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Edge Manipulation and Chromatic Trends: A Study of Vertex and Edge Domination in Semigraphs

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Abstract

Graph coloring is a significant aspect within graph theory, offering various techniques tailored to different graph types. The exploration of graph domination represents another pivotal domain in graph theory. This paper delves into the nuanced realms of vertex domination and edge domination, specifically within the context of semigraph coloring. Additionally, our investigation extends to examining the dynamic alterations in color energy within semigraphs resulting from targeted edge deletions.

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Keyword: *Semigraph, coloring of semigraphs, domination with respect to colors in semigraphs, color energy of semigraphs*

1. Introduction

Graph theory serves as a fundamental field in mathematics, with graph coloring and domination standing out as key areas of study. The process of assigning colors to graph components and understanding dominance relationships contributes significantly to unraveling the intricate nature of graphs. This paper delves into the intersection of these two domains by focusing on vertex domination, edge domination, and their connection to the coloring of semigraphs.

Graph coloring, a well-established concept, involves assigning colors to vertices or edges under certain constraints. Different types of graphs necessitate varied coloring techniques, each offering insights into the underlying structure of the graph. Simultaneously, the exploration of domination in graphs, which centers around the control and influence exerted by specific vertices or edges, represents another cornerstone in graph theory.

In this study, we specifically direct our attention to semigraphs—a versatile class of graphs where edges may lack direction. Our objective is to investigate the intricacies of vertex domination and edge domination within the context of semigraph coloring. By delving into these interwoven concepts, we aim to enhance our understanding of the relationships between graph coloring and dominance structures.

Furthermore, our exploration extends beyond conventional analyses as we delve into the dynamic realm of color energy within semigraphs. Specifically, we investigate how the removal of edges influences the overall color energy of semigraphs. This nuanced examination provides a fresh perspective on the dynamic nature of semigraphs and their response to structural alterations.

Through this interdisciplinary exploration, we seek to contribute novel insights to the broader field of graph theory, shedding light on the complex interplay between graph coloring, domination, and the dynamic changes induced by edge deletions in semigraphs.

2. In this section we recall the definitions of graphs, semigraphs, coloring of graphs and domination in graphs. From [1, 2, and 3] we introduce new definitions in coloring of semigraphs and color domination of semigraphs with respect to vertices and edges.

2.1.1 A Graph G is an ordered pair (V, E) where the vertex set V is non empty and the edge set E may be empty or non-empty. We say that the edge is incident with the vertices x and y if x and y are the end points of E.[4]

2.1.2 A Semigraph S is a pair (V, X) where V is a nonempty set whose elements are observed as vertices of S and X is a set of ordered n-tuples $n \geq 2$ of prescribed vertices called edges of S satisfying the following conditions:

(i) Any two edges have at most one vertex in common place.

(ii) Two edges E1 = (u1, u2, um) and E2 = (v1, v2, vn) are said to be identical if (a) m = n and (b) either ui = vi or ui = vn−i+1 for $1 \leq i \leq n$

The vertices in a semigraph are splitted into three types namely end vertices, middle vertices and middle-end vertices, based on their positions in an edge. The end vertices are represented by thick dots, middle vertices are represented by small circles, a small tangent is drawn at small circles to represent middle-end vertices. [4]

2.1.3. Coloring of graphs: Graph Coloring problem is to assign colors to certain elements of a graph subject to certain constraints. Vertex coloring is the most common graph coloring problem. The problem is, given m colors, find a way of coloring the vertices of a graph such that no two adjacent vertices are colored using same color. The other graph coloring problems like Edge Coloring (No vertex is incident to two edges of same color) and Face Coloring (Geographical Map Coloring) can be transformed into vertex coloring.

2.1.4 Domination in graphs: We introduce the concept of dominating set in graphs. A set S, a subset of V of vertices in a graph G= (V, E) is a dominating set if every vertex v in V is an element of S or adjacent to an element of S. We can also say that if S is a subset of C, then N(S) = V (G). A dominating set S is a minimal dominating set if no proper subset S' of S is a dominating set. The dominating number of G is the minimal cardinality of the dominating set of G. [3]

3. In this section we introduce new definitions in semigraphs with respect to coloring and domination in semigraphs with respect to vertices and edges.

3.1.1 Coloring in semigraphs is defined as the set of colors assigned to vertices and edges in such a way that no two adjacent vertices and no two adjacent edges have the same color. The minimum number of colors used for vertices is called as the vertex chromatic number of the semigraph denoted by $\chi_v(G)$ and the minimum number of colors used for edges is called as the edge chromatic number of the semigraph denoted by $\chi_e(G)$. The chromatic number of the graph is the maximum number of colors used for vertices or and edges.

3.1.2 Equal coloring in semigraph: If the number of colors used for vertices and edges are same in a semigraph then it is called as equal coloring in a semigraph. That is the vertex chromatic number $\chi_v(G)$ and edge chromatic number $\chi_e(G)$ are same.

3.1.3 Uniform coloring in a semigraph: If the end vertices and all the middle vertices and middle end vertices of every edge receives the same color then it is defined as uniform coloring in a semigraph.

3.1.4 Strong coloring in a semigraph: If the colors assigned for vertices and edges are repeated more than once then we have strong coloring in a semigraph.

3.1.5 Bipartite semigraph: A bipartite semi graph is a graph whose vertices can be divided into disjoint vertex sets comprising end vertices and middle vertices such that no edge connect the vertices of the same set. A balanced bipartite semi graph is the one that has equal number of left vertices, right vertices, middle vertices, left end middle vertices and right end middle vertices.

4. In this section, we study the relation between the chromatic numbers of the semigraph, vertex chromatic number and edge chromatic number through the colors assigned to the vertices and edges of the semigraph. The chromatic number of the graph, vertex and edges of the few types of semigraphs are shared in the following table.

S.No.	Type of Graph	Chromatic number of graph	Vertex Chromatic number	Edge Chromatic number
1	Path	4	2	2
2	Cycle C3	3	3	3
	C4	4	2	2
	C5	4	3	4
	C6	4	2	2
	C7	5	3	5
	C8	4	2	2
	C9	4	3	4
3	Complete graph K2	2	2	1
	K3	3	3	3
	K4	5	4	5
4	Pan graph P3	4	3	4
	P4	4	3	4
	P5	5	2	5
5	Sun graph S4	5	3	5
6	Star $K_{1,n}$	4	2	4
7	Particular eye graph	4	2	4
8	Particular Arch	3	2	1
9	Particular Caterpillar graph(1)	4	2	4
10	Particular Caterpillar graph(2)	6	2	6
11	Particular Snake graph with 2 vertices	6	2	6
12	Traingular snake graph with 3 vertices	5	2	5
13	Particular Lollipop graph	4	2	4
14	Complete bipartite graph K2,2	4	2	4
15	Bipartite graph K1,2	4	3	4
16	Particular Lollipop	4	3	4

	semigraph			
17	Particular Lollipop semigraph	5	2	5

We have found the chromatic number of the graph, vertex chromatic number and edge chromatic number of few kinds of semigraphsby considering the following:

1. We fix the number of the colors for the vertices in such a way that no two adjacent vertices receives the same color.

2.We give colors to the edges in such a way that no edge shares the color assigned to the adjacent vertices and the adjacent edges.

3. The chromatic number of the graph is the maximum number of colors used.

The minimum chromatic number is same as vertex chromatic number in most of the graphs and the maximum chromatic number is same as edge chromatic number in most of the graphs.

We find that color domination can be taken either with respect to vertex chromatic number or with respect to edge chromatic number.

We find that for most of the graphs the chromatic number of the graph is same as the edge chromatic number of the graph.

5. In this section, we discuss about singular value inequality and we introduce new definitions to discuss the relation with respect to the color energy of semigraphs due to edge deletion.

5.1.1 Matrix sum inequality with respect to the singular values

Let the n by n complex matrix be X and let us denote its singular values by

$s_1(X) \geq s_2(X) \geq s_3(X) \geq \ldots\ldots\ldots \geq s_n(X) \geq 0$. Let there be only real eigenvalues in X, that is $\lambda_1(X) \geq \lambda_2(X) \geq \ldots.. \geq \lambda_n(X)$. Consider positive semi definite $|X| = \sqrt{XX^*}$ where $\lambda_i(X) = s_i(X)$ for all i. Then the Matrix sum inequality with respect to the singular values is $\sum_{i=1}^{n} s_i(A+B) \leq \sum_{i=1}^{n} s_i(A) + \sum_{i=1}^{n} s_i(B)$

5.1.2 Adjacency Matrix of a Color Semigraph

Consider a semigraph SG (V, X). Let $V = \{1,2,\ldots.p\}$ be vertex set and X={$e_1,e_2,\ldots e_q$}, the edge set where $e_j = (i_1, i_2, \ldots.. i_j)$ and $i_1, i_2, \ldots\ldots i_j$ are distinct elements of V, then the $p \times p$ matrix A is the Adjacency matrix of semigraph $SG(V, X)$ whose entries are given by

a_{ij} = 1, if v_i and v_j are adjacent

= 0, otherwise.

5.1.3 Color Energy of a graph

Consider the graph G not directed, not infinite and not multiple graph with number of vertices n and number of edges m. Consider the adjacency matrix $A = (a_{ij})$ of graph G, then the eigenvalues assumed in non-increasing order $\lambda_1, \lambda_2, \ldots..\lambda_n$ of $A(G)$, are the graph eigenvalues. Then $CE(G)$ color graph energy G, is defined as $CE(G) = \sum_{i=1}^{n} |\lambda_i|$. The spectrum G is the set $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$ denoted by Spec G. If G has distinct eigenvalues say, $\lambda_1 > \lambda_2 > \ldots. > \lambda_n$ and if their multiplicities are $m(\lambda_i)$ then

$$Spec\, G = \begin{pmatrix} \lambda_1 & \lambda_2 & .. & .. & \lambda_n \\ m(\lambda_1) & m(\lambda_2) & .. & .. & m(\lambda_n) \end{pmatrix}$$

The spectrum of the graph does not depend on the labeling of the vertex set of the graph. Since we have the matrix symmetric and real with the trace zero, we have sum of the real eigenvalues to be zero.

5.1.4 A kind of color energy with respect to the distance in a semigraph

If SG is a simple connected semigraph and the vertices are labelled as $v_1, v_2, \cdots v_n$. then the matrix of a semigraph SG with respect to the distance, is given by a square matrix $D(G) = (d_{ij})$ in which the entries are the distance between the vertices v_i and v_j in SG. The eigenvalues of the matrix we consider $\mu_1, \mu_2, \cdots \mu_n$ are said to be the **distance eigenvalues**. As we have a matrixwhich is symmetric, we have real eigenvalues in order $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. The energy with respect to distance of a color semigraph $CE_D(SG)$ is defined as

$$CE_D(SG) = \sum_{i=1}^{n} |\mu_i|.$$

5.1.5 Color Energy of a Semigraph

Let SG be not directed, not infinite and not a multiple semigraph with number of vertices n and number of edges m. Consider $A = (a_{ij})$ the adjacency matrix of color semigraph SG. The eigenvalues $\eta_1, \eta_2, .\ .\ \eta_n$ of $A(SG)$, taken not increasing order, are the eigenvalues of the color semigraph CSG. Color *energy* of a semigraph SG, denoted by $CE(SG)$ is defined as $CE(SG) = \sum_{i=1}^{n} |\eta_i|$. The set $\eta_1, \eta_2, \ldots. \eta_n$ is the spectrum of color semigraph and is denoted by Spec CSG. If the eigenvalues of SG are distinct say, $\eta_1 > \eta_2 > \ldots. > \eta_n$ with their multiples $m(\eta i)$ then we write

$$Spectrum C\, SG = \begin{pmatrix} \eta_1 & \eta_2 & .. & .. & \eta_n \\ m(\eta_1) & m(\eta_2) & .. & .. & m(\eta_n) \end{pmatrix}$$

The spectrum of the above graph does not depend on the labeling of the vertex set of the graph. Since we have the matrix symmetric and real with the trace zero, we have sum of the real eigenvalues to be zero.

5.1.6 Theorem

If CSH is a non-empty induced color subsemigraph of a simple connected regular semigraph CSG then

$$CE(CSG) - CE(CSH) \leq CE(CSG') \leq CE(CSG) + CE(CSH)$$

Proof

CSG is a connected simple semigraph. CSH be an induced subsemigraph of CSG ,containing all edges of CSG connecting two vertices of CSH. Let $CSG - CSH$ be the semigraph, having got from CSG removing all vertices of CSH and the edges incident with CSH. If $CSG1$ *and* $CSG2$ are the two semigraphs with out any vertices in common and if we consider $CSG1 \oplus CSG2$ as the semigraph with vertex set and the edge set $V(CSG1) \cup V(CSG2)$; $E(CSG1) \cup E(CSG2)$ respectively. Hence

$$A(CSG1 \oplus CSG2) = A(CSG1) \oplus A(CSG2)$$

$$A(CSG) = \begin{pmatrix} A(CSH) & X^T \\ X & A(CSG - CSH) \end{pmatrix}$$

$$= \begin{pmatrix} A(CSH) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & X^T \\ X & 0 \end{pmatrix}$$

in which the edges joining CSH and $CSG - CSH\ is\ X$.

Also if $A(CSG') = \begin{pmatrix} 0 & X^T \\ X & 0 \end{pmatrix}$

Using the inequality theorem of matrices with respect to singular values, we get

$CE(SG) \leq CE(SH) + CE(SG')$, which gives one part of the inequality

$$CE(SG) - CE(SH) \leq CE(SG')$$

$$A(CSG') = A(CSG) + \begin{pmatrix} -A(CSH) & 0 \\ 0 & 0 \end{pmatrix}$$

By the inequality theorem with respect to singular values,

$$CE(SG') \leq CE(SG) + CE(SH)$$

$$CE(SG') \leq CE(SG) + C(SH) - - - ii$$

From i and ii ,itfollows

$$CE(SG) - CE(SH) \leq CE(SG') \leq CE(SG) + CE(SH)$$

Both the left and right equality holds when $CE(SH) = \phi$

5.1.7 Theorem

If CSH is a non-empty induced color sub-semigraph of a simple connected semigraph SG. Then $CE_D(SG) - CE_D(SH) \leq E(CSG') < CE_D(SG) + CE_D(SH)$

Proof

CSG is a connected simple color semigraph. CSH be an induced color subsemigraph of CSG, containing all edges of CSG joining two vertices of CSH Let $CSG - CSH$ denote the semigraph, having got from SG removing all vertices of CSH and the edges that are incident with CSH. If $CSG1\ and\ CSG2$ are the two semigraphs with out any vertices in common and if we consider $CSG1 \oplus CSG2$ as the semigraph with vertex set and the edge set $V(CSG1) \cup V(CSG2)$; $E(CSG1) \cup E(CSG2)$ respectively. Hence

$$A(CSG1 \oplus CSG2) = A(CSG1) \oplus A(CSG2)$$

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$$D(CSG) = \begin{pmatrix} D(CSH) & X^T \\ X & D(CSG - CSH) \end{pmatrix} = \begin{pmatrix} D(CSH) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & X^T \\ X & D(CSG - CSH) \end{pmatrix}$$

X Represents theedges connecting CSH and $CSG - CSH$.

$$D(CSG) = \begin{pmatrix} D(CSH) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & X^T \\ X & A(CSG - CSH) \end{pmatrix}$$

Also if $A(CSG') = \begin{pmatrix} 0 & X^T \\ X & A(CSG - CSH) \end{pmatrix}$

$$D(CSG) = \begin{pmatrix} D(CSH) & 0 \\ 0 & 0 \end{pmatrix} + A(CSG')$$

By singular value inequality theorem

$$CE_D(SG) \leq CE_D(SH) + CE(SG')$$
$$CE_D(SG) - CE_D(SH) \leq CE(SG') --- i$$
$$A(CSG') = D(CSG) + \begin{pmatrix} -D(CSH) & 0 \\ 0 & 0 \end{pmatrix}$$

By singular value inequality theorem,

$$CE(SG') \leq CE_D(SG) + CE_D(SH)$$
$$CE(SG') \leq CE_D(SG) + CE_D(SH) --- ii$$

From i and ii, we have

$$CE_D(SG) - CE_D(SH) \leq CE(SG') \leq CE_D(SG) + CE_D(SH)$$

Both the left and right equality holds when $CE_D(SH) = \phi$

From i and ii,we have

$$CE_D(SG) - CE_D(SH) \leq CE(SG') \leq CE_D(SG) + CE_D(SH)$$

Both the left and right equality holds when $CE_D(SH) = \phi$

6. Conclusion

We find that color energy of semigraphs changes due to edge deletion. We can study the changes in color energy of semigraph due to vertex deletion. We can study the relation with other forms of color energies of semigraphs due to edge deletion and vertex deletion

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