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A Fuzzy Logic-Based Approach to Nonlinear Optimisation Problems Involving Linear Approximation Relationships.

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Abstract:A fuzzy logic-based approach to nonlinear optimisation issues including linear approximation relationships is developed in this paper. The suggested approach begins by using Taylor's series to linearly approximate the nonlinear objective function and the constrained functions that make up the feasible regions around an initial feasible point. From there, the nonlinear optimisation problem is subtracted to a linear optimisation or linear programming problem. The optimal fuzzy solution to the problem is found by iterative application of the fuzzy decision-making approach.

Keywords: fuzzy logic-based approach, nonlinear optimisation problem, linear approximation technique, LP problem

1. Introduction

Many fields have found practical uses for fuzzy set theory's advanced principles, including optimisation, clustering, data processing, engineering design, regional policy, logistics, algebra, graph theory, topology, and many more. The formal theory of fuzzy sets has evolved into a more complex and well-defined framework in recent decades, thanks to the incorporation of novel notions and ideas and the generalisation (fuzzification) of classical optimisation and clustering techniques. As a result, fuzzy set theory evolved into a potent modelling tool capable of handling a significant portion of the uncertainties present in real-world scenarios. Its flexibility allows it to be applied to various contexts and to modify the original ideas of formal classical set and fuzzy set theories depending on those contexts.

2. Review of Literature

One of the most common optimisation methods is the linear programming problem, where the variables $f(x)$ and $g_i\ (x)$ are linear functions of x_js. An approach for solving the general linear programming problem, the simplex method was developed by G.B. Dantzig in 1947. Hartley (1985) and Walsh (1977) are just two of many texts that have explained the associated theory. Methods such as the sequential unconstrained minimization methodology developed by Carroll (1961), the gradient projection method, penalty function methods [Walsh (1977)], and the Method of Langrange multipliers are available for solving restricted non-linear optimisation problems. Another example is the Kuhn-Tucker theory. Several works, such as Wash (1977) and Box (1965), have addressed the aforementioned approaches.

3. Results and Discussion

This research applies fuzzy logic to the solution of nonlinear optimisation problems including linear approximation relationships. A nonlinear optimisation problem with constraints is considered. The nonlinear optimisation problem is transformed into a linear optimisation or linear programming problem by first using Taylor's series to linearly approximate the nonlinear objective function and the constrained functions that make up the feasible regions around an initial feasible point. Finding the fuzzy optimal solution to the problem is done iteratively using the notion of fuzzy decision making [e.g. see Bellman &Zadeh (1970), Warners (1987), Feng (1987), Zhang et.al. (2013), Chamani et.al. (2013)]. Let us consider a system of constraints:

$$\begin{aligned} g_i(x) &\le b_i \;\; (i=1,2,\ldots,u) \\ &\ge b_i \;\; (i=u+1,\; u+2,\ldots,v) \\ &= b_i \;\; (i=v+1, v+2,\ldots,p) \\ & x \ge 0,\; x \in R^n \end{aligned}$$ ....(1.1)

We write the constraints (1.1) in the form:

$$\begin{aligned} g_i(x) &\ge 0 \;\; (i=1,2,\ldots,v) \\ &= 0 \;\; (i=v+1, v+2,\ldots,p) \end{aligned}$$ ....(1.2)

Starting at some arbitrary point x_0, we find the feasible constraints and non-feasible constraints at x_0. Then, we construct the unconstrained function Z which is to be minimized:

$$Z = -\sum_{i=1}^{v}{}' g_i(x) + \sum_{i=v+1}^{p} \left[g_i(x) \right]^2 .$$ ....(1.3)

where $\sum'$ indicates that the summation is taken over only those constraints that are violated at the current point x_0.

An alternative method for finding the feasible starting point can be used as given below: i). Write the constraints of (1.1) in the form :

$$g_i(x) \le 0, \quad i=1,2,\ldots,p \; .$$

ii). Choose an arbitrary point x_0 and evaluate the constraints $g_i\,(x)$ at the point x_0. Since the point x_0 is arbitrary, it may violate some constraints.

If k out of p constraints are violated, we identify the constraints such that the last k constraints will become the violated ones that is

$$\left.\begin{aligned} g_i(x_\circ) &< 0 \;\; i=1,2,\ldots,p-k \\ g_i(x_\circ) &\ge 0 \;\; i=p-k+1, p-k+2,\ldots,p \end{aligned}\right\} .$$ ....(1.4)

iii). We identify the constraints which is violated most at the point x_0, that is, we find the integer r such that

$$g_r(x_\circ) = \max\left[g_i(x_\circ) \right] \text{ for } i=p-k+1, p-k+2,\ldots,p \, .$$ ....(1.5)

iv). We now formulate a new optimization problem as:

$$\begin{aligned} &\text{Min } g_r(x) \\ &\text{subject to} \quad g_i(x) \le 0, \quad i=1,2,\ldots,p-k. \\ &\qquad g_i(x) - g_r(x_\circ) \le 0, \;\; i=p-k+1,\; p-k+2,\ldots,r-1,\; r+1,\ldots,p. \end{aligned}$$ ....(1.6)

v). With x_0 as a possible initial value, solve the optimisation problem (1.6) using an unconstrained optimisation methodology, ideally the interior penalty function method (see Rao (1992) and Walsh (1977) for more information). As the objective function $g_r(x)$ approaches zero, it is possible to end this optimisation process. Hence, an additional constraint will be satisfied by the given solution x_* compared to x_0.

vi). Ifall the constraints are not satisfied at the point x_*, then set the new starting point as $x_o = x_*$ and renumber the constraints such that the last k constraints will be the unsatisfied ones, and go to step (iii).

The above procedure is repeated until all the constraints are satisfied and thus a point $x_o = x_*$ is obtained for which $g_i(x_o) < 0$ $i=1, 2,..., m$.

Note that if the constraints are consistent, it should be possible to obtain, by applying the above procedure, a point x_o that satisfies all the constraints.

However, the solution of the problem formulated in step (*iv*) gives a local minimum of $g_r(x)$ with a positive value . In such case, we are to start afresh with a new point x_o from step (*i*) onwards.

4. An Approach to Mathematical Formulation and Solution Using Concepts from Fuzzy Decision Making

Consider a constrained nonlinear optimization problem:

$$\begin{aligned} &\text{Max } z = f(x) \\ &\text{subject to } g_i(x) \le, =, or \ge b_i \quad i = 1.2, ..., p \\ &\qquad\qquad x \ge 0 \end{aligned} \qquad \text{....................(1.7)}$$

Where, $x \epsilon R_n$, $b_i(i=1,2,...,p)$ are constants. Suppose x_o is an initial feasible point of problem (1.7)

By using the linear approximation technique [Walsh (1997)] about the point x_o and by change of variables, we obtain the following LP problem:

$$\begin{aligned} &Max\ \hat{z} = c^T(w - m) \\ &\text{subject to } a_i^T w \le, =, or \ge \hat{b}_i + a_i^T m \\ &\qquad w_j' \le w_j \le w_j'' \\ &\qquad i = 1, 2, ..., p;\ j = 1, 2, ..., n \end{aligned} \qquad \text{................(1.8)}$$

$$\left.\begin{aligned} &\text{where } w = [w_1, w_2, ..., w_n] \ (n\text{-dimensional column vector}) \\ &m = [m_1, m_2, ..., m_n], \text{ vector of step length} \\ &w_j' = \max\{x_j' - x_{\circ j} + m_j, 0\}. \\ &w_j'' = \min\{x_j'' - x_{\circ j} + m_j, 2m_j\}. \\ &x_j' \text{ and } x_j'' \text{ are the lower and upper bounds of } x_j \\ &(\text{ the } j^{\text{th}} \text{component of } x). \\ &x_{\circ j} \text{ is the } j^{th} \text{component of } x_\circ. \\ &a_i = \nabla g_i(x_\circ),\ (n - \text{dimensional column vector}) \\ &\hat{b}_i = b_i - g_i(x_\circ) \\ &c = \nabla f(x_\circ),\ (n - \text{dimensional column vector}) \\ &\hat{z} = z - f(x_\circ) \end{aligned}\right\} \qquad \text{................(1.9)}$$

By omitting the constant term $c^T m$ in problem (1.7), we obtain the following LP problem:

$$\begin{aligned} &\text{Max } \hat{z} = c^T w \\ &\text{subject to } a_i^T w \le, =, or \ge \hat{b}_i + a_i^T m \\ &\qquad w_j' \le w_j \le w_j'', w \ge 0 \\ &\qquad i = 1, 2, ..., p;\ j = 1, 2, ..., n \end{aligned} \qquad \text{...................(1.10)}$$

Now our problem is to construct a fuzzy LP model which is to be solved iteratively. Here, we shall present a model that is particularly suitable for the type of linear programming model (1.10) in fuzzy environment which seem to have some advantages [Werners (1987)]. Werner (1987) suggested the following definition.

Definition 1.7 [Werner (1987)]. Let $:f\ X\rightarrow R'$ be an objective function, $\tilde{R}$ = fuzzy feasible region, $S(\tilde{R})$ =support of $\tilde{R}, \tilde{R}_1$ =1-level cut of $\tilde{R}$. The membership function of this goal (objective function) in the given solution space is then defined as

$$\mu_{\tilde{G}}(x)=\begin{cases}0 & \text{if } f(x)\le \sup\limits_{\tilde{R}} f\\ \dfrac{f(x)-\sup\limits_{\tilde{R}} f}{\sup\limits_{S(\tilde{R})} f-\sup\limits_{\tilde{R}_1} f} & \text{if } \sup\limits_{\tilde{R}_1} f<f(x)<\sup\limits_{S(\tilde{R})} f\\ 1 & \text{if } \sup\limits_{S(\tilde{R})} \mathrm{f}\le f(x)\end{cases}$$

The corresponding membership in functional space is then

$$\mu_{\tilde{G}}(r)=\begin{cases}\sup\limits_{x\in f^{-1}(r)} \mu_{\tilde{G}}(r) & \text{if } r\in R, f^{-1}(r)\neq 0\\ 0 & \text{otherwise}\end{cases}$$

We extend the above definition as follows:

Definition 1.8. Let $\hat{z} : W\rightarrow R^1$ be an objective function, $\tilde{R}$= fuzzy feasible region, $S(\tilde{R})$=

support of $\tilde{R}, \tilde{R}_1$ = 1-level cut of $\tilde{R}$and $\tilde{G}$ = fuzzy goal. The membership function of this goal $\tilde{G}$in the given solution space is then defined as

$$\mu_{\tilde{G}}(w)=\begin{cases}0 & \text{if } \hat{z}(w)\le \sup\limits_{\tilde{R}_1} \hat{z}\\ \dfrac{\hat{z}-\sup\limits_{\tilde{R}_1} \hat{z}}{\sup\limits_{S(\tilde{R})} \hat{z}-\sup\limits_{\tilde{R}_1} \hat{z}} & \text{if } \sup\limits_{\tilde{R}_1} \hat{z}<\hat{z}(w)<\sup\limits_{S(\tilde{R})} \hat{z}\\ 1 & \text{if } \sup\limits_{S(\tilde{R})} \hat{z}\le \hat{z}(w)\end{cases}$$

The corresponding membership in functional space is then

$$\mu_{\tilde{G}}(r)=\begin{cases}\sup\limits_{w\in \hat{z}^{-1}(r)} \mu_{\tilde{G}}(w) & \text{if } r\in R,\ \hat{z}^{-1}(r)\neq 0\\ 0 & \text{otherwise}\end{cases}$$

We assume that each of p rows of (1.10) shall be represented by a fuzzy set, the membership functions of which is given by $\mu_i(x)$, i=1,2,...p.

Let the membership functions of the fuzzy sets representing the fuzzy constraints forming the fuzzy feasible region $\tilde{R}$

$$\left(a_i^T w \lesssim, \; \cong \text{ or } \gtrsim \hat{b}_i + a_i^T m, \; i = 1, 2, \dots, p\right)$$

be defined as

$$\begin{aligned} &\text{Maximize } \hat{z} = c^T w \\ &\quad \text{subject to } a_i^T w \le, = \text{ or } \ge \hat{b}_i + a_i^T m \\ &\quad\quad w \ge 0 \end{aligned}$$

$$\text{yielding } \sup_{\tilde{R}_1} \hat{z} = \left(c_w^T\right)_{optimum} = \hat{z}_1; \text{ and}$$

$$\begin{aligned} &\text{Maximize } \hat{z} = c^T w \\ &\quad \text{subject to } a_i^T w \le, \; =, \; \ge \hat{b}_i + a_i^T m + k_i \\ &\quad\quad w \ge 0 \end{aligned}$$

$$\text{yielding } \sup_{\tilde{R}_1} \hat{z} = \left(c_w^T\right)_{optimum} = \hat{z}_\circ .$$

..................(1.11)

wherek_i is thej^{th} component of tolerance vector $k \in R^p$.

The membership function of objective function of fuzzy solution can be determined by solving the following LP problems:

................(1.12)

$$\mu_i(x) = \begin{cases} 1 & \text{if } a_i^T w \le, \; = \text{ or } \gtrsim \hat{b}_i + a_i^T m \\ \dfrac{\hat{b}_i + k_i - a_i^T w}{k_i} & \text{if } \hat{b}_i + a_i^T m \le a_i^T w \le \hat{b}_i + a_i^T m + k_i \\ 0 & \text{if } a_i^T w \ge \hat{b}_i + a_i^T m + k_i \end{cases}$$

..................(1.13)

Therefore, the membership function of fuzzy goal $\tilde{G}$ is given by

$$\mu_{\tilde{G}}(w) = \begin{cases} 1 & \text{if } \hat{z}_\circ \le c^T w \\ \dfrac{c^T w - \hat{z}_1}{\hat{z}_\circ - \hat{z}_1} & \text{if } \hat{z}_1 < c^T w < \hat{z}_\circ \\ 0 & \text{if } c^T w \le \hat{z}_1 \end{cases}$$

........................(1.14)

By applying the concept of fuzzy decision set $\tilde{D}$ [see Bellman &Zadeh (1970)], the membership function of $\tilde{D}$ is defined as

$$\mu_{\tilde{D}}(w) = \min\left(\mu_{\hat{G}}(w), \mu_i(w)\right)$$ ....(1.15)

Introducing a new variable, λ , which corresponds essentially to (5.3.8), we arrive at:

$$\begin{aligned}&\text{Maximize } \lambda\\&\quad \text{subject to } \lambda(\hat{z}_{\circ} - \hat{z}_1) - c^T w \le -\hat{z}_1\\&\qquad\qquad \lambda k_i + a_i^T w \le, = or \ge \hat{b}_i + a_i^T m + k_i\\&\qquad\qquad\quad \lambda \le 1,\ \lambda, w \ge 0\end{aligned}$$ ....(1.15)

Model (1.15) is to be solved for each iteration. If w_{*r} is the optimal solution of problem (1.15) in the r^{th}iteration(which corresponds with trial pointx_r), then, the next trial point is

$$x_{r+1} = x_r + w_r^* - m.$$ ....(1.16)

The iteration terminates when where is the value of obtained in the iteration and is a very small positive numerator.

In order to illustrate the efficient of the above proposed model, we consider the following nonlinear optimization problem:

$$\begin{aligned}\text{Maximize}\quad & z = 2x_1^2 - x_1x_2 + 3x_2^2\\\text{subject to}\quad & 3x_1 + 4x_2 \le 12\\& x_1^2 - x_2^2 \ge 1\\& x_1,\ x_2 \ge 0\end{aligned}$$ ....(1.17)

5. Conclusion and Solution:

We set x_o = [x_{o1}, x_{o2}] = [2,1] [(where the first subscript denotes the iteration number) as the initial feasible solution and $m = \left[\frac{1}{2}, \frac{1}{2}\right]$ Using step 3, we calculate

$$a_1 = [3,4],\ a_1 = [4,-2]$$

$$\hat{b}_1 = 2,\ \hat{b}_2 = -2$$

$$c = [7,4]$$

Then, construct the LPP as

$$\begin{aligned}&\text{Max } \hat{z} = 7w_1 + 4w_2\\&\text{subject to } 3w_1 + 4w_2 \le 5.5\\&\qquad\qquad 4w_1 - 2w_2 \ge -1\\&\qquad\qquad 0 \le w_1 \le 1,\ 0 \le w_2 \le 1\\&\text{yielding } \hat{z} = 9.5,\ w_1 = 1,\ w_2 = 0.63.\end{aligned}$$ ....(1.18)

We set tolerance vector k = [1.5,3] and solve the LP problem (1.18) and

$$\begin{aligned} &\text{Max } \hat{z} = 7w_1 + 4w_2 \\ &\text{subject to } 3w_1 + 4w_2 \le 7 \\ &\qquad 4w_1 - 2w_2 \ge 2 \\ &\qquad 0 \le w_1 \le 1,\ 0 \le w_2 \le 1 \end{aligned}$$

yielding $\hat{z} = 11,\ w_1 = 1,\ w_2 = 1$. ....(1.19)

By calculating the membership function (see (1.18)), we have the parametric LPP:

$$\begin{aligned} &\text{Max } \lambda = 7w_1 + 4w_2 \\ &\text{subject to } 1.5\lambda\text{-}3w_1 - 4w_2 \le -9.5 \\ &\qquad 1.5\lambda + 3w_1 + 4w_2 \le 7 \\ &\qquad 3\lambda + 4w_1 - 2w_2 \ge 2 \\ &\qquad 0 \le w_1 \le 1,\ 0 \le w_2 \le 1 \end{aligned}$$

yielding $\lambda_\circ^* = 0.5,\ w_1 = 1,\ w_2 = .81$. Thus, we get- $x_{11} = 2.5,\ x_{12} = 1.31$.

Setting $x_1 = [x_{11}, x_{12}] = [2.5, 1.31]$

as initial trial point, we repeat the same procedure. In this iteration, we get

$\lambda_1^* = 0.5, w_1 = 0.5,\ w_2 = 0.63$

yielding $x_{11} = 2.5,\ x_{12} = 1.44$

Since $|\lambda_1^* - \lambda_\circ^*| < \delta, \delta = 0.00001$,

therefore the required optimal solution of the given problem is x_1= 2.5, x_2 =1.44, = z=15.1208.

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