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An Operator Defined on Hadamard Product Pertaining to Generalized Hurwitz-Lerch Zeta Function with Conical Section

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Abstract:- The author's goal is to highlight the most recent developments in the research on the study of complex-valued functions as seen through an understanding of geometric function theory. Contributions will be required for any aspect of the Hadamard product associated to the generalized Hurwitz-Lerch Zeta function convoluted with the theory of functions that are univalent. In the present investigation, the author focused into the inclusion relations of a few subclasses of the k-starlike functions, k-uniformly convex functions, and k-quasi-convex functions which together make up the generalized Hurwitz-Lerch Zeta function.

Key words: Starlike function, convex function, k - uniformly starlike functions, k - uniformly convex functions, quasi-convex functions, generalized Hurwitz-Lerch Zeta Function.

1. Introduction

The class of all holomorphic functions f in $\mathcal{A}$, of the method

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

defined on

$$\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}.$$

Let g be given as

$$g(z) = z + \sum_{n=p+1}^{\infty} b_n z^n,$$

their convolution is

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

If $f \in \mathcal{A}$ satisfies

$$\Re(T(Z)-\alpha) \geq k|T(Z)-1| \quad (z \in \mathbb{U})$$

where T(Z)=$\left(\frac{zf'(z)}{f(z)}\right)$ for certain $\alpha(0 \leq \alpha < 1)$ with $k(0 \leq k < \infty)$, then we affirm that f is *k-uniformly starlike* of order α. This category is denoted as $k-\mathcal{ST}(\alpha)$. If $f \in \mathcal{A}$ gratifies

$$\Re(1+M(Z)-\alpha) \geq k|M(Z)| \quad (z \in \mathbb{U})$$

where M(Z)=$\left(\frac{zf''(z)}{f'(z)}\right)$ for certain $\alpha(0 \leq \alpha < 1)$ with $k(0 \leq k < \infty)$, then we affirm that f is k-uniformly convex of order α. This class is denoted as $k-\mathcal{UCV}(\alpha)$. (see also Kharasani and Hijari). While $k=0$ inequalities [1.2] and [1.3] diminish to the already established starlike($\mathcal{S}^*$) and convex($\mathcal{C}$) respectively. While $k=1$, [1.3] leads to the class $\mathcal{UCV}$ proposed by Goodman and studied further by Rønning , Ma and Minda . While $k=1$, [1.2] leads to the class $\mathcal{ST}$ studied by Rønning . Conical sections were putforth by Kanas, Kanas and Wiśniowska –. For $0 \leq k < \infty$ define the domain $\Omega_{k,\alpha}$ as

$$\Omega_{k,\alpha} = \{u+iv : (u-\alpha)^2 > k^2(u-1)^2 + k^2v^2\}.$$

for $0 < k < 1$,

$$\Omega_{k,\alpha} = \left\{u+iv : \left(\frac{u+\frac{k^2-\alpha}{1-k^2}}{k\left(\frac{1-\alpha}{1-k^2}\right)}\right)^2 - \left(\frac{v}{\frac{1-\alpha}{\sqrt{1-k^2}}}\right)^2 > 1\right\},$$

and for $k > 1$,

$$\Omega_{k,\alpha} = \left\{u+iv : \left(\frac{u+\frac{k^2-\alpha}{k^2-1}}{k\left(\frac{1-\alpha}{k^2-1}\right)}\right)^2 + \left(\frac{v}{\frac{1-\alpha}{\sqrt{k^2-1}}}\right)^2 < 1\right\}.$$

The conspicuous representation of the connecting extremal function $\mathbb{U}$ onto $\Omega_{k,\alpha}$ is given by

$$Q_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z} & k=0 \\ 1+\frac{2(1-\alpha)}{\pi^2}\log^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), & k=1 \\ 1+\frac{2(1-\alpha)}{1-k^2}\sinh^2\left(A(k)\operatorname{arctanh}\sqrt{z}\right), & 0<k<1 \\ \frac{(1-\alpha)}{k^2-1}\sin^2\left(\frac{\pi}{2\kappa(t)}\zeta\left(\frac{\sqrt{z}}{\sqrt{t}},t\right)\right)+\frac{k^2-\alpha}{k^2-1} & k>1 \end{cases}$$

where $A(k)=\frac{2}{\pi}\arccos k$, $\zeta(\omega,t)$ is Legendre's elliptic integral

$$\zeta(\omega,t) = \int_0^{\omega} \frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}, \quad \kappa(t) = \zeta(1,t)$$

and $t \in (0,1)$ is selected that $k=\cosh\frac{\pi\kappa'(t)}{4\kappa(t)}$, maps $\mathbb{U}$ onto the conic domain. The image of $\mathbb{U}$ under $Q_{k,\alpha}(z)$ for various values of α is given by the figures 1-3 By feature of

$$p(z) = T(z) \prec Q_{k,\alpha}(z) \text{ and } p(z) = 1+M(z) \prec Q_{k,\alpha}(z)$$

By the characteristics of domains, we claim

$$\Re(p(z)) > \Re\left(Q_{k,\alpha}(z)\right) > \frac{k+\alpha}{k+1}.$$

Express $\mathcal{UCC}(k,\alpha,\beta)$, let $f\in\mathcal{A}$ that satisfies

$$\eta(z)\prec Q_{k,\alpha}(z)\quad \text{for certain}\quad g\in k-\mathcal{ST}(\alpha).$$

In the same way , we define $\mathcal{UQC}(k,\alpha,\beta)$, let $f\in\mathcal{A}$ that satisfies

$$\eta'(z)\prec Q_{k,\alpha}(z),\quad \text{for certain}\quad g\in k-\mathcal{UCV}(\alpha).$$

The category of close-to-convex and quasi-convex univalent functions of order α and type β are $\mathcal{UCC}(0,\alpha,\beta)$ and $\mathcal{UQC}(0,\alpha,\beta)$ respectively.

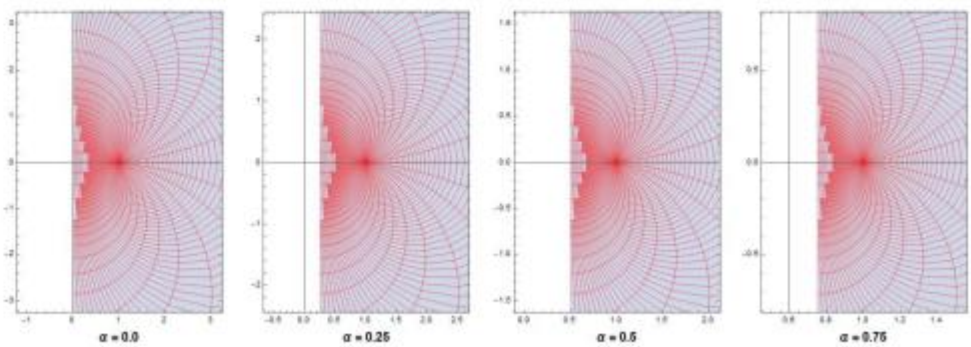

FIGURE 1. The image of $\mathbb{U}$ under $Q_{k,\alpha}(z)$ for $k=0$ with various values of α

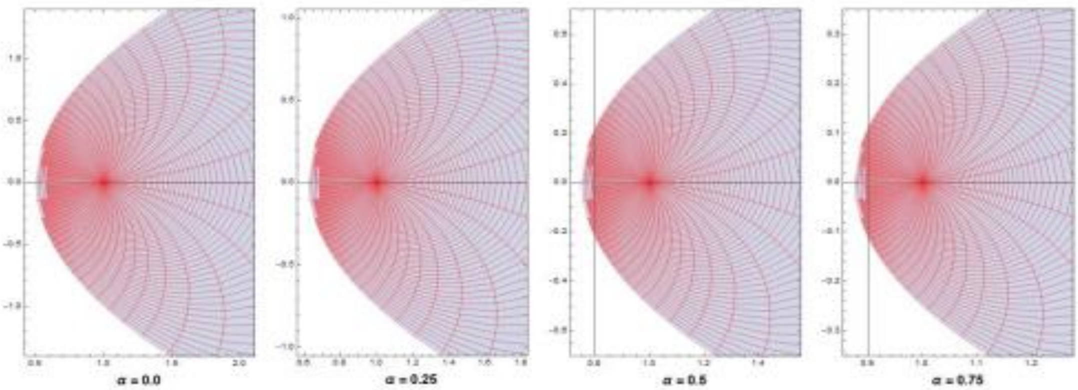

FIGURE 2. The image of $\mathbb{U}$ under $Q_{k,\alpha}(z)$ for $k=1$ with various values of α

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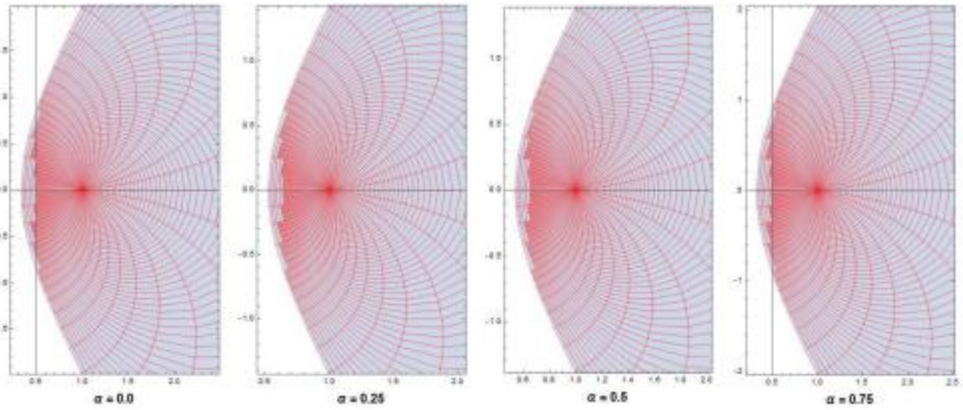

FIGURE 3. The image of $\mathbb{U}$ under $Q_{k,\alpha}(z)$ for $k = 0.5$ with various values of α

2 Preliminaries

Now we study about some subclasses of Generalized Hurwitz-Lerch Zeta Function considered by Mohammed and Darus.

Denote by $D^{\lambda}: \mathcal{A} \to \mathcal{A}$ the operator labelled as

$$D^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \quad \lambda > -1.$$

More precisely, $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$ and,

$$D^m f(z) = \frac{z\left(z^{m-1} f(z)\right)^{(m)}}{m!}, \quad m \in N_0 = N \cup 0.$$

we spot that

$$D^m f(z) = z + \sum_{k=2}^{\infty} C\,(m,k) a_k z^k$$

where $C(m,k) = \frac{(k+m-1)!}{(m!)(k-1)!}$.
$D^n f$ is termed as the n-th order Ruscheweyh.
$I_n: \mathcal{A} \to \mathcal{A}$ the operator is labelled as:

$$f_n(z) * f_n^{-1}(z) = \frac{z}{1-z}, \quad \text{where} \quad f_n(z) = \frac{z}{(1-z)^{n+1}}, \quad n \in N_0.$$

Then,

$$f_n^{-1}(z) = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)}$$

$$I_n f(z) = z + \sum_{k=2}^{\infty} \frac{n!\,k!}{(k+n-1)!} a_k z^k$$

Also $I_0 f(z) = f(z)$ and $I_1 f(z) = zf'(z)$. I_n is termed as Noor integral operator defined and studied by Noor and Noor.

For $f \in \mathcal{A}$, Salageanfamiliarized the following operator

$$D^n f(z) = f(z) * \left(z + \sum_{k=2}^{\infty} k^n z^k \right), \quad m \in N_0 = N \cup 0.$$

Note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$.

Latterly, Shaqsi and Darus putforththe linear operator

$$D_\lambda^n f(z) = \left(G(n,z)\right)^{(-1)} * f(z),$$

$G(n,z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ and

$$\sum_{k=1}^{\infty} \frac{z^k}{k^n} * \left(G(n,z)\right)^{(-1)} = \frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1} \quad \lambda > -1.$$

where

$$\left(G(n,z)\right)^{(-1)} = \sum_{k=1}^{\infty} k^n \frac{(k+\lambda-1)!}{\lambda!\,(k-1)!} a_k z^k.$$

Hence

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} k^n \frac{(k+\lambda-1)!}{\lambda!\,(k-1)!} a_k z^k, \quad n, \lambda \in N_0.$$

Now, we consider

$$\phi_\mu(z,s,\sigma) = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \frac{z^k}{(k+\sigma)^\mu}, z \in \mathbb{C}, |z| < 1, \sigma \in \mathbb{C}\,\{0,-1,-2,\dots\}, \mu, s \in \mathbb{C},$$

the generalized Hurwitz-Lerch zeta function, putforth by Goyal and Laddha.

$$(\mu)_k = \frac{\Gamma(\mu+k)}{\Gamma(\mu)} = \mu(\mu+1)\dots(\mu+k-1) \quad \text{for} k = 1,2,3,\dots \quad \mu \in R \quad (\mu)_0 = 1$$

Apparently, the distinct cases of Hurwitz-Lerch zeta function were investigated by numerous authors like Lin and Srivastava and Kanemitsu et al..

when $\sigma = 1$, the generalized Hurwitz-Lerch zeta function diminishes to

$$z\phi_\mu(z,s,1) = \sum_{k=1}^{\infty} \frac{(\mu)_{k-1} z^k}{(k-1)!\, k^s}$$

Now introduced a function $\left[z\phi_\mu(z,s,1)\right]^{(-1)}$ given by:

$$\left[z\phi_\mu(z,s,1)\right] * \left[z\phi_\mu(z,s,1)\right]^{(-1)} = \frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1},$$

and gained a linear operator:

$$\theta_\mu^{\lambda,s} f(z) = \left[z\phi_\mu(z,s,1)\right]^{(-1)} * f(z)$$

From [2.2] we attained

$$^{(-1)} = \sum_{k=1}^{\infty} \frac{(\lambda+1)_{k-1}}{(\mu)_{k-1}} k^s z^k,$$

For $s, \lambda \in \mathbb{N}_0$ and $\mu \in \mathbb{N}$, we observe

$$\theta_\mu^{\lambda,s} f(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda+1)_{k-1}}{(\mu)_{k-1}} k^s a_k z^k,$$

putforth by Mohammed and Darus or

$$\theta_\mu^{\lambda,s} f(z) = z + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)(\mu-1)!}{\lambda!\,(k+\mu-2)!} k^s a_k z^k,$$

which is analogous to:

$$\theta_\mu^{\lambda,s} f(z) = z + \sum_{k=2}^{\infty} \frac{C(\lambda,k)}{\delta(\mu,k)} k^s a_k z^k,$$

where

$$C(\lambda,k) = \frac{(1+\lambda)_{k-1}}{(\lambda)!} \quad and \quad \delta(\mu,k) = \frac{(\mu)_{k-1}}{(\mu-1)!}.$$

We observe that

1. Ruscheweyh introduced the derivative operator $\theta_1^{\lambda,0} f(z)$,
2. Salagean introduced the derivative operator $\theta_1^{1,s} f(z)$
3. Noor and Noor introduced the integral operator $\theta_{\mu+1}^{0,0} f(z)$
4. Shaqsi and Darus introduced $\theta_1^{k,s} f(z)$

More precisely, $\theta_1^{0,0} f(z) = f(z)$ and $\theta_1^{0,1} f(z) = zf'(z)$.
In view of [1.1] and [2.4] we obtain:

$$z\left(\theta_\mu^{\lambda,s} f(z)\right)' = (\lambda+1)\theta_\mu^{\lambda+1,s} f(z) - \lambda\theta_\mu^{\lambda,s} f(z)$$

and

$$z\left(\theta_\mu^{\lambda,s} f(z)\right)' = \mu\theta_\mu^{\lambda,s} f(z) - (\mu-1)\theta_{\mu+1}^{\lambda,s} f(z).$$

The relation [2.9] play vital role in obtaining our results.

Lemma 1. *Raghib Nadeem Let* h *be convex holomorphic function in* $\mathbb{U}$ *and* $q(0) = 1$, $\Re(\nu q(z) + \mu) > 0$ $(\nu, \mu \in \mathbb{C})$. *If* p *isholomorphic in* $\mathbb{U}$ *and* $p(0) = 1$ *Hence* $p(z) + \frac{zp\prime(z)}{\nu p(z)+\mu} \prec q(z)$, $(z \in \mathbb{U}) \Rightarrow p(z) \prec q(z)$ $(z \in \mathbb{U})$.

Lemma 2. *S.S. Miller and P.T. Mocanu Let q be convex in $\mathbb{U}$ and $E \geq 0$. Suppose B is holomorphic in $\mathbb{U}$ and $\Re(B(z)) > 0$. If g is holomorphic in $\mathbb{U}$ with $q(0) = g(0)$. Hence $Ez^2g''(z) + B(z)g(z) \prec q(z) \Rightarrow g(z) \prec q(z)$.*

Eventually, we recollect the Bernardi-Libera-Livingston integral operator given by

$$L_\gamma(f(z)) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f(t)dt, \quad \gamma > -1.$$

3 Main Results

Theorem 1. *Let $f \in \mathcal{A}$. If $\theta_\mu^{\lambda,s} f(z) \in k - \mathcal{ST}(\alpha)$. Then $\theta_\mu^{\lambda+1,s} f(z) \in k - \mathcal{ST}(\alpha)$.*

Proof. Let

$$s(z) = z\frac{\left(\theta_\mu^{\lambda,s} f(z)\right)'}{\theta_\mu^{\lambda,s} f(z)} \quad (z \in \mathbb{U})$$

where s is holomorphic in $\mathbb{U}$ and $s(0) = 1$. Utilizind [2.9], the following is obtained

$$s(z) + \lambda = (\lambda+1)\frac{\theta_\mu^{\lambda+1,s} f(z)}{\theta_\mu^{\lambda,s} f(z)}$$

Differentiating both sode logarithmically w.r.to z and multiplying with z, we attain

$$s(z) + \frac{zs'(z)}{s(z)+\lambda} = \frac{z\left(\theta_\mu^{\lambda+1,s} f(z)\right)'}{\theta_\mu^{\lambda+1,s} f(z)}.$$

From this argument, we affirm

$$s(z) + \frac{zs'(z)}{s(z)+\lambda} \prec Q_{k,\alpha}(z).$$

Using Lemma 1 and [1.3a], $Q_{k,\alpha}(z)$ is injevtive holomorphic function on an open subset of the complex plane and convex in $\mathbb{U}$, also $\Re\left(Q_{k,\alpha}(z)\right) > \frac{k+\alpha}{k+1}$. □

Theorem 2. *Suppose $f \in \mathcal{A}$. If $\theta_\mu^{\lambda,s} f(z) \in k - \mathcal{UCV}(\alpha)$, then $\theta_\mu^{\lambda+1,s} f(z) \in k - \mathcal{UCV}(\alpha)$.*

Proof. From equations [1.2] and [1.3] and the Theorem 1 we attain

$$\begin{aligned}
\theta_\mu^{\lambda,s} f(z) \in k - \mathcal{UCV}(\alpha) &\Leftrightarrow \quad z\left(\theta_\mu^{\lambda,s} f(z)\right)' \in k - \mathcal{ST}(\alpha) \\
&\Leftrightarrow \quad \theta_\mu^{\lambda,s} zf'(z) \in k - \mathcal{ST}(\alpha) \\
&\Rightarrow \quad \theta_\mu^{\lambda+1,s} zf'(z) \in k - \mathcal{ST}(\alpha) \\
&\Leftrightarrow \quad \theta_\mu^{\lambda+1,s} f(z) \in k - \mathcal{UCV}(\alpha)
\end{aligned}$$

□

Theorem 3. *Suppose* $f \in \mathcal{A}$. *If* $\theta_\mu^{\lambda,s} f \in \mathcal{UCC}(k,\alpha,\beta)$, *then* $\theta_\mu^{\lambda+1,s} f \in \mathcal{UCC}(k,\alpha,\beta)$.

Proof. Given

$$\theta_\mu^{\lambda,s} f(z) \in \mathcal{UCC}(k,\alpha,\beta)$$

$$\frac{z\left(\theta_\mu^{\lambda,s} f(z)\right)'}{k(z)} \prec Q_{k,\alpha}(z), for \quad certain \quad k(z) \in k-\mathcal{ST}(\beta).$$

For $g(z)$, $\theta_\mu^{\lambda,s} g(z) = k(z)$ we attain

$$\frac{z\left(\theta_\mu^{\lambda,s} f(z)\right)'}{\theta_\mu^{\lambda,s} g(z)} \prec Q_{k,\alpha}(z).$$

Letting

$$h(z) = \frac{z\left(\theta_\mu^{\lambda+1,s} f(z)\right)'}{\theta_\mu^{\lambda+1,s} g(z)} \quad and \quad H(z) = \frac{z\left(\theta_\mu^{\lambda+1,s} g(z)\right)'}{\theta_\mu^{\lambda+1,s} g(z)},$$

Hence h, H are holomorphic in $\mathbb{U}$ with $h(0) = H(0) = 1$.
Using Theorem 1,

$$\theta_\mu^{\lambda+1,s} g(z) \in k-\mathcal{ST}(\beta) \quad with \quad \Re\big(H(z)\big) > \frac{k+\beta}{k+1}.$$

Also

$$z\left(\theta_\mu^{\lambda+1,s} f(z)\right)' = \left(\theta_\mu^{\lambda+1,s} g(z)\right) h(z)$$

Differentiating [neweq] on both sides w.r.to z, we attain

$$\frac{z\left(z\left(\theta_\mu^{\lambda+1,s} f(z)\right)'\right)'}{\theta_\mu^{\lambda+1,s} g(z)} = \frac{z\left(\theta_\mu^{\lambda+1,s} g(z)\right)'}{\theta_\mu^{\lambda+1,s} g(z)} h(z) + zh'(z) = H(z).h(z) + zh'(z).$$

Using [2.9], we attain

$$\begin{aligned}
\frac{z\left(\theta_\mu^{\lambda,s}\right)'}{\theta_\mu^{\lambda,s} g(z)} &= \frac{\theta_\mu^{\lambda,s}\left(zf'(z)\right)}{\theta_\mu^{\lambda,s} g(z)} \\
&= \frac{z\left(\theta_\mu^{\lambda+1,s} zf'(z)\right)' + \lambda\theta_\mu^{\lambda+1,s}\left(zf'(z)\right)'}{z\left(\theta_\mu^{\lambda+1,s} g(z)\right)' + \lambda\theta_\mu^{\lambda+1,s} g(z)} \\
&= \frac{\dfrac{z\left(\theta_\mu^{\lambda+1,s} zf'(z)\right)'}{\theta_\mu^{\lambda+1,s} g(z)} + \lambda\dfrac{\theta_\mu^{\lambda+1,s}\left(zf(z)\right)'}{\theta_\mu^{\lambda+1,s} g(z)}}{\dfrac{z\left(\theta_\mu^{\lambda+1,s} g(z)\right)'}{\theta_\mu^{\lambda+1,s} g(z)} + \lambda} \\
&= \frac{H(z)h(z) + zh'(z) + \lambda h(z)}{H(z) + \lambda} \\
&= h(z) + \frac{zh'(z)}{H(z) + \lambda}.
\end{aligned}$$

Using [3.4], [3.5] and above equation, we conclude

$$h(z) + \frac{zh'(z)}{H(z)+\lambda} \prec Q_{k,\alpha}(z)$$

When $E = 0$ with $B(z) = \frac{1}{H(z)+\lambda}$, we attain

$$\Re\big(B(z)\big) = \frac{\Re(H(z)+\lambda)}{|H(z)+\lambda|^2} > 0$$

the above mentioned inequality well pleased the constraints prescribed in Lemma 1. Therefore

$$h(z) \prec Q_{k,\alpha}(z)$$

. □

Using analogous argument in Theorem 3, we can verify the succeding theorems.

Theorem 4. *Suppose* $f \in \mathcal{A}$, *if* $\theta_\mu^{\lambda,s} f \in \mathcal{UQC}(k,\alpha,\beta)$, *then* $\theta_\mu^{\lambda,s} f(z) \in \mathcal{UQC}(k,\alpha,\beta)$.

Theorem 5. *Suppose* $\gamma > -\frac{k+\alpha}{k+1}$, *if* $\theta_\mu^{\lambda,s} f \in k - \mathcal{UCV}(\alpha)$ *so is* $\theta_\mu^{\lambda,s} L_\gamma\big(f(z)\big)$.

Theorem 6. *Suppose* $\gamma > -\frac{k+\alpha}{k+1}$, *if* $\theta_\mu^{\lambda+1,s} f \in k - \mathcal{UCC}(\alpha,\beta)$ *so is* $\theta_\mu^{\lambda,s} L_\gamma\big(f(z)\big)$.

Proof. By the definition, we have

$$K(z) = \theta_\mu^{\lambda,s} g(z) \in k - \mathcal{ST}(\beta)$$

Hence

$$\frac{z\left(\theta_\mu^{\lambda+1,s}\big(f(z)\big)\right)'}{\theta_\mu^{\lambda+1,s}\big(g(z)\big)} \prec Q_{k,\alpha}(z)(z \in \mathbb{U}).$$

Now from [2.8] we have

$$\begin{aligned}\frac{z\big(\theta_\mu^{\lambda+1,s} f\big)'}{\theta_\mu^{\lambda+1,s}\big(g(z)\big)} &= \frac{z\left(\theta_\mu^{\lambda+1,s} L_\gamma(zf')\right)' + \gamma\theta_\mu^{\lambda+1,s} L_\gamma\big(zf'(z)\big)}{z\left(\theta_\mu^{\lambda+1,s} L_\gamma\big(g(z)\big)\right)' + \lambda\theta_\mu^{\lambda+1,s} L_\gamma\big(g(z)\big)} \\ &= \frac{\dfrac{z\left(\theta_\mu^{\lambda+1,s}\big(zf'(z)\big)\right)'}{\theta_\mu^{\lambda+1,s} L_\gamma\big(g(z)\big)} + \dfrac{\gamma\theta_\mu^{\lambda+1,s}\big(zf'(z)\big)}{\theta_\mu^{\lambda+1,s} L_\gamma\big(g(z)\big)}}{\dfrac{z\left(\theta_\mu^{\lambda+1,s} L_\gamma\big(g(z)\big)\right)'}{\theta_\mu^{\lambda+1,s} L_\gamma\big(g(z)\big)} + \gamma}.\end{aligned}$$

Since $\theta_\mu^{\lambda+1,s} g \in k - \mathcal{ST}(\beta)$, by Theorem 4, we have $L_\gamma\big(\theta_\mu^{\lambda+1,s} g\big) \in k - \mathcal{ST}(\alpha)$. Taking

$$\frac{z\left(\theta_\mu^{\lambda+1,s} L_\gamma\big(g(z)\big)\right)'}{\theta_\mu^{\lambda+1,s} L_\gamma(g)} = H(z)$$

We observe $\Re\big(H(z)\big) > \frac{k+\beta}{k+1}$. Also

$$h(z) = \frac{z\left(\theta_\mu^{\lambda+1,s} L_\gamma\big(f(z)\big)\right)'}{\theta_\mu^{\lambda+1,s} L_\gamma\big(g(z)\big)}$$

we obtain

$$z\left(\theta_{\mu}^{\lambda+1,s}L_{\gamma}(f(z))\right)' = h(z)\theta_{\mu}^{\lambda+1,s}L_{\gamma}(g(z)).$$

Differentiating [eq3.18] both sides w.r.to z, we attain

$$\frac{z\left(\theta_{\mu}^{\lambda+1,s}\left(zL_{\gamma}(f)\right)'\right)'}{\theta_{\mu}^{\lambda+1,s}L_{\gamma}(g)} = zh'(z) + h(z)\frac{z\left(\theta_{\mu}^{\lambda+1,s}L_{\gamma}(g)\right)'}{\theta_{\mu}^{\lambda+1,s}L_{\gamma}(g)} = zh'(z) + H(z)h(z).$$

Using [eq3.17] and [eq3.19], we attain

$$\frac{z\left(\theta_{\mu}^{\lambda+1,s}f(z)\right)'}{\theta_{\mu}^{\lambda+1,s}g} = \frac{zh'(z) + H(z)h(z) + \gamma h(z)}{H(z)+\gamma}$$

Also using [eq3.16], we attain

$$h(z) + \frac{zh'(z)}{H(z)+\gamma} \prec Q(k,\alpha)(z).$$

We proceed $B(z) = \frac{1}{H(z)+\gamma}$ in [eq3.21] and observing that $\Re(B(z)) > 0$ with $\gamma > -\frac{k+\alpha}{k+1}$. Now for $A = 0$ and B Also the suitable conditions of Lemma 2 are satisfied, the proof is concluded. □

An analogous argument leads to the following theorem

Theorem 7. *Let* $\gamma > -\frac{k+\alpha}{k+1}$. *If* $\theta_{\mu}^{\lambda+1,s}f(z) \in \mathcal{UQC}(k,\alpha,\beta)$ *so is* $\theta_{\mu}^{\lambda,s}L_{\gamma}(f(z))$.

References

[1] H. A. Al-Kharsani and S. S. Al-Hajiry, Subordination results for the family of uniformly convex p-valent functions, J. Inequal. Pure Appl. Math. **7** (2006), no. 1, Article 20, 11 pp. (electronic).

[2] P. Eenigenburg et al., On a Briot Bouquet differential subordination, in *General inequalities, 3 (Oberwolfach, 1981)*, 339–348, Internat. Schriftenreihe Numer. Math., 64, Birkhäuser, Basel.

[3] A. W. Goodman, On uniformly starlike functions, J. Math. Anal. Appl. **155** (1991), no. 2, 364–370.

[4] S.P. Goyal and R. K. Laddha, *On the generalized Riemann zeta functions and the generalized Lambert transform*, Ganita Sandesh 11: 99-108.

[5] S. Kanas, Alternative characterization of the class $k - UCV$ and related classes of univalent functions, Serdia Math. J., **25** (1999), 341–350.

[6] S. Kanas, Techniques of the differential subordination for domains bounded by conic sections, Internat. J. Math. Math. Sci., **38** (2003), 2389–2400.

[7] S. Kanas, Differential subordination related to conic sections, J. Math. Anal. Appl., **317** (2006), 650–658.

[8] S. Kanas, Subordination for domains bounded by conic sections, Bull. Belg. Math. Soc. Simon Stevin, **15** (2008), 589–598.

[9] S. Kanas, Norm of pre-Schwarzian derivative for the class of k-uniform convex and k-starlike functions, Appl. Math. Comput, **215** (2009), 2275–2282.

[10] S. Kanas and H. M. Srivastava, Linear operators associated with k-uniform convex functions, Integral Transforms Spec. Function., **9** (2000), 121–132.

[11] S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, II, Zeszyty Nauk.Politech. Rzeszowskiej Mat., **22** (1998), 65–78.

[12] S. Kanas and A. Wiśniowska, Conic regions and k-uniform convexity, J. Comput. Appl. Math. **105** (1999), no. 1-2, 327–336.

[13] S. Kanas and A. Wiśniowska, Conic regions and k-starlike function, Rev. Roumania Math. Pures Appl., **45** (2000), 647–657.

[14] S. Kanemitsu, M. Katsurada and M. Yoshimoto, *On the Hurwitz-Lerch zeta-function*, Aequationes Mathematics 59: 1-19.

[15] F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), 8–12.

[16] S. Lin and H. M. Srivastava, *Some families of the Hurwitz-Lerch zeta functions and associated fractional derivative and other integral representations*, Appl. Math. and Comp. 154: 725-733.

[17] S. S. Miller and P. T. Mocanu, Differential subordinations and inequalities in the complex plane, J. differential equations **67** (1978), 199-211.

[18] A.Mohammed and M.Darus, *An Operator defined by convolution involving the generalised Hurwitz-Lerch Zeta function*, Sains Malaysiana 41(12)(2012): 1657-1661.

[19] K. I. Noor, *On new classes of integral operators*,Jour. of Nat. Geo. 16: 71-80.

[20] K. I. Noor and M. A. Noor, *On integral operators*, Jour. of Math. Anal. and Appl. 238: 341-352.

[21] D. V. Prokhorov and J. Szynal, *Inverse coefficients for (α,β)-convex functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **35** (1981), 125–143 (1984).

[22] S.D. Purohit and R.K. Raina, Certain subclasses of analytic functions associated with fractional q-calculus operators, Mathematica Scandinavica, 109 1 (2001), pp. 55-77.

[23] Raghib Nadeem, Analytical properties of the Hurwitz–Lerch zeta function, Advances in Difference Equations, (2020) 2020:466, 1-15.

[24] F. Rønning, A survey on uniformly convex and uniformly starlike functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A **47** (1993), 123–134.

[25] S. Ruscheweyh, *New criteria for univalent functions*, Proc of Amer. Math. Soc. 49: 109-115.

[26] W. C. Ma and D. Minda, Uniformly convex functions, Ann. Polon. Math. **57** (1992), no. 2, 165–175.

[27] G. S.Salagean,*Subclasses of univalent functions* Lecture Notes in Math. 1013: 262-372.

[28] K. Al-Shaqsi and M. Darus, *An operator defined by convolution involving the polylogarithms functions*, Jour. of Math. and Stat. 4: 46-50.