The Split (D, C) Number of a Graph and Related Properties

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Abstract: In this paper, we introduce the concept of dominating chromatic set [(D, C)-set in short] and split dominating chromatic set of a connected graph G, it leads to an associated color variable, namely (D, C) number and split (D, C) number of G. It characterizes the parameter, along with optimized dominating sets. Also, the split (D, C) number of some standard graphs are identified. We study the realization problem of K-coloring graph G. For any three integers λ, p and q such that 2 ≤ λ ≤ q ≤ p, there exists a connected graph G such that \( γ_{Ds}(G) = λ \) and \( |V(G)| = p \)

Keywords: Chromatic Set, Dominating Chromatic Set, Split (D, C) Set, (D, C) Number, Split (D, C) Number.

1. Introduction

A graph \( G = (V, E) \) is an ordered pair of two sets \( V \) and \( E \), where \( V \) is called vertex set and \( E \) is called an edge set of \( G \). The cardinality of vertex set \( V \) and edge set \( E \) is called order and size of a graph \( G \) respectively. A graph is said to be acyclic if it has no cycles. A connected acyclic graph \( G \) is called a tree. A support vertex in a tree is defined as a vertex adjacent to a leaf and a leaf is a vertex of degree one in a tree. A vertex \( v \) is said to be cut vertex of \( G \) if \( G - v \) is disconnected. The set \( \text{Cut}(G) \) is the set of all cut vertices of the graph \( G \). For the basic terminology and graph theoretical notation, one can refer to [3].

In this article, a graph \( G \) means that an undirected, connected, finite and simple graph. A set \( D \subseteq V \) in a connected graph \( G \) is said to be a dominating set if every vertex \( v \in V - D \) is adjacent to some vertex in \( D \). That is, closed neighbourhood\( N[D] = V \). The dominating number \( γ(G) \) is the minimum size of dominating sets of \( G \), refer to [4, 5].

A dominating set \( D \) of a graph \( G \) is a split dominating set if the induced subgraph \( (V - D) \) is disconnected. The split domination number \( γ_s(G) \) is the minimum cardinality of a split dominating set, refer to [8]. A vertex coloring of a graph \( G \) is a function \( f: V \rightarrow N \) such that \( f(u) = f(v) \Rightarrow \{u, v\} \notin E(G) \) for all \( u, v \in V \). A graph \( G \) is said to be \( k \)-colorable if it has proper \( k \)-vertex coloring. The least such positive integer \( k \) such that \( G \) is \( k \)-colorable is called chromatic number \( χ(G) \) of a graph \( G \), refer to [2, 6]. A \( k \)-chromatic graph \( G \) means that \( G \) is a connected graph with \( χ(G) = k \). Next, define the notion of chromatic sets of graphs, by choosing the vertices from each color classes of \( G \), studied in [2]. Let \( G \) be \( k \)-chromatic graph. A set \( C \subseteq V \) of vertices of \( G \) is said to
be chromatic set of $G$ if $C$ is the set of all vertices of $G$ having different colors in $G$. The minimum size among all chromatic sets of a graph $G$ is known as the chromatic number $\chi(G)$ of the graph $G$. That is,

$$\chi(G) = \min\{|C|: C \text{ is a chromatic set of } G\}.$$ 

In this paper, we propose a novel approach to the fusion of dominating set and a chromatic set of connected graph, known as dominating chromatic set or $(D, C)$ set.

2. THE $(D, C)$-NUMBER & SPLIT $(D, C)$-NUMBER OF A GRAPH

**Definition 2.1**

A set $S \subseteq V$ of vertices in a connected graph $G$ is said to be a $(D, C)$ set of $G$ if it is both dominating and chromatic set of $G$. The minimum cardinalities among the $(D, C)$ sets of $G$ is called $(D, C)$-number of the $G$ graph, denoted by $\gamma_{(D, C)}(G)$.

$$\text{i.e., } \gamma_{(D, C)}(G) = \min\{|S|: S \text{ a } (D, C)\text{set of } G\}.$$ 

Obviously, the parametery$(G)$ and $\chi(G)$ may be distinct or not. In general, $1 \leq \gamma_{(D, C)}(G) \leq p$, where $p = |V|$ is the order of the graph $G$.

**Definition 2.2**

A set $S \subseteq V$ of vertices in a connected graph $G$ is said to be a split $(D, C)$ set of $G$ if $S$ is a $(D, C)$ set and the induced subgraph $(V - S)$ is disconnected. The split $(D, C)$ set of minimum order is called minimum split $(D, C)$ number of the $G$ graph and its cardinality is called split $(D, C)$-Number of $G$, denoted by $\gamma_{(D, C)}(G)$.

$$\text{i.e., } \gamma_{(D, C)}(G) = \min\{|S|: S \text{ a split } (D, C)\text{set } G\}.$$ 

We establish this by an example.

**Definition 2.3**

A split $(D, C)$ set $S \subseteq V$ of minimum cardinality is called $\gamma_{(D, C)}$-set of the graph $G$. That is, $\gamma_{(D, C)}(G)$ is the size of the $\gamma_{(D, C)}$-set of $G$.

**Example 1.**

Consider a 2-coloring graph $G$. i.e., $\chi(G) = 2$ which is given in the Figure 2.1.

![Figure 2.1](image)

The 2-element chromatic sets of $G$ are $S_1 = \{v_1, v_2\}$, $S_2 = \{v_1, v_4\}$, $S_3 = \{v_2, v_3\}$, $S_4 = \{v_2, v_5\}$, $S_5 = \{v_3, v_4\}$ and $S_6 = \{v_4, v_5\}$. These sets are also dominating. So that $(D, C)$-number $\gamma_{(D, C)}(G) = 2$. But the induced subgraph $(V - S_i)$ is not disconnected for $i = 1$ to 6, we treat the split $(D, C)$-number $\gamma_{(D, C)}(G)$ as zero. i.e., $\gamma_{(D, C)}(G) = 0$. 

If $G$ is upgrade to 3-colorable pattern, then the possible three elements chromatic sets are $T_1 = \{v_2, v_2, v_4\}$ and $T_2 = \{v_2, v_4, v_5\}$. Clearly, $T_1$ and $T_2$ are also dominating sets of $G$. The induced subgraph $(V - T_1)$ is disconnected. Therefore, the split $(D, C)$-number $\gamma_{xs}(G) = 3$.

![Figure 2.2](image)

Example 2.

Consider a tree $T_4$ on four vertices, which is given in the Figure 2.3

![Figure 2.3](image)

The two element chromatic sets of $T_4$ are $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_2, v_4\}$ respectively. These sets are also dominating in nature. If we choose $S = \{v_1, v_2\}$ or $\{v_2, v_3\}$ or $\{v_2, v_4\}$, then $(V - S)$ is disconnected. Therefore, $S$ is $\gamma_{xs}$-set of $G$ and $\gamma_{xs}(T_4) = |S| = 2$. If we add one or two vertices to any of the tree leafs of the given tree $T_4$, then we obtain a new tree on five or six vertices denoted by $T_5$ or $T_6$. Therefore,

$$\gamma_{xs}(T_5) = \gamma_{xs}(T_6) = 2.$$

Example 3.

Consider a 2-coloring tree $G$ which is given in the figure 2.4

![Figure 2.4](image)
The set $S = \{v_1, v_2, v_3\}$ is the only dominating set of $G$ but it is not chromatic set because $G$ is 2-colorable. Also $(V - S)$ is disconnected. Therefore, $\gamma(G) = 3$ and $\gamma_{X_S}(G) = 0$. If we allow tree $G$ to three color pattern, then we can find the split $(D, C)$-number of $G$, $\gamma_{X_S}(G) = 3$.

3. Basic Observations

In this section we propose a few basic results.

**Proposition 3.1**

For a given $k$-coloring $G$, not all dominating sets are $(D, C)$ sets. But the converse always true.

Let $G = K_p$ be a complete graph. Then each $\{v_i\}$ is a dominating set of $G$ for $i = 1, 2, \ldots, p$. So $\gamma(G) = 1$. But these sets are not $(D, C)$ sets, since all vertices adjacent. i.e., the set $\{v_1, v_2, \ldots, v_p\}$ is unique $(D, C)$ set of $G$. So that $\gamma_x(G) = p$. In this case the split $(D, C)$-number, $\gamma_{X_S}(G) = 0$.

**Proposition 3.2**

For a given $k$-coloring $G$, not all $(D, C)$ sets are split $(D, C)$ sets. But the converse always true.

**Theorem 3.1**

For any connected graph $G$,

(i) $\gamma_x(G) \geq \max\{\gamma(G), \chi(G)\}$

(ii) $\gamma_{X_S}(G) \geq \max\{\gamma_x(G), \chi(G)\}$

**Proof:**

(i) Due to the Proposition 3.1, $\gamma_x(G) \geq \gamma(G)$. Since the $(D, C)$ set of $G$ is always a chromatic set, $\gamma_x(G) \geq \chi(G)$. It follows that the $(D, C)$ number bounded below by $\max\{\gamma(G), \chi(G)\}$. Therefore, the result (i) follows.

(ii) Due to the Proposition 3.2, $\gamma_{X_S}(G) \geq \gamma(G)$ and the split $(D, C)$ set is always a chromatic set and $V - C$ is disconnected $\Rightarrow \gamma_{X_S}(G) \geq \chi(G)$. It follows that the split $(D, C)$-number bounded below by $\max\{\gamma_x(G), \chi(G)\}$. Therefore, the result (ii) follows.

Next result gives the upper bound of any connected graph $G$ with $\text{Cut}(G) = \phi$ where $\text{Cut}(G)$ is the set of all cut vertices of $G$.

4. Characterization Results

**Theorem 4.1**

Let $G$ be a connected graph with $n$ vertices. Then $\gamma_{X_S}(G) = 2$ if and only if there exists a vertex $u \in \text{Cut}(G)$ such that $d(u)$ is almost $n - 1$.

**Proof:**

Assume $\gamma_{X_S}(G) = 2$, there exists a $\gamma_{X_S}$-set $S = \{u, v\}$ such that $(V - S)$ is disconnected. Therefore, we cannot find another vertex $v \in \text{Cut}(G)$. So $d(u)$ must be almost $n - 1$. Conversely assume that a vertex $u \in \text{Cut}(G)$ and $d(u) = n - 1$. Then we can find $u \in V$ such that $\{u, v\}$ is a split $(D, C)$ set due to the disconnectedness of $(V - \{u, v\})$. Therefore, $\gamma_{X_S}(G) = 2$.

**Corollary 4.1**

For a connected graph $G$ of order $n > 4$ with $\text{Cut}(G) \neq \phi$, $\gamma_{X_S}(G) \geq 2$. 

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Theorem 4.2
For any connected graph $G$ with $\text{Cut}(G) \neq \emptyset$, then $\gamma_{xG}(G) \leq \gamma(G) + \chi(G) - \lambda$ where $\lambda$ is the cardinality of the optimised dominating set.

Proof:
Consider a $k$-coloring graph $G$, a minimum dominating set $S \subseteq V$ turns into an optimized dominating set only when $S$ contains maximum number of vertices of chromatic sets. For the time being $S_1$ denotes optimized version of minimum dominating sets. Since $\text{Cut}(G) \neq \emptyset$, there exist at least one vertex $u$ such that $G - u$ is disconnected. It follows that, split $(D, C)$-number coincide with $(D, C)$-number. Therefore, it is enough to prove that $\gamma_{xG}(G) \leq \gamma(G) + \chi(G) - \lambda$.

We claim that $A = S_1 \cup S$ is a $(D, C)$ set where $S$ is the chromatic set such that, $S \not\subseteq S_1$. Since $S_1 \subseteq A$, $A$ is a dominating set. Therefore, it is a chromatic set also. That is, $S_1 \cup A$ is a $(D, C)$ set. Thus

$$A = S_1 \cup S = |A| = |S_1 \cup S|$$

$$= |S_1| + |S| - |S_1 \cap S|$$

$$\leq \gamma(G) + (\chi(G) - \lambda) - 0$$

$$\gamma_{xG}(G) \leq \gamma(G) + \chi(G) - \lambda$$

Hence the result follows.

Theorem 4.3
For a any $k$-coloring of $G$, split $(D, C)$-number

$$\gamma_{xG}(G) = \begin{cases} 
\gamma_x(G) & \text{if } \text{Cut}(G) \neq \emptyset \\
0 & \text{if } \text{Cut}(G) = \emptyset
\end{cases}$$

Proof: Let $C = \{v_1, v_2, \ldots, v_m\}$ be a split $(D, C)$ set of $G$. Then $\gamma_{xG}(G) \leq m$. Since $G$ is $k$-colorable, then $m \geq k$. Since $C$ is a split $(D, C)$-set of $G$, $N[C] = V, C$ is chromatic and $(V - C)$ is disconnected.

Case 1: $\text{Cut}(G) \neq \emptyset$

We have $\gamma_{xG}(G) \leq |C|$. It follows that, $\gamma_{xG}(G) \leq \gamma_x(G)$. But $\gamma_x(G) \leq \gamma_{xG}(G)$ is always true, due to the Theorem 3.1. Hence, $\gamma_{xG}(G) = \gamma_x(G)$.

Case 2: $\text{Cut}(G) = \emptyset$

Since $\text{Cut}(G) = \emptyset$, $(V - C)$ is disconnected. Therefore $\gamma_{xG}(G) = 0$. Hence the result follows.

Theorem 4.4
A tree $T$ has a support adjacent to more than one pendant vertex or $T$ has a non-support if and only if every $(D, C)$ set is also a split $(D, C)$ set, that is,

$$\gamma_x(G) = \gamma_{xG}(G).$$

Proof: Let $S$ be a $(D, C)$ set of $T$ with $\gamma_x(G) = |S|.$

Case 1: $T$ has a support vertex $v$ such that $v$ is adjacent to more than one pendant vertex. Therefore, the support vertex $v$ must belongs to the $(D, C)$ set and $(V - S)$ is disconnected. So, $\gamma_x(G) = \gamma_{xG}(G)$.

Case 2: $T$ has a non-support vertex $w$, then $(D, C)$ set $S$ contains either $w$ or at least one support or one non-support adjacent to $w$. Therefore $(V - S)$ is disconnected. Hence $\gamma_x(G) = \gamma_{xG}(G)$. 

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5. Main Results

Theorem 5.1
If \( G \) is a path graph of order \( n \), then
\[
\gamma_{xs}(G) = \begin{cases} 
\left\lceil \frac{n}{3} \right\rceil, & n \geq 4 \\
0, & \text{otherwise}
\end{cases}
\]

Proof: Let \( G = P_n \) be a path graph with vertices \( \{v_1, v_2, \ldots, v_n\} \). Consider the following cases on different orders \( n \).

Case 1: \( \text{when } n \leq 3 \)
If \( n = 1 \), we cannot find split \((D,C)\)-number from \( G \). So \( \gamma_{xs}(G) = 0 \). If \( n = 2 \), all the vertices of \( G \) are pendant, it means that set of all pendant vertices is itself a chromatic set. But its complement does not induce a disconnected graph. i.e., \( \gamma_{xs}(G) = 0 \). Finally, \( n = 3 \), all the vertices are not pendant. So the chromatic set contain at least one pendant vertex and the remaining are internal vertices. So the \((D,C)\)-number becomes \( 2 \). But its complement does not induce a disconnected graph. Hence, \( \gamma_{xs}(G) = 0 \).

Case 2: \( \text{when } n \geq 4 \)
In this case maximum degree of \( P_n, \Delta(P_n) = 2 \). It shows that any given vertex of \( P_n \) may only dominate at most two vertices. i.e., \( \left\lceil \frac{n}{3} \right\rceil \) becomes the lower bound of the dominating sets. If we starting with second vertex at anyone of the pendant vertex of \( P_n \) and choose the third vertex there after. This shows that the upper bound reaches to \( \left\lceil \frac{n}{3} \right\rceil \) where \( n \equiv 0 \pmod{3} \). Therefore, we can easily upgrade the chromatic set to dominating sets. Hence the split \((D,C)\) Number of \( P_n \) coincide with \( \left\lceil \frac{n}{3} \right\rceil \). That is, \( \gamma_{xs}(G) = \left\lceil \frac{n}{3} \right\rceil \) for \( n \geq 4 \). \[\blacksquare\]

In the case of cycle graph \( G = C_n \) for \( n = 3, 4 \) we cannot find split \((D,C)\) number of \( G \), so we treat it as \( \gamma_{xs}(G) = 0 \). In this case \( n = 5, 6 \) the split \((D,C)\) number coincide with its chromatic number.
\[
\gamma_{xs}(G) = \chi(G) = \begin{cases} 
3 & \text{if } n = 5 \\
2 & \text{if } n = 6
\end{cases}
\]

When \( n \geq 7 \), we can upgrade the chromatic set of \( G \) to its dominating set. Therefore, split \((D,C)\)-number coincide with chromatic number of \( G \).

i.e., \( \gamma_{xs}(G) = \chi(G) = \left\lceil \frac{n}{3} \right\rceil \) for \( n \geq 7 \).

Therefore,

Theorem 5.2
Let \( G = C_n \) be a cycle graph of order \( n \), then
\[
\gamma_{xs}(G) = \begin{cases} 
0 & \text{if } n = 3, 4 \\
3 & \text{if } n = 5 \\
2 & \text{if } n = 6 \\
\left\lceil \frac{n}{3} \right\rceil & \text{if } n \geq 7
\end{cases}
\]

Theorem 5.3
For a complete graph \( G = K_n \), \( \gamma_{xs}(K_n) = 0 \).
Proof: Since $G$ is complete, $V$ is the only chromatic set in $G$. So $V$ is the $(D, C)$ set and $\gamma_x(G) = n$. So split $(D, C)$ set does not exist. That is, $\gamma_{xS}(G) = 0$.

Theorem 5.4
For a complete bipartite graph $G = K_{m,n}$, $\gamma_{xS}(G) = 0$.

Proof: Since $G$ is complete bipartite, we can have two partitions $V_1$ and $V_2$. Let $S = \{u, v\}$ such that $u \in V_1$ and $v \in V_2$, then $S$ is a $(D, C)$ set and $(V - S)$ is connected. Therefore, $\gamma_{xS}(G) = 0$.

The split $(D, C)$ Number of some standard graphs are given in the Table 1.

Table 1: Standard Graphs with split $(D, C)$-number

<table>
<thead>
<tr>
<th>Graphs $G$</th>
<th>$\chi(G)$</th>
<th>$\gamma(G)$</th>
<th>$\gamma_x(G)$</th>
<th>$\gamma_{xS}(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb graph $C_{n,n}$ ($n \geq 3$)</td>
<td>2</td>
<td>$n$</td>
<td>0</td>
<td>$n$</td>
</tr>
<tr>
<td>Star graph $S_n$ ($n \geq 4$)</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Friendship graph $F_n$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Wheel graph $W_n$ ($n = 4$)</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Wheel graph $W_n$ ($n &gt; 4$)</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

6. Realization Result

Theorem 6.1
For any three integers $\lambda, p$, and $q$ with $2 \leq \lambda \leq q \leq p$, there exists a connected graph $G$ such that $\gamma_{xS}(G) = \lambda$ and $|V(G)| = p$.

Proof: Consider a cycle graph $C_q$ with vertex set $V(C_q) = \{x_1, x_2, \ldots, x_q\}$. Join one copy of complete bipartite graph $K_{1,p-q}$ to any one of the $x_i$ in $V(C_q)$ where $i = 1, 2, 3, \ldots, q$, we get a new graph $G$ (See Figure 6.1) with order

$$|V(G)| = |V(C_q)| + |V(K_{1,p-q})|$$

$$= q + p - q = p$$

Figure 6.1: A connected graph $G$ with $\gamma_{xS}(G) = \lambda$ and $|V(G)| = p$. 
If \( q \) is even, the chromatic number of \( C_q \) is always 2 and for odd, it will become 3. In both cases domination number and chromatic number is at most \( q \). i.e., there exist a \((D,C)\) set \( S \) such that \( |S| = \lambda \leq q \) and \( \lambda \geq 2 \). It follows that \((V - S)\) is also disconnected. So the split \((D,C)\) Number, \( \gamma_{S}(G) = \lambda \). Hence the result follows.

7. Conclusion

The split domination chromatic number of a graph is a relatively new parameter in Graph Theory, but it has already been shown to have many applications in different fields, such as network design, facility location, social network analysis, and wireless mesh networks. It is also related to other well-studied parameters in graph theory, such as the domination number and the chromatic number of a graph. This research article has presented a number of new results on the split domination chromatic number of graphs. We have provided bounds on the split domination chromatic number in terms of other graph parameters, such as the domination number, the chromatic number, and the maximum degree. We have also characterized the split domination chromatic number of some special classes of graphs, such as trees, cycles, and complete graphs.

References