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Hub Dominating Sets and Hub Domination Polynomials of the Lollipop Graph L_(n,1)

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Abstract

Let G=(V,E)be a simple graph. Let HD(G,j) be the family of hub dominating sets in G with cardinality j. Then, the polynomial,

HD(G,x)=∑_(j=hd(G))^|V(G)|▒〖hd(G,j)x^j 〗

is called the hub domination polynomial of G where hd(G,j) is the number of hub dominating sets of G of cardinality j and hd(G) is the hub domination number of G . Let L_(n,1) denotes the Lollipop graph with n+1 vertices and HD(L_(n,1),j) denotes the family of hub dominating sets of L_(n,1)with cardinality j. Then, the polynomial,

HD(L_(n,1),x)=∑_(j=hd(L_(n,1)))^|V(L_(n,1)) ▒〖hd(L_(n,1),j) x^j 〗

is called the hub domination polynomial of L_(n,1)where hd(L_(n,1),j) is the number of hub dominating sets of L_(n,1)of cardinality j and hd(L_(n,1))is hub domination number of L_(n,1).In this paper, we obtain a recursive formula for hd(L_(n,1),j). Using this recursive formula, we construct the hub domination polynomial ofL_(n,1) as,

HD(L_(n,1),x)=∑_(j=1)^(n+1)▒〖hd(L_(n,1),j)x^j 〗

where hd(L_(n,1),j) is the number of hub dominating sets of L_(n,1)of cardinality j and some of the properties of this polynomial also have been studied.

Keywords*:Lollipop Graph, Hub Dominating Sets, Hub Domination Number, Hub Domination Polynomial.*

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1. Introduction

Let $G = (V, E)$ be a simple graph. The number of vertices in G is called the order of G and the number of edges in G is called the size of G. A graph G is called a complete graph if any

two distinct vertices of G are adjacent. A path graph is a tree with two nodes of vertex degree1 and the other nodes of vertex degree 2. Let K_n denotes the complete graph with n vertices and P_m denotes the path graph with m vertices.

The Lollipop graph $L_{n,m}$ is the graph obtained by joining a complete graph K_n to a path graph P_m with a bridge and it is denoted by $L_{n,m}$.A set $D \subseteq V$ is a dominating set of G if $N[D] = V$ or equivalently, every vertex in $V - D$ is adjacent to atleast one vertex in D. The domination number of a graph G is defined as the minimum cardinality taken over all the dominating sets D of vertices in G and is denoted by $\gamma(G)$.

In the next section, we construct the families of hub dominating sets of $L_{n,1}$ by recursive method. In section III, we study about the hub domination polynomials of the complete bipartite graph $L_{n,1}$ using the results which we obtained in section II.

For n to j combination, we use $\binom{n}{j}$ as usual. In addition, we denote the set $\{1,2,\dots\dots.n\}$by $[n]$.

$v$path of length two in same direction. Orient the left out $u-v$ path in opposite direction. It is strongly connected and is called a Diglobe. It is denoted as $\overrightarrow{Gl(n)}$.

Ii. Hub Dominating Sets of the Lollipop Graph $L_{n,1}$

The hub domination number for the Lollipop graph $L_{n,1}$and a few of the properties of the hub dominating sets of $L_{n,1}$ are discussed in this section.

Definition 2.1: The Lollipop graph $L_{n,1}$ is the graph obtained by joining a complete graph on n vertices and a path graph on 1 vertex with a bridge.

Definition 2.2: Let G be a simple graph of order n with no isolated vertices. A set $D \subseteq V$ is said to be a hub dominating set if every vertex in $V-D$ is adjacent to atleast one vertex in D and every pair of vertices in $V-D$ has a path in G such that all the internal vertices of the path are in D.

Definition 2.3: The hub dominaton number of a graph G is defined as the minimum cardinality taken over all the hub dominating sets D of vertices in G and is denoted by $\hbar d(G)$.

Lemma 2.4: For all $n \in Z^+$, $\binom{n}{j} = 0$ if $j > n$ or $j < 0$.

Theorem 2.5: Let $L_{n,1}$ be the Lollipop graph with $n+1$ vertices.

Then $hd(L_{n,1}j) = \begin{cases} \binom{n+1}{j} - \binom{n}{j} \ when\ j = 1 \\ \binom{n+1}{j} - \binom{n-1}{j} \quad if\ 2 \le j \le n+1 \end{cases}$

Proof: Let $L_{n,1}$ be the Lollipop graph with $n+1$ vertices and $n \ge 4$. Let $v_1, v_2, v_3 \dots v_n, v_{n+1}$ be the vertices of $L_{n,1}$, where v_n is the vertex of degree n and v_{n+1} is the vertex of degree $n+1$. Since $L_{n,1}$ contains $n+1$ vertices, the number of subsets of $L_{n,1}$ with cardinality j is $\binom{n+1}{j}$. Each time $\binom{n-1}{j}$ number of subsets of $L_{n,1}$ with cardinality $j$are not hub dominating sets. Hence, $L_{n,1}$ contains $\binom{n+1}{j} - \binom{n-1}{j}$ number of subsets of hub dominating sets with cardinality j. When the cardinality is 1, $\{v_n\}$ is the only hub dominating sets.

Therefore, $hd(L_{n,1}j) = \begin{cases} \binom{n+1}{j} - \binom{n}{j} \ when\ j = 1 \\ \binom{n+1}{j} - \binom{n-1}{j} \quad if\ 2 \le j \le n+1 \end{cases}$

Theorem 2.6: Let $L_{n,1}$ be the Lollipop graph with $n+1$ vertices. Then,

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(i) $hd(L_{n,1}, j) = 1$if $j = 1$

(ii) $hd(L_{n,1}, j) = hd(L_{n-1,1}, j) + hd(L_{n-1,1}, j-1) + 1$ if $j = 2$

(iii) $hd(L_{n,1}, j) = hd(L_{n-1,1}, j) + hd(L_{n-1,1}, j-1)$if $3 \le j \le n+1$

Proof:

(i) When $j = 1$, by Theorem 2.5

$hd(L_{n,1},1) = \binom{n+1}{1} - \binom{n}{1}$

$= n + 1 - n$

$hd(L_{n,1},1) = 1$ when $j = 1$.

(ii) When $j = 2$, by Theorem 2.5

we have $hd(L_{n,1},2) = \binom{n+1}{2} - \binom{n-1}{2}$

$hd(L_{n-1,1},2) = \binom{n}{2} - \binom{n-2}{2}$

and $hd(L_{n-1,1},1) = \binom{n}{1} - \binom{n-1}{1}$

Consider, $hd(L_{n,1},2) = \binom{n+1}{2} - \binom{n-1}{2}$

$= \frac{(n+1)n}{2} - \frac{(n-1)(n-2)}{2}$

$= \frac{n^2+n}{2} - \frac{n^2-3n+2}{2}$

$= \frac{4n-2}{2}$

$hd(L_{n,1},2) = 2n - 1$

Consider $hd(L_{n-1,1},2) + hd(L_{n-1,1},1) = \binom{n}{2} - \binom{n-2}{2} + \binom{n}{1} - \binom{n-1}{1}$

$= \frac{n(n-1)}{2} - \frac{(n-2)(n-3)}{2} + n - (n-1)$

$= \frac{n^2-n}{2} - \frac{n^2-5n+6}{2} + 1$

$= \frac{4n-6+2}{2}$

$= 2n - 2$

$= 2n - 1 - 1$

$hd(L_{n-1,1},2) + hd(L_{n-1,1},1) = hd(L_{n,1},2) - 1$

Therefore, $hd(L_{n,1},2) = hd(L_{n-1,1},2) + hd(L_{n-1,1},1) + 1$

Hence $hd(L_{n,1},j) = hd(L_{n-1,1},j) + hd(L_{n-1,1},j-1) + 1$ if $j = 2$

(iii) By Theorem 2.5, we have,

$hd(L_{n,1}j) = \binom{n+1}{j} - \binom{n}{j}$ for all $3 \le j \le n+1$

$$hd(L_{n-1,1},j) = \binom{n}{j} - \binom{n-1}{j}$$

$hd(L_{n-1,1},j-1) = \binom{n}{j-1} - \binom{n-1}{j-1}$

Consider, $hd(L_{n-1,1},j) + hd(L_{n-1,1},j-1) = \binom{n}{j} - \binom{n-1}{j} + \binom{n}{j-1} - \binom{n-1}{j-1}$

$$= \left[\binom{n}{j} + \binom{n}{j-1}\right] - \left[\binom{n-1}{j} + \binom{n-1}{j-1}\right]$$

$$= \binom{n+1}{j} - \binom{n}{j}$$

Therefore, $hd(L_{n-1,1}, j) + hd(L_{n-1,1}, j-1) = hd(L_{n,1} j)$

Hence $hd(L_{n,1}, j) = hd(L_{n-1,1}, j) + hd(L_{n-1,1}, j-1)$ if $3 \le j \le n+1$

III. Hub Domination Polynomials of the Lollipop Graph $L_{n,1}$

Definition 3.1: Let $L_{n,1}$ denotes the Lollipop graph with $n+1$ vertices and $HD(L_{n,1}, j)$denotes the family of hub dominating sets of $L_{n,1}$with cardinality j. Then, the polynomial,

$$HD(L_{n,1}, x) = \sum_{j=\hbar d(L_{n,1})}^{|V(L_{n,1})|} hd(L_{n,1}, j)x^j$$

is called the hub domination polynomial of $L_{n,1}$where $hd(L_{n,1}, j)$ is the number of hub dominating sets of $L_{n,1}$ of cardinality j and $\hbar d(L_{n,1})$is the hub domination number of $L_{n,1}$.

Remark 3.2: $\hbar d(L_{n,1}) = 1$.

Proof: Let $L_{n,1}$be the Lollipop graph with $n+1$ vertices and $n \ge 4$. Label the vertices of $L_{n,1}$ by $v_1, v_{2,} v_3, \dots, v_n, v_{n+1}$ where $v_1, v_2, v_{3,} \dots, v_{n-1}$ are the vertices of degree $n-1, v_n$ is a vertex of degree n and v_{n+1} is a vertex of degree 1. There is a path between any two vertices of $v_1, v_2, v_{3,} \dots, v_n$because they are adjacent to one another.In addition, the path between the vertices of $\{v_1, v_{2,} v_3, \dots, v_{n-1}\}$ and v_{n+1} uses v_n as its internal vertex.. Hence $\{v_n\}$ is a hub dominating set of cardinality 1.

Hence, $\hbar d(L_{n,1}) = 1$.

Theorem 3.3: Let$L_{n,1}$ be the Lollipop graph with $n+1$ vertices. Then, the connected hub polynomial of $L_{n,1}$ is $HD(L_{n,1}, x) = (1+x)HD(L_{n-1,1,} x) + x^2$ with initial value

$HD(L_{3,1,} x) = x + 5x^2 + 4x^3 + x^4$.

Proof: We have, $HD(L_{n,1,} x) = \sum_{j=1}^{n+1} hd(L_{n,1}, j)x^j$

$$HD(L_{n,1,} x) = hd(L_{n,1,} 1)x + hd(L_{n,1,} 2)x^2 + \sum_{j=3}^{n+1} hd(L_{n,1,} j)x^j$$

$$= hd(L_{n,1,} 1)x + [hd(L_{n-1,1}, 2) + hd(L_{n-1,1}, 1) + 1]x^2 + \sum_{j=3}^{n+1} [hd(L_{n-1,1}, j) + hd(L_{n-1,1}, j-1)]x^j$$

$$= hd(L_{n,1,} 1)x + x^2 + \sum_{j=2}^{n+1} [hd(L_{n-1,1}, j) + hd(L_{n-1,1}, j-1)]x^j$$

$$= \sum_{j=1}^{n+1} [hd(L_{n-1,1}, j) + hd(L_{n-1,1}, j-1)]x^j + x^2$$

$$= \sum_{j=1}^{n+1} hd(L_{n-1,1}, j)x^j + \sum_{j=1}^{n+1} hd(L_{n-1,1,} j-1)x^j + x^2$$

$$= \sum_{j=1}^{n+1} hd(L_{n-1,1}, j)x^j + x\sum_{j=1}^{n+1} hd(L_{n-1,1}j-1)x^{j-1} + x^2$$

$$= HD(L_{n-1,1,}\ x) + xHD(L_{n-1,1,}\ x) + x^2$$

$$= (1+x)HD(L_{n-1,1,}\ x) + x^2$$

Hence, $HD(L_{n,1}, x) = (1+x)HD(L_{n-1,1,}\ x) + x^2$ with initial value

$HD(L_{3,1,}x) = x + 5x^2 + 4x^3 + x^4$.

Example 3.4

Consider the Lollipop graph $L_{5,1}$ with order 6 given in Figure 1.

Figure 1

$$HD(L_{4,1,}x) = x + 7x^2 + 9x^3 + 5x^4 + x^5$$

By Theorem 3.3, we have,

$$HD(L_{5,1,}x) = (1+x)HD(L_{4,1,}\ x) + x^2$$

$$= (1+x)(x + 7x^2 + 9x^3 + 5x^4 + x^5) + x^2$$

$$= x + 9x^2 + 16x^3 + 14x^4 + 6x^5 + x^6$$

Theorem 3.5: Let $L_{n,1}$ be the Lollipop graph with $n \geq 3$.Then

$HD(L_{n,1}, x) = \sum_{j=1}^{n+1} \binom{n+1}{j} x^j - \sum_{j=1}^{n+1} \binom{n-1}{j} x^j - x.$

Proof: Proof follows from Theorem 2.5, Theorem 2.6 and the definition of Hub Domination Polynomial.

For $3 \leq n \leq 10$ and $1 \leq j \leq 15$, we obtain $hd(L_{n,1}, j)$ as shown in Table 1.

***$HD(L_{n,1}, j)$*, Hub Dominating Sets of $L_{n,1}$ with cardinality j.**

j \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$L_{3,1}$	1	5	4	1											

$L_{4,1}$	1	7	9	5	1										
$L_{5,1}$	1	9	16	14	6	1									
$L_{6,1}$	1	11	25	30	20	7	1								
$L_{7,1}$	1	13	36	55	50	27	8	1							
$L_{8,1}$	1	15	49	91	105	77	35	9	1						
$L_{9,1}$	1	17	64	140	196	182	112	44	10	1					
$L_{10,1}$	1	19	81	204	336	378	294	156	54	11	1				
$L_{11,1}$	1	21	100	285	540	714	672	450	210	65	12	1			
$L_{12,1}$	1	23	121	385	825	1254	1386	1122	660	275	77	13	1		
$L_{13,1}$	1	25	144	506	1210	2079	2640	2508	1782	935	352	90	14	1	
$L_{14,1}$	1	27	169	650	1716	3289	4719	5148	4290	2717	1287	442	104	15	1

Table 1

In the following Theorem we obtain some properties of $HD(L_{n,1}, j)$.

Theorem 3.6: The following properties hold for the coefficients of $HD(L_{n,1}, j)$ for all n.

(i) $hd(L_{n,1}, 1) = 1$, for every $n \geq 3$.

(ii) $hd(L_{n,1}, 2) = 2n - 1$,for every $n \geq 3$.

(iii) $hd(L_{n,1}, n+1) = 1$, for every $n \geq 3$.

(iv) $hd(L_{n,1}, n) = n + 1$, for every $n \geq 3$.

(v) $hd(L_{n,1}, n-1) = \frac{1}{2}[n^2 + n - 2]$, for every $n \geq 3$.

(vi) $hd(L_{n,1}, n-2) = \frac{1}{6}[n^3 - 7n + 6]$, for every $n \geq 4$.

(vii) $hd(L_{n,1}, n-3) = \frac{1}{24}[n^4 - 2n^3 - 13n^2 + 38n - 24]$, for every $n \geq 5$.

Proof:

(i) Since there is only one hub dominating set of cardinality one we have the result.

(ii) To prove $hd(L_{n,1}, 2) = 2n - 1$, for every $n \geq 3$,

we apply induction on n.

When $n = 3$,

L.H.S = $hd(L_{3,1}, 2) = 5$ (from the Table 1)

R.H.S = 6 − 1 =5

The result holds true for $n = 3$.

Now, suppose that the result is true for all numbers less than n and we prove it for n.

$$hd(L_{n,1}, 2) = hd(L_{n-1,1}, 2) + hd(L_{n-1,1}, 1) + 1$$
$$= 2[n-1] - 1 + 1 + 1$$

$= 2n - 2 + 1$

$$hd(L_{n,1}, 2) = 2n - 1$$

Hence, the result is true for all n.

(iii) Since, $HD(L_{n,1}, n+1) = [n+1]$, we have the result.

(iv) Since, $HD(L_{n,1}, n) = \{[n+1] - x / x \in [n+1]\}$, we have the result.

(v) To prove $\hbar d(L_{n,1}, n-1) = \frac{1}{2}[n^2 + n - 2]$, for every $n \geq 3$,

we apply induction on n.

When $n = 3$,

L.H.S = $\hbar d(L_{3,1}, 2) = 5$ (from the Table 1)

R.H.S = $\frac{1}{2}(3^2 + 3 - 2)$ =5

The result holds true for $n = 3$.

Now, suppose that the result is true for all numbers less than n and we prove it for n.

$$\hbar d(L_{n,1}, n-1) = \hbar d(L_{n-1,1}, n-1) + \hbar d(L_{n-1,1}, n-2)$$
$$= n + \frac{1}{2}[(n-1)^2 + (n-1) - 2]$$
$$= 2n + \frac{1}{2}(n^2 - 2n + 1 + n - 1 - 2)$$
$$= \frac{1}{2}(2n + n^2 - n + 2)$$
$$\hbar d(L_{n,1}, n-1) = \frac{1}{2}(n^2 + n - 2)$$

Hence, the result is true for all n.

(vi) To prove $\hbar d(L_{n,1}, n-2) = \frac{1}{6}[n^3 - 7n + 6]$, for every $n \geq 4$,

we apply induction on n.

When $n = 4$,

L.H.S = $\hbar d(L_{4,1}, 2) = 7$ (from the Table 1)

R.H.S = $\frac{1}{6}(4^3 - 28 + 6)$ =7

The result holds true for $n = 4$.

Consider the case when we prove the conclusion for n and it holds true for all numbers less than n.

$$\hbar d(L_{n,1}, n-2) = \hbar d(L_{n-1,1}, n-2) + \hbar d(L_{n-1,1}, n-3)$$
$$= \frac{1}{2}[(n-1)^2 + (n-1) - 2] + \frac{1}{6}[(n-1)^3 - 7(n-1) + 6]$$
$$= \frac{1}{2}[n^2 - n - 2] + \frac{1}{6}[n^3 - 3n^2 - 4n + 12]$$
$$\hbar d(L_{n,1}, n-2) = \frac{1}{6}[n^3 - 7n + 6]$$

Hence, the result is true for all n.

(vii) To prove $\hbar d(L_{n,1}, n-3) = \frac{1}{24}[n^4 - 2n^3 - 13n^2 + 38n - 24]$, for every $n \geq 5$,

we apply induction on n.

When $n = 5$,

L.H.S = $hd(L_{5,1},1) = 9$ (from the Table 1)

R.H.S = $\frac{1}{24}(625 - 250 - 325 + 190 - 24) = 9$

The result holds true for $n = 5$.

Consider the case when we prove the conclusion for n and it holds true for all numbers less than n.

$$hd(L_{n,1}, n-3) = hd(L_{n-1,1}, n-3) + hd(L_{n-1,1}, n-4)$$
$$= \frac{1}{6}[(n-1)^3 - 7(n-1) + 6] + \frac{1}{24}[(n-1)^4 - 2(n-1)^3 - 13(n-1)^2 + 38(n \quad -1) - 24]$$
$$= \frac{1}{6}[n^3 - 3n^2 - 4n + 12] + \frac{1}{24}[n^4 - 6n^3 - n^2 + 54n - 72]$$
$$hd(L_{n,1}, n-3) = \frac{1}{24}[n^4 - 2n^3 - 13n^2 + 38n - 24]$$

Hence, the result is true for all n

2. Conclusion

This work deduces the hub domination polynomials of the Lollipop graph $L_{n,1}$by finding its hub dominating sets. We can also use cardinality j to characterise the hub dominating sets.

Any Lollipop graph $L_{n,m}$can be used as a generalisation of this study, and several intriguing properties can be discovered.

3. References

[1] Walsh, Matthew. "The hub number of a graph." Int. J. Math. Comput. Sci 1, no. 1 (2006): 117-124.

[2] Veettil, RagiPuthan, and T. V. Ramakrishnan. "Introduction to hub polynomial of graphs." Malaya Journal of Matematik (MJM) 8, no. 4, 2020 (2020): 1592-1596.

[3] S. Alikhani and Y.H. Peng. "Dominating sets and domination polynomials of paths." International journal of Mathematics and mathematical sciences, 2009.

[4] S. Alikhani and Y.H. Peng. "Introduction to domination polynomial of a graph. "arXiv preprint arXiv:0905.2251, 2009.

[5] Sahib.Sh.Kahat, Abdul JalilM.Khalaf and RoslanHasni. "Dominating sets and Domination Polynomials of Wheels." Asian Journal of Applied Sciences (ISSN:2321− 0893), volume 02− Issue 03, June 2014.

[6] Sahib.Sh.Kahat, Abdul JalilM.Khalaf and RoslanHasni. "Dominating sets and Domination Polynomials of Stars". Australian Journal of Basics and Applied Science,8(6) June 2014, pp 383−386.

[7] A.Vijayan,T.Anitha Baby, G.Edwin. "Connected Total Dominating Sets and Connected Total Domination Polynomials of Stars and Wheels." IOSR Journal of Mathematics,VolumeII, pp 112−121.

[8] [8] B. Basavanagoud and MahammadsadiqSayyed. "Hub Polynomial of a Graphs" International Journal of Applied Engineering Research ISSN 0973-4562, Volume 16, Number 3 (2021) pp. 166-173.