**ℓ-HILBERT MEAN LABELING OF SOME PATH RELATED GRAPHS**


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**Abstract:**

Let G be a graph with p vertices and q edges. The q\textsuperscript{th} Hilbert number is denoted by H\textsubscript{q} and is defined by H\textsubscript{q} = 4(q − 1) + 1 where q ≥ 1. A ℓ- hilbert mean labeling is an injective function f: V(G) → \{0, 1, 2, ..., H\textsubscript{ℓ}(q−1)\}, where ℓ ≥ 1 that induces a bijection f*: E(G) → \{H\textsubscript{ℓ}, H\textsubscript{ℓ}+1, H\textsubscript{ℓ}+2, ..., H\textsubscript{ℓ}(q−1)\} defined by

\[ f^*(uv) = \begin{cases} \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \\ \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \end{cases} \]

for all uv ∈ E(G). A graph which admits such labeling is called a ℓ- hilbert mean graph. In this paper, a new type of labeling called ℓ- hilbert mean labeling is introduced and the path related graphs is studied.

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1. **Introduction**

Let G = (V, E) be a graph with p vertices and q edges. The graph considered in this paper are simple, finite, undirected and without loops or multiple edges. Terms not defined here are used in the sense of Harary [4]. For number theoretic terminology [1] is followed. A graph labeling is an assignment of integers to the vertices or the edges or both subject to certain conditions. If the domain of the mapping is a set of vertices (edges/both) then the labeling is called a vertex (edge/total) labeling. A dynamic survey of graph labeling is regularly updated by Gallian [2] and it is published by Electronic Journal of Combinatorics. The concept of mean labeling was introduced by S.Somasundaram and R.Ponraj [6]. For Triangular mean labeling and k- Mean labeling refer [5] and [3] and Hilbert mean labeling was introduced in [7].

2. **Preliminaries**

**Definition 2.1:** A path \( P_n \) is obtained by joining \( u_i \) to the consecutive vertices \( u_{i+1} \) for \( 1 \leq i \leq n - 1 \).

**Definition 2.2:** The graph obtained by joining a single pendant edge to each vertex of a path \( P_n \) is called a comb. It is denoted by \( P_n \square K_1 \) (or) \( P_n^* \).

**Definition 2.3:** Bistar is the graph obtained by joining the apex vertices of two copies of star \( K_{1,n} \).

**Definition 2.4:** The H-graph of path \( P_n (n \geq 3) \) is the graph obtained from two copies of \( P_n \) with the vertices \( u_1, u_2, \ldots, u_n \) and \( v_1, v_2, \ldots, v_n \) by joining the vertices \( \left( \frac{u_{n+1}}{2}, \frac{v_n+1}{2} \right) \) if \( n \) is odd and \( \left( \frac{u_n}{2}, \frac{v_n}{2} \right) \) if \( n \) is even. It is denoted by \( H(P_n) \).
**Definition 2.5**: A F-tree $F(P_n)$ is a graph obtained from path on $n \geq 3$ vertices by appending two pendant edges one to an end vertex and other to vertex adjacent to an end vertex.

**Definition 2.6**: The ladder graph $L_n$ is a planar, undirected graph with $2n$ vertices and $3n - 2$ edges. The ladder graph $L_n$ is a graph obtained as the Cartesian product of $P_2$ and $P_n$.

**Definition 2.7**: The slanting ladder $SL_n$ is a graph that consists of two copies of $P_n$ having vertex set $\{u_i; 1 \leq i \leq n\}$ or $\{v_i; 1 \leq i \leq n\}$ and the edge set is formed by adjoining $u_i+1$ and $v_i$ for all $1 \leq i \leq n - 1$.

**Definition 2.8**: The graph obtained from a path by attaching exactly two pendant edges to each internal vertex of the path is called a Twig graph and it is denoted by $T(n)$.

**Definition 2.9**: A graph $G$ with $p$ vertices and $q$ edges is called a mean graph if there is an injective function $f$ from the vertices of $G$ to $\{0, 1, 2, ..., q\}$ in such a way that when each edge $e = uv$ is labeled by $(f(u) + f(v)) / 2$ if $f(u) + f(v)$ is even and $(f(u) + f(v) + 1) / 2$ if $f(u) + f(v)$ is odd, then the resulting edge labels are distinct. Here $f$ is called a mean labeling of $G$. If $G$ is a mean graph, then the edges get labels $1, 2, ..., q$.

**Definition 2.10** [3]: A $(p, q)$ graph $G$ is set to have $k$-mean labeling, if there is an injective function $f$ from the vertices of $G$ to $\{0, 1, 2, ..., k + q - 1\}$ such that the induced map $f^*$ defined on $E$ by

$$f^*(uv) = \begin{cases} \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \\ \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \end{cases}$$

is a bijection from $E$ to $\{k, k + 1, k + 2, ..., k + q - 1\}$. A graph that admits a $k$-mean labeling is called a $k$-mean graph.

**Definition 2.11**: The $n^{th}$ hilbert number $H_n$ is given by the formula $4(n -1) + 1$ for $n \geq 1$. The first few hilbert numbers are $1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57$, etc.

**Definition 2.12**: Let $G$ be a graph with $p$ vertices and $q$ edges. The $n^{th}$ hilbert number is denoted $H_n$ and is defined by $H_n = 4(n -1) + 1$ where $n \geq 1$. A hilbert mean labeling is an injective function $f: V(G) \rightarrow \{0, 1, 2, ..., H_q\}$, where $H_q$ is the $q^{th}$ hilbert number that induces a bijection $f^*: E(G) \rightarrow \{H_1, H_2, ..., H_q\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \\ \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \end{cases}$$

for all $uv \in E(G)$. A graph which admits such labeling is called a hilbert mean graph.

### 3. Main Results

**Definition 3.1**: Let $G$ be a graph with $p$ vertices and $q$ edges. The $q^{th}$ hilbert number is denoted by $H_q$ and is defined by $H_q = 4(q -1) + 1$ where $q \geq 1$. A $\ell^{th}$ hilbert mean labeling is an injective function $f: V(G) \rightarrow \{0, 1, 2, ..., H_{\ell+q-1}\}$ where $\ell \geq 1$, that induces a bijection $f^*: E(G) \rightarrow \{H_\ell, H_{\ell+1}, H_{\ell+2}, ..., H_{\ell+q-1}\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \\ \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \end{cases}$$

for all $uv \in E(G)$. A graph which admits such labeling is called a $\ell^{th}$ hilbert mean graph.

**Theorem 3.2**: $P_m$ is a $\ell^{th}$ hilbert mean graph, where $m \geq 2$.

**Proof**: Let $G = P_m$. Let $V(G) = \{x_i; 1 \leq i \leq m\}$ and $E(G) = \{x_i, x_{i+1}; 1 \leq i \leq m-1\}$. We observe that $G$ has $m$ vertices and $m - 1$ edges.

Define $f: V(G) \rightarrow \{0, 1, 2, ..., H_{\ell+2m-2}\}$ as follows.

**Case 1**: $m$ is odd and $\ell \geq 2$, $f(x_{\ell}) = 4\ell - 7$.

For $2 \leq i \leq m$,

$$f(x_i) = \begin{cases} 4(\ell + i - 3) + 1, & \text{i is odd} \\ 4(\ell + i - 2), & \text{i is even} \end{cases}$$

**Case 2**: $m$ is even and $\ell \geq 1$,
For $1 \leq i \leq m$,
\[
f(x_i) = \begin{cases} 
4(\ell + i - 2), & \text{i is odd} \\
4(\ell + i - 3) + 1, & \text{i is even}
\end{cases}
\]
Clearly $f$ is injective and the induced edge labeling $f^*: E(G) \to \{H, H_{\ell+1}, ..., H_{\ell+m-2}\}$ is defined as follows.
f*(x_i x_{i+1}) = H_{\ell+2(i-1)}, where $1 \leq i \leq m - 1$
Thus, we get the induced edge labels as $H, H_{\ell+1}, ..., H_{\ell+m-2}$.
Hence $P_m^\ell$ is a $\ell$- hilbert mean graph, where $m \geq 2$.

**Example 3.3:** The $\ell$- hilbert mean labeling of $P_3$ is shown in figure 1.

**Theorem 3.4:** $P_m^\ell$ is a $\ell$- hilbert mean graph, where $m \geq 3$.

**Proof:** Let $G = P_m^\ell$. Let $V(G) = \{x_i, y_i: 1 \leq i \leq m\}$ and $E(G) = \{y_i y_{i+1}: 1 \leq i \leq m - 1\} \cup \{x_i y_i: 1 \leq i \leq m\}$.
We observe that $G$ has $2m$ vertices and $2m - 1$ edges.
Define $f: V(G) \to \{0, 1, 2, ..., H_{\ell+2m-2}\}$ as follows.
f(x_i) = 4(\ell + 2i - 3), \quad 1 \leq i \leq m 

\[
f(y_i) = 4(\ell + 2i - 3) + 1, \quad 1 \leq i \leq m
\]
Clearly $f$ is injective and the induced edge labeling $f*: E(G) \to \{H, H_{\ell+1}, ..., H_{\ell+2m-2}\}$ is defined as follows.
f*(x_i x_{i+1}) = H_{\ell+2(i-1)}, where $1 \leq i \leq m - 1$

\[
f*(y_i y_{i+1}) = H_{\ell+2(i-1)}, where $1 \leq i \leq m
\]
Thus, we get the induced edge labels as $H, H_{\ell+1}, ..., H_{\ell+2m-2}$.
Hence $P_m^\ell$ is a $\ell$- hilbert mean graph, where $m \geq 3$.

**Example 3.5:** The $\ell$- hilbert mean labeling of $P_m^\ell$ is shown in figure 2.

**Theorem 3.6:** $B_{n,n}$ is a $\ell$- hilbert mean graph, where $n \geq 2$.

**Proof:** Let $G = B_{n,n}$, where $n \geq 2$.
Let $V(G) = \{x, y, x_i, y_i: 1 \leq i \leq n\}$ and $E(G) = \{xy, xx_i, yy_i: 1 \leq i \leq n\}$.
We observe that $G$ has $2n + 2$ vertices and $2n + 1$ edges.
Define $f: V(G) \to \{0, 1, 2, ..., H_{\ell+2n}\}$ as follows.
f(x) = 4(\ell - 1), f(x_i) = 4(\ell + 2i - 3) + 1, \quad 1 \leq i \leq n

\[
f(y) = 4(\ell + 2i - 1), \quad 1 \leq i \leq n, f(y_i) = 4(\ell + 2n - 1) + 1.
\]
Clearly $f$ is injective and the induced edge labeling $f^*$ is defined as follows.
f^*(xx_i) = H_{\ell+2(i-1)} where $1 \leq i \leq n$. f^*(xy) = H_{\ell+n}.
f^*(yy_i) = H_{\ell+2(n+1)} where $1 \leq i \leq n
Thus, we get the induced edge labels as $H, H_{\ell+1}, ..., H_{\ell+2n}$.
Hence $B_{n,n}$ is a $\ell$- hilbert mean graph, where $n \geq 3$.

**Example 3.7:** $\ell$- hilbert mean labeling of $B_{4,4}$ is shown in figure 3.
Theorem 3.8: $H(P_n)$ is a $\ell$-hilbert mean graph, where $n \geq 3$.

Proof: Let $G = H(P_n)$. Let $V(G) = \{x_i, y_i: 1 \leq i \leq n\}$ and $E(G) = \{x_iy_i: 1 \leq i \leq n\} \cup \{x_iy_{i+1}, y_iy_{i+1}: 1 \leq i \leq n - 1\}$

Let $G$ has $2n$ vertices and $2n - 1$ edges.

We define a labeling $f : V(G) \rightarrow \{0, 1, 2, ..., H_{f+2n-2}\}$ as follows.

**Case 1:** $n$ is odd. For $1 \leq i \leq n$,

$$f(x_i) = \begin{cases} 4(\ell + i - 2), & \text{i is odd} \\ 4(\ell + i - 3) + 1, & \text{i is even} \end{cases}$$

$$f(y_i) = \begin{cases} 4(\ell + n + i - 3) + 1, & \text{i is odd} \\ 4(\ell + n + i - 2), & \text{i is even} \end{cases}$$

**Case 2:** $n$ is even. For $1 \leq i \leq n$,

$$f(x_i) = \begin{cases} 4(\ell + i - 2), & \text{i is odd} \\ 4(\ell + i - 3) + 1, & \text{i is even} \end{cases}$$

$$f(y_i) = \begin{cases} 4(\ell + n + i - 2), & \text{i is odd} \\ 4(\ell + n + i - 3) + 1, & \text{i is even} \end{cases}$$

Clearly $f$ is injective and the induced edge labeling $f^*$ is defined as follows.

$$f^*(\frac{x_{n+1}}{2}, \frac{y_{n+1}}{2}) = H_{f+2(n-1)}$$, where $n$ is odd

$$f^*(\frac{x_{n+2}}{2}, \frac{y_n}{2}) = H_{f+2(n-1)}$$, where $n$ is even

$$f^*(x_iy_{i+1}) = H_{f(1-\ell)}$$, where $1 \leq i \leq n - 1$

$$f^*(y_iy_{i+1}) = H_{f(n+1-\ell)}$$, where $1 \leq i \leq n - 1$

Thus, we get the induced edge labels as $H_p, H_{f+1} \ldots, H_{f+2n-2}$.

Hence $H(P_n)$ is a $\ell$-hilbert mean graph, where $n \geq 3$.

**Example 3.9:** $\ell$-hilbert mean labeling of $H(P_3)$ is shown in figure 4.

Theorem 3.10: $F(P_n)$ is a $\ell$-hilbert mean graph, where $n$ is even and $n \geq 4$.

Proof: Let $G = F(P_n)$. Let $V(G) = \{x_i: i = 1, 2\} \cup \{y_i: 1 \leq i \leq n\}$ and $E(G) = \{x_iy_i: i = 1, 2\} \cup \{y_iy_{i+1}: 1 \leq i \leq n - 1\}$

We observe that $G$ has $n + 2$ vertices and $n + 1$ edges.

We define a labeling $f : V(G) \rightarrow \{0, 1, 2, ..., H_{f+n}\}$ as follows.
Example 3.1

Hence

Clearly

For

Case 2:

Let

Proof

Theorem 3.1

Thus, we get the induced edge labels as $H_r, H_{t+1} \ldots, H_{t+n}$.

Example 3.11: $\ell$- hilbert mean labeling of $F(P_n)$ is shown in figure 5.

Figure - 5

Theorem 3.12: $L_n$ is a $\ell$- hilbert mean graph, where $n \geq 3$.

Proof: Let $G = L_n$. Let $V(G) = \{x_i, y_i : 1 \leq i \leq n\}$ and $E(G) = \{x_i, y_i : 1 \leq i \leq n\} \cup \{x_{i+1}, y_{i+1} : 1 \leq i \leq n - 1\}$

Let $G$ has $2n$ vertices $3n - 2$ edges. Define $f : V(G) \rightarrow \{0, 1, 2, \ldots, H_{t+3n-3}\}$ as follows.

Case 1: $n$ is odd and $\ell \geq 1$,

For $1 \leq i \leq n$,

\[ f(x_i) = \begin{cases} 4(\ell + i - 2), & \text{if } i \text{ is odd} \\ 4(\ell + i - 3), & \text{if } i \text{ is even} \end{cases} \]

\[ f(y_i) = \begin{cases} 4(\ell + 2n + i - 4) + 1, & \text{if } i \text{ is odd} \\ 4(\ell + 2n + i - 3), & \text{if } i \text{ is even} \end{cases} \]

Case 2: $n$ is even and $\ell \geq 2$, $f(x_1) = 4\ell - 7$,

For $2 \leq i \leq n$,

\[ f(x_i) = \begin{cases} 4(\ell + i - 3) + 1, & \text{if } i \text{ is odd} \\ 4(\ell + i - 2), & \text{if } i \text{ is even} \end{cases} \]

\[ f(y_i) = \begin{cases} 4(\ell + 2n + i - 3), & \text{if } i \text{ is odd} \\ 4(\ell + 2n + i - 4) + 1, & \text{if } i \text{ is even} \end{cases} \]

Clearly $f$ is injective and the induced edge labeling $f^*$ is defined as follows.

\[ f^*(x_i x_{i+1}) = H_{t+i-1}, \text{ where } 1 \leq i \leq n - 1, \]

\[ f^*(y_i y_{i+1}) = H_{t+2n+i-2}, \text{ where } 1 \leq i \leq n - 1, \]

\[ f^*(x_i y_i) = H_{t+3n-3}, \text{ where } 1 \leq i \leq n \]

Thus, we get the induced edge labels as $H_r, H_{t+1} \ldots, H_{t+3n-3}$.

Hence $L_n$ is a $\ell$- hilbert mean graph, where $n \geq 3$.

Example 3.13: $\ell$- hilbert mean labeling of $L_3$ is shown in figure 6.
Theorem 3.14: $SL_n$ is a $\ell$-hilbert mean graph, where $n$ is even and $n \geq 3$.

Proof: Let $G = SL_n$. Let $V(G) = \{x_i, y_i: 1 \leq i \leq n\}$ and $E(G) = \{x_ix_{i+1}, y_iy_{i+1}, x_{i+1}y_i: 1 \leq i \leq n - 1\}$.

Let $G$ has $2n$ vertices $3n - 3$ edges. Define $f: V(G) \rightarrow \{0, 1, 2, ..., H_{\ell+3n-4}\}$ as follows.

**Case 1**: $n$ is odd and $\ell \geq 2$.

For $2 \leq i \leq n$,

$f(x_i) = \begin{cases} 4(\ell + i - 3) + 1, & \text{if } i \text{ is odd} \\ 4(\ell + i - 2), & \text{if } i \text{ is even} \end{cases}$

$f(y_i) = \begin{cases} 4(\ell + 2n + i - 5), & \text{if } i \text{ is odd} \\ 4(\ell + 2n + i - 4) + 1, & \text{if } i \text{ is even} \end{cases}$

**Case 2**: $n$ is even and $\ell \geq 2$.

For $1 \leq i \leq n$,

$f(x_i) = \begin{cases} 4(\ell + i - 2), & \text{if } i \text{ is odd} \\ 4(\ell + i - 3) + 1, & \text{if } i \text{ is even} \end{cases}$

$f(y_i) = \begin{cases} 4(\ell + 2n + i - 4), & \text{if } i \text{ is odd} \\ 4(\ell + 2n + i - 5) + 1, & \text{if } i \text{ is even} \end{cases}$

Clearly $f$ is injective and the induced edge labeling $f^*$ is defined as follows.

$f^*(x_ix_{i+1}) = H_{\ell+(i-1)}$, where $1 \leq i \leq n - 1$

$f^*(y_iy_{i+1}) = H_{\ell+(2n+i-3)}$, where $1 \leq i \leq n - 1$

Thus, we get the induced edge labels as $H_{\ell}, H_{\ell+1}, ..., H_{\ell+3n-4}$.

Hence $SL_n$ is a $\ell$-hilbert mean graph, where $n$ is even and $n \geq 3$.

**Example 3.15**: $\ell$-hilbert mean labeling of $SL_4$ is shown in figure 7.
Theorem 3.16: $T(n)$ is a $\ell$-hilbert mean graph, where $n$ is even and $n \geq 4$.

Proof: Let $G = T(n)$. Let $V(G) = \{x_i : 1 \leq i \leq n\} \cup \{y_i, z_i: 1 \leq i \leq n - 2\}$

$E(G) = \{x_i x_{i+1} : 1 \leq i \leq n - 1\} \cup \{y_i z_i: 1 \leq i \leq n - 2\} \cup \{x_i z_i: 1 \leq i \leq n - 2\}$

Let $G$ has $3n - 4$ vertices and $3n - 5$ edges.

Define a function $f : V(G) \rightarrow \{0, 1, 2, ..., H_{\ell + 2n - 2}\}$ as follows.

For $1 \leq i \leq n$,

$f(x_i) = \begin{cases} 4(\ell + 3i - 4), & \text{if } i \text{ is odd} \\ 4(\ell + 3i - 7) + 1, & \text{if } i \text{ is even} \end{cases}$

$f(y_i) = \begin{cases} 4(\ell + 3i - 2), & \text{if } i \text{ is odd} \\ 4(\ell + 3i - 5) + 1, & \text{if } i \text{ is even} \end{cases}$

$f(z_i) = \begin{cases} 4(\ell + 3i), & \text{if } i \text{ is odd} \\ 4(\ell + 3i - 3) + 1, & \text{if } i \text{ is even} \end{cases}$

Clearly $f$ is injective and the induced edge labeling $f^*$ is defined as follows.

$f^*(x_i x_{i+1}) = H_{\ell + (i - 1)}$, where $1 \leq i \leq n - 1$

$f^*(x_i y_i) = H_{\ell + (3i - 2)}$, where $1 \leq i \leq n - 2$

$f^*(x_i z_i) = H_{\ell + (3i - 1)}$, where $1 \leq i \leq n - 2$

Thus, we get the induced edge labels as $H_{\ell}, H_{\ell + 1}, ..., H_{\ell + 2n - 2}$.

Hence $T(n)$ is a $\ell$-hilbert mean graph, where $n$ is even and $n \geq 4$.

Example 3.17: $\ell$-hilbert mean labeling of $T(4)$ is shown in figure 8.

![Figure 8](image)

4. Conclusion

In this paper, we have introduced $\ell$-hilbert mean labeling and studied $\ell$-hilbert mean labeling of some path related graphs. This work contributes several new results to the theory of graph labeling.

References


