ (a\vee b)\wedge(a\vee c)\wedge(a\vee d)=a$ for every a $\in$ L.

2.10 Definition[11]

$S(B_n)=(\overline{B_2}\times\overline{B_n})\cup\{(1,1)\}$ where $\overline{B_2}=B_2-\{1\}\ and\ \overline{B_n}=B_n-\{1\}$ and B_n is the Boolean lattice of rank n.

$S(B_3)=(\overline{B_2}\times\overline{B_3})\cup\{(1,1)\}$ where B_3 is the Boolean lattice of rank 3.

As 0-supermodularity condition is not an identity, we can not immediately conclude that CS(L) is 0-supermodular if L is 0-supermodular. Therefore a natural question is:as to for what 0-supermodular lattices L,CS(L) will be 0-supermodular?In this paper, we prove independently that CS[S(B_n)] is 0-supermodular.

2.11 SimplicialPoset

Let P be a Poset with 0. P is said to be simplicial if for every element t∈P, [0,t] is Boolean. A dual simplicialposet is defined dually.

For example, S(B_n) is Simplicial.

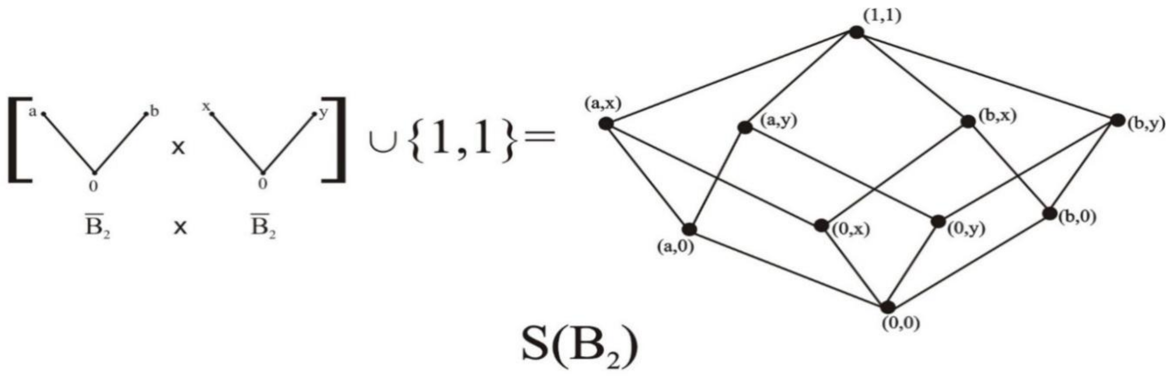

Figure 1

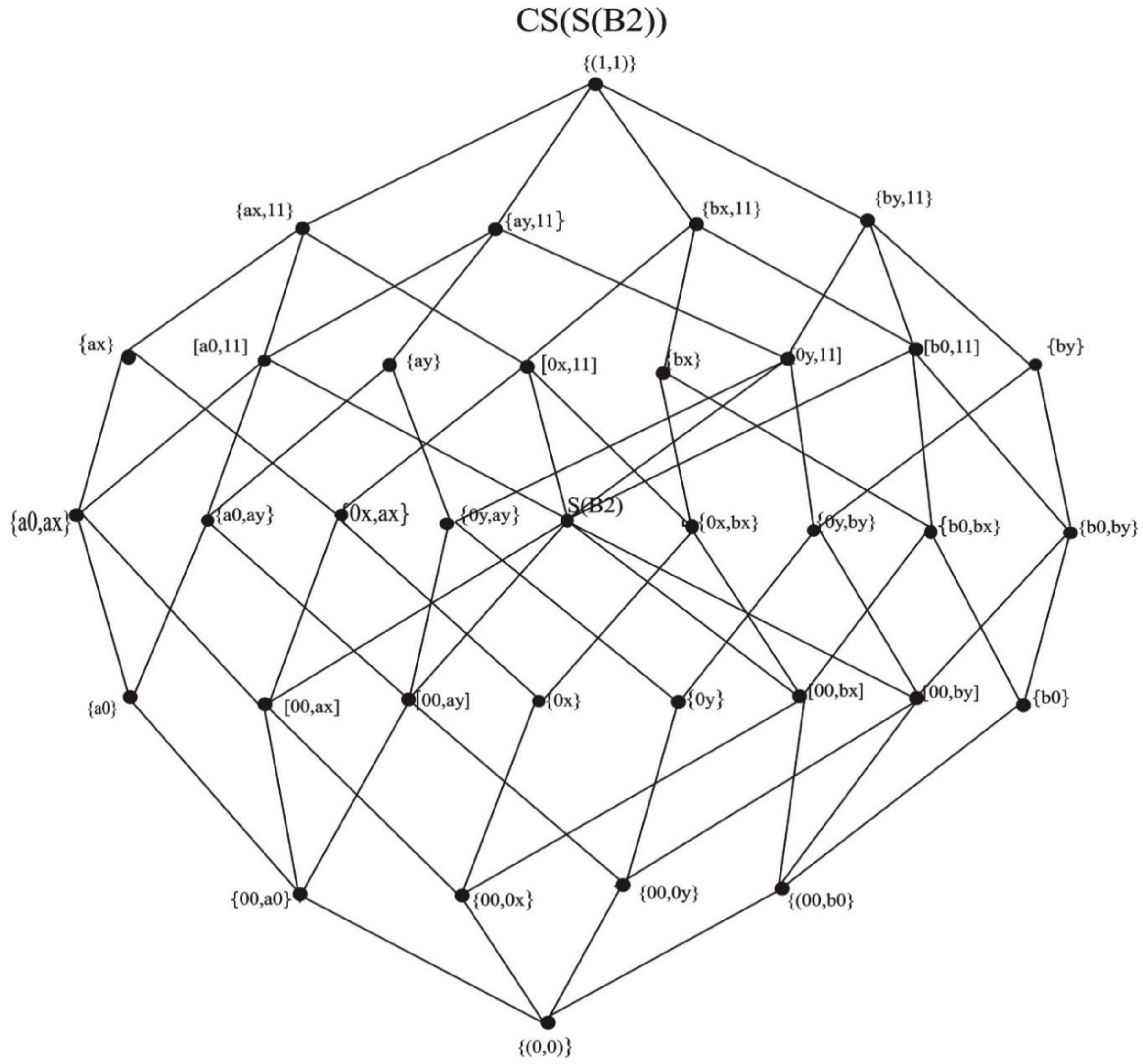

Figure 1.a

2.13 Remark:

In fig1 we give the lattice structure of $S(B_2)$ and in figure 1a, we produce the lattice structure of $CS[S(B_2)]$so that one can get a clear understanding of the structure of $CS[S(B_n)]$, for n>2.Since the diagram of $CS[S(B_4)]$is big, we only give in table1,below the rankwise.

The rankwise classification of the elements of $CS[S(B_4)]$ specifying the types in the respective ranks to get a glimpse of elements in $CS[S(B_n)]$.We note that the rank of $S(B_4) = 5$;

Therefore, the rank of $CS[S(B_4)] = 2x5=10$

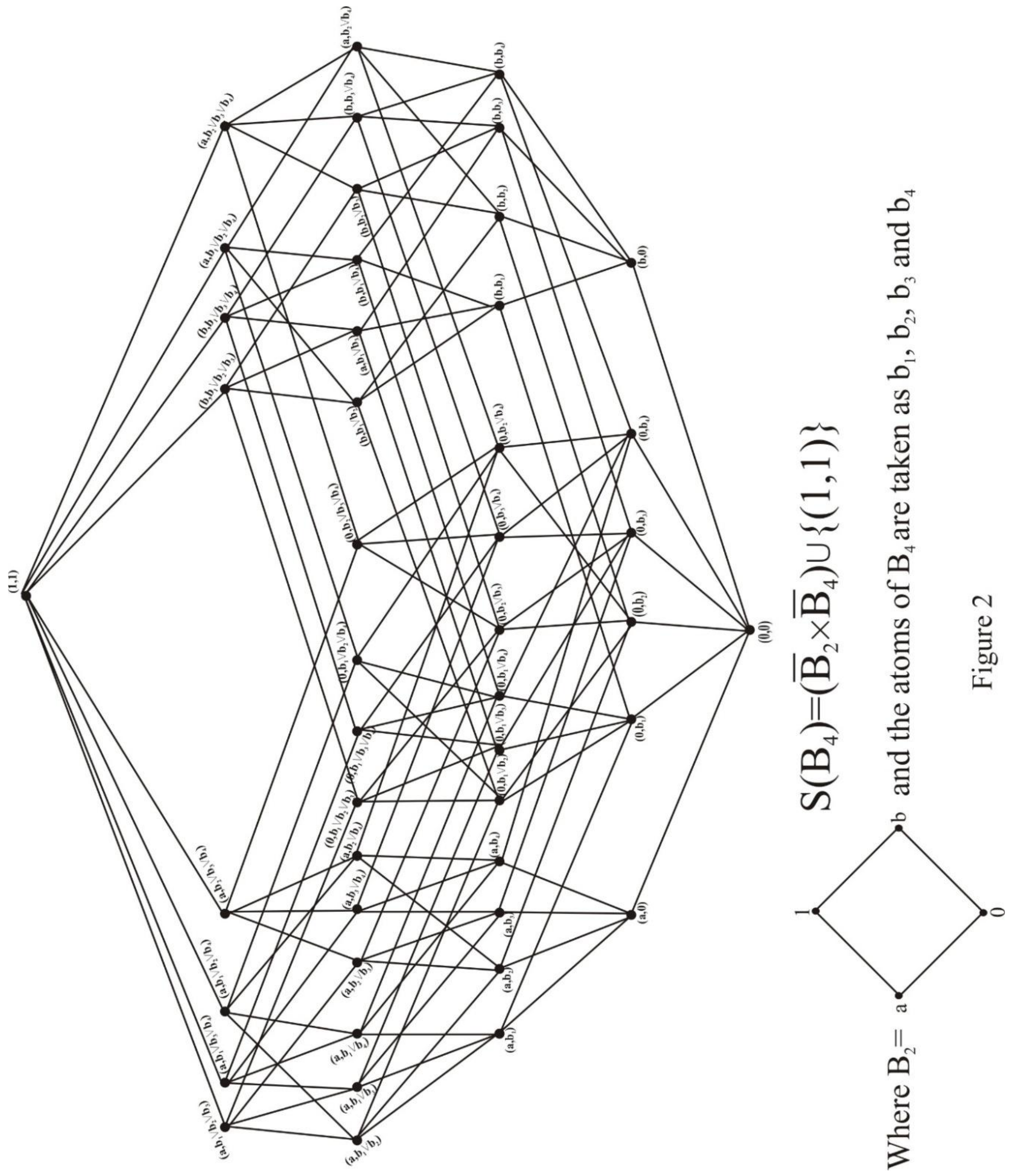

Figure 2

Table 1: Rankwise classification of the elements of CS[S(B_4)]

S.No	Rank	Number of types	Types of convex sublattices of S(B_4)
1	0	1	$\{(0,0)\}$
2	1	1	Edges from (0,0) to an element of rank 1(atom)For example, the setsof the form $((0,b_i)]\ or\ ((a,0)]\ or\ ((b,0)]$
3	2	2	Singletons of elements of rank 1 and intervals of the

			form[(0,0),a rank2 element] $\cong B_2$. For example,the sets of the form $\{(0,b_i)\}$ or $\{(a,0)\}$ or $\{(b,0)\}$ and $((a,b_i),(b,b_i)]$ or $((0,b_i \vee b_j)]$
4	3	2	Edges from an element of rank 1 to an element of rank 2 and intervals of the form[(0,0),a rank3 element] $\cong B_3$. For example,the sets of the form $[(a,0),(a,b_i)]$ or $[(b,0),(b,b_i)]$ or $[(0,b_i),(b_i,b_i \vee b_j)]$
5	4	3	Singletons of rank 2 element and the intervals of the form[a rank 1 element, a rank 3 element]and [(0,0), a rank 4 element] $\cong B_4$. For example,the sets of the form $\{(a,b_i)\}$ or $\{(b,b_i)\}$ or $[(a,0),(a,b_i \vee b_j)]$or $[(0,b_i),(0,b_i \vee b_j \vee b_k)]$or $((a,b_i \vee b_j \vee b_k)]$ or $((b,b_i \vee b_j \vee b_k)]$
6	5	3	Edges from an element of rank 2 to an element of rank 3 and the intervals of the form[a rank 1 element, a rank 4 element] and $S(B_4)$. For example,the sets of the form $[(a,b_i),(a,b_i \vee b_j)]$ or $[(b,b_i),(b,b_i \vee b_j)]$ or $[(0,b_i \vee b_j),(0,b_i \vee b_j \vee b_k)]$or $[(a,0),(a,b_i \vee b_j \vee b_k)]$or $[(0,b_i),(a,b_i \vee b_j \vee b_k)]$
7	6	3	Singletons of rank 3 elements and theupper intervals froman element of rank2 to an element of rank 4 and the upper intervals froman atom. For example,the sets of the form $\{(a,b_i \vee b_j)\}$or $\{(0,b_i \vee$ *bj* ∨*bk*or *b,bi* ∨*bj* or *a,0,a,bi* ∨*bj* ∨*bk*or *0,bi,a,bi* ∨*bj* ∨*bk*
8	7	2	Edges from an element of rank 3 to an element of rank4 and the upper intervals from an element of rank 2. For example,the sets of the form $[(a,b_i \vee b_j),(a,b_i \vee b_j \vee b_k)]$ or $[(0,b_i \vee b_j \vee b_k),(a,b_i \vee b_j \vee b_k)]$or $[(a,b_i))$ or $[(0,b_i \vee b_j \vee b_k))$
9	8	2	Singletons of rank 4 element and the upper intervals from an element of rank 3. For example,the sets of the form $\{(a,b_i \vee b_j \vee b_k)\}$ or $\{(b,b_i \vee b_j \vee b_k)\}$ or $[(a,b_i \vee b_j))$
10	9	1	The upper intervals from an element of rank 4

			$\left[\left(a,b_i\vee b_j\vee b_k\right)\right)$
11	10	1	$\{(1,1)\}$

Section 3:0-Supermodularity of CS[S(B_n)]

Theorem3.1CS[S(B_n), ≤] is 0-Supermodular

Proof:

To prove that CS[S(B_n)] is 0-Supermodular,we need to prove that

(A∨B)∧(A∨C)∧(A∨D)=A, for every three mutually disjoint elementsB,C,D∈CS[S(B_n)] and for every A∈CS[S(B_n)] $----\rightarrow(a)$

If all of A,B,C,D∈ [{0,0},S(B_n)] ≅ S(B_n),then (a) is true as S(B_n) is 0-Supermodular.

If one of B,C,D is in [S(B_n),{(1,1)}],then there is no choice for other mutual disjoint elements, Therefore we avoid that cases.

Let us take the n atoms of $B_n$as $b_1,b_2,\dots,b_n$.

We note that rank of S(B_n) = n+1, therefore rank of CS[S(B_n)] = 2n+2 by [7]

Choose an element B of rank k in CS[S(B_n)]

Let us assume that k is odd and k>n+1

Let k = n+1+s for some s such that $1\leq s< n+1$

We observe that the convex sublattices of S(B_n) with rank k in CS[S(B_n)] are of the following types:

(1) :The highest rank convex sublattice is an upper interval from an element of rank s in S(B_n) (or a dual ideal)

(2) : The next highest rank convex sublattice is an interval from an element of rank s+1to an element of rank n in S(B_n) and so on.

(r+1) : Convex sublattice of rank n+1-2r-s is an interval from an element of rank s+r to an element of rank n-r+1

Finally , an edge from an element of rank (k-1)/2 to an element of rank (k+1)/2

Let us choose one of the mutual disjoint elements B to have the rank k.

Choose B to be an interval from an element of rank s+r to an element of rank n–r+1

Assume without loss of generality that

B = $\left[\left(a,b_1\vee b_2\dots\vee b_{s+r-1}\right),\left(a,b_1\vee b_2\dots\vee b_{s+r-1}\vee b_{s+r}\vee\dots\vee b_{n-r}\right)\right]$

Next we have to choose C such that C∧B={(0,0)}

Then $C\in CS\left[(0,0),\left(b,b_{n-r+1}\vee\dots\vee b_n\right)\right]$

Rank of $\left[(0,0),\left(b,b_{n-r+1}\vee\dots\vee b_n\right)\right]=r+1 ----\rightarrow(I)$

Then the rank of $CS\left[(0,0),\left(b,b_{n-r+1}\vee\dots\vee b_n\right)\right]=2(r+1)$

Choose C such that the rank of C is m_1, where 1< m_1 <2(r+1)

Case(i) let m_1 be even, let m_1 = $2r_1$ where we choose $2r_1 \leq$ r +1

1. The highest rank convex sublattice is an upper interval from an element of rank 0 to $2r_1$ (or an ideal)
2. The next highest rank convex sublattice is an interval from an element of rank 1 to an element of rank $2r_1 - 1$ and so on.

The convex sublattice in the k_1th step is an interval from an element of rank k_1 to an element of rank $2r_1 - k_1$

Finally since C is of even rank $2r_1$, C may be a singleton of an element of rank r_1 in $S(B_n)$

C= $\left[\left(0, b_{n-r+1} \vee \ldots \vee b_{n-r+k_1}\right), \left(0, b_{n-r+1} \vee \ldots \vee b_{n-r+k_1} \vee \ldots \vee b_{n-r+2r_1-1}\right)\right]$

Now we have to choose D such that D∧B={(0,0)} and D∧C={(0,0)} then

$$D \in CS\left[(0,0), \left(b, b_{n-r+2r_1} \vee \ldots \vee b_n\right)\right]$$

Rank of $\left[(0,0), \left(b, b_{n-r+2r_1} \vee \ldots \vee b_n\right)\right]$ = $r + 2r_1 + 1 \text{-----}\rightarrow (II)$

Then the rank of $CS\left[(0,0), \left(b, b_{n-r+2r_1} \vee \ldots \vee b_n\right)\right]$ = $2\left(r + 2r_1 + 1\right)$

Choose D such that rank of D be m_2, where 1< m_2 <$2\left(r + 2r_1 + 1\right)$

Case(i)let m_2 be even, let m_2 = $2r_2$ where we choose $2r_2 \leq \left(r + 2r_1 + 1\right)$

1. The highest rank convex sublattice is an interval from an element of rank 0 to $2r_2$(or an ideal)
2. The next highest rank convex sublattice is an interval from an element of rank 1 to an element of rank $2r_2$-1 and so on.

The convex sublattice in the k_2th step is an interval from an element of rank k_2 to an element of rank $2r_2 - k_2$

Finally since D is of even rank $2r_2$, D may be a singleton set of an element of rank r_2,

Let D= $\left[\left(0, b_{n-r+2r_1} \vee \ldots \vee b_{n-r+2r_1+k_2-1}\right), \left(b, b_{n-r+2r_1+k_2} \vee \ldots \vee b_{n-r+2r_1+2r_2-1}\right)\right]$

So the remaining atoms in B_n are $b_{n-r+2r_1+2r_2}, b_{n-r+2r_1+2r_2+1}, \ldots, b_n$

There are $r - 2r_1 - 2r_2 + 1$ remaining number of atoms

Now we have to choose the 4th element A, which is incomparable with any of B,C or D which can be chosen in two ways.

$(I)\, A \in CS\left[(0,0), \left(b, b_{n-r+2r_1+2r_2} \vee \ldots \vee b_n\right)\right]$ (or)

(II) A may contain some of the atoms coming in B,C(or) D

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In Case(I) Let A = $\left[\left(0,b_{n-r+2r_1+2r_2}\right),\left(b,b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\right], 1\le t\le r-2r_1-2r_2+1$

B = $\left[\left(a,b_1\vee b_2\ldots\vee b_{s+r-1}\right),\left(a,b_1\vee b_2\ldots\vee b_{s+r-1}\vee b_{s+r}\vee\ldots\vee b_{n-r}\right)\right]$

C= $\left[\left(0,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\right),\left(0,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\vee\ldots\vee b_{n-r+2r_1-1}\right)\right]$

D= $\left[\left(0,b_{n-r+2r_1}\vee\ldots\vee b_{n-r+2r_1+k_2-1}\right),\left(b,b_{n-r+2r_1+k_2}\vee\ldots\vee b_{n-r+2r_1+2r_2-1}\right)\right]$

Now A∨B= $\left[\left(0,b_{n-r+2r_1+2r_2}\right),\left(b,b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\right]\vee$ $\left[\left(a,b_1\vee b_2\vee\ldots\vee b_{s+r-1}\right),\left(a,b_1\vee b_2\vee\ldots\vee b_{s+r-1}\vee b_{s+r}\vee\ldots\vee b_{n-r}\right)\right]$

$=\left[\left(a,b_1\vee b_2\vee\ldots\vee b_{s+r-1}\vee b_{n-r+2r_1+2r_2}\right),(1,1)\right]$

A∨C= $\left[\left(0,b_{n-r+2r_1+2r_2}\right),\left(b,b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\right]\vee$ $\left[\left(0,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\right),\left(0,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\vee\ldots\vee b_{n-r+2r_1-1}\right)\right]$

$$=\left[\begin{array}{l}\left(0,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\vee b_{n-r+2r_1+2r_2}\right),\\ \left(b,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\vee\ldots\vee b_{n-r+2r_1-1}\vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\end{array}\right]$$

A∨D

$=\left[\left(0,b_{n-r+2r_1+2r_2}\right),\left(b,b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\right]\vee$ $\left[\left(0,b_{n-r+2r_1}\vee\ldots\vee b_{n-r+2r_1+k_2-1}\right),\left(b,b_{n-r+2r_1}\vee\ldots\vee b_{n-r+2r_1+2r_2-1}\right)\right]$

$$=\left[\begin{array}{l}\left(0,b_{n-r+2r_1}\vee\ldots\vee b_{n-r+2r_1+k_2-1}\vee b_{n-r+2r_1+2r_2}\right),\\ \left(b,b_{n-r+2r_1+k_2}\vee\ldots\vee b_{n-r+2r_1+2r_2-1}\vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\end{array}\right]$$

(A∨B)∧(A∨C) = $\left[\left(a,b_1\vee b_2\vee\ldots\vee b_{s+r-1}\vee b_{n-r+2r_1+2r_2}\right),(1,1)\right]\wedge$

$$\begin{bmatrix}\left(0,b_{n-r+1}\vee\ldots\vee b_{n-r+2r_1+2r_2}\right)\\ \left(b,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\vee\ldots\vee b_{n-r+2r_1-1}\vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\end{bmatrix}$$

$$=\begin{bmatrix}\left(0,b_{n-r+2r_1+2r_2}\right),\\ \left(b,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\vee\ldots\vee b_{n-r+2r_1-1}\vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\end{bmatrix}$$

(A∨B)∧(A∨C)∧(A∨D)=

$$\begin{bmatrix}\left(0,b_{n-r+2r_1+2r_2}\right),\\ \left(b,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\vee\ldots\vee b_{n-r+2r_1-1}\vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\end{bmatrix}\wedge$$

$$\begin{bmatrix}\left(0,b_{n-r+2r_1}\vee\ldots\vee b_{n-r+2r_1+k_2-1}\vee b_{n-r+2r_1+2r_2}\right),\\ \left(b,b_{n-r+2r_1+k_2}\vee\ldots\vee b_{n-r+2r_1+2r_2-1}\vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\end{bmatrix}$$

$$=\left[\left(0,b_{n-r+2r_1+2r_2}\right),\left(b,b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+t}\right)\right]$$

=A

Case(II)

Let

A=

$$\begin{bmatrix}(0,0),\\ \left(b,b_{i_1}\vee b_{i_2}\ldots b_{i_{l_1}}\vee b_{j_1}\vee b_{j_2}\ldots\vee b_{j_{l_2}}\vee b_{n_1}\vee b_{n_2}\ldots\vee b_{n_{l_3}}\vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+u}\right)\end{bmatrix}$$

B=$\left[\left(a,b_1\vee b_2\vee\ldots\vee b_{s+r-1}\right),\left(a,b_1\vee b_2\vee\ldots\vee b_{s+r-1}\vee b_{s+r}\vee\ldots\vee b_{n-r}\right)\right]$

C=$\left[\left(0,b_{n-r+1}\vee\ldots b_{n-r+k_1}\right),\left(0,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\vee\ldots\vee b_{n-r+2r_1-1}\right)\right]$

D=$\left[\left(0,b_{n-r+2r_1}\vee\ldots\vee b_{n-r+2r_1+k_2-1}\right),\left(b,b_{n-r+2r_1+k_2}\vee\ldots\vee b_{n-r+2r_1+2r_2-1}\right)\right]$

Now (A∨B) =$\left[\left(a,b_1\vee b_2\vee\ldots\vee b_{s+r-1}\right),(1,1)\right]$

A∨C=

$$\left[\left(0,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\right),\begin{pmatrix} b,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\vee\ldots\vee b_{n-r+2r_1} \\ \vee b_{i_1}\vee b_{i_2}\ldots\vee b_{i_{l_1}}\vee b_{j_1}\vee b_{j_2}\ldots b_{j_{l_2}}\vee b_{n_1}\vee b_{n_2}\ldots\vee b_{n_{l_3}} \\ \vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+u} \end{pmatrix}\right]$$

A∨D=

$$\left[\left(0,b_{n-r+2r_1}\vee\ldots\vee b_{n-r+2r_1+k_2-1}\right),\begin{pmatrix} b,b_{n-r+2r_1+k_2}\vee\ldots\vee b_{n-r+2r_1+2r_2-1} \\ \vee b_{i_1}\vee b_{i_2}\ldots\vee b_{i_{l_1}} \\ \vee b_{j_1}\vee b_{j_2}\ldots b_{j_{l_2}}\vee b_{n_1}\vee b_{n_2}\ldots\vee b_{n_{l_3}} \\ \vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+u} \end{pmatrix}\right]$$

$$(A\vee B)\wedge(A\vee C)=\left[\begin{matrix}(0,0), \\ \begin{pmatrix} b,b_{n-r+1}\vee\ldots\vee b_{n-r+k_1}\vee\ldots\vee b_{n-r+2r_1-1} \\ \vee b_{i_1}\vee b_{i_2}\ldots b_{i_{l_1}}\vee b_{j_1}\vee b_{j_2}\ldots\vee b_{j_{l_2}}\vee b_{n_1}\vee b_{n_2}\ldots\vee b_{n_{l_3}} \\ \vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+u} \end{pmatrix}\end{matrix}\right]$$

(A∨B)∧(A∨C)∧(A∨D) =

$$\left[(0,0),\begin{pmatrix} b,b_{i_1}\vee b_{i_2}\ldots\vee b_{i_{l_1}} \\ \vee b_{j_1}\vee b_{j_2}\ldots\vee b_{j_{l_2}}\vee b_{n_1}\vee b_{n_2}\ldots\vee b_{n_{l_3}}\vee b_{n-r+2r_1+2r_2}\vee\ldots\vee b_{n-r+2r_1+2r_2+u} \end{pmatrix}\right]$$

=A

Hence in case(i) when m_1 is even , m_2 is even and $m_1 \le r+1$, $m_2 < r+2r_1+1$, we have proved that the equation (a) is true.

The other cases are

Case(ii) m_1 is even and m_2 is even and $m_1 \le r+1$, $m_2 > r+2r_1+1$

Case(iii) m_1 is even and m_2 is even and $m_1 > r+1$, $m_2 \le r+2r_1+1$

Case(iv) m_1 is even and m_2 is even and $m_1 > r+1$, $m_2 > r+2r_1+1$

Case(v) m_1 is even and m_2 is odd and $m_1 \le r+1$, $m_2 \le r+2r_1+1$

Case(vi) m_1 is even and m_2 is odd and $m_1 \le r+1$, $m_2 > r+2r_1+1$

Case(vii) m_1 is even and m_2 is odd and $m_1 > r+1$, $m_2 \le r+2r_1+1$

Case(viii) m_1 is even and m_2 is odd and $m_1 > r+1$, $m_2 > r+2r_1+1$

Case(ix) m_1 is odd and m_2 is even and $m_1 > r+1$, $m_2 \leq r+2r_1+1$

Case(x) m_1 is odd and m_2 is even and $m_1 > r+1$, $m_2 > r+2r_1+1$

Case(xi) m_1 is odd and m_2 is even and $m_1 \leq r+1$, $m_2 \leq r+2r_1+1$

Case(xii) m_1 is odd and m_2 is even and $m_1 \leq r+1$, $m_2 > r+2r_1+1$

Case(xiii) m_1 is odd and m_2 is odd and $m_1 > r+1$, $m_2 \leq r+2r_1+1$

Case(xiv) m_1 is odd and m_2 is odd and $m_1 > r+1$, $m_2 > r+2r_1+1$

Case(xv) m_1 is odd and m_2 is odd and $m_1 \leq r+1$, $m_2 \leq r+2r_1+1$

Case(xvi) m_1 is odd and m_2 is odd and $m_1 \leq r+1$, $m_2 > r+2r_1+1$

In all the above cases, the argument is the same as in case(i),since the convex sublattices of $S(B_n)$ are in a particular interval form only.

Hence in all the cases (a) is true.

Hence we conclude that $CS[S(B_n)]$ is 0-supermodular.

Conclusion:

There is a scope that the above result can be extended to other weaker conditions like 0-modularity,0-semi modularity, pseudo 0-distributivityetc.

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