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Lieb's Inequality for Continuous Modulated Shearlet Transform on LCA Groups

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Abstract

We have proved a version of Lieb's inequality for continuous modulated shearlet transform on locally compact Abelian (LCA) groups. An alternative proof for the inequality has also been provided using Riesz-Thorin interpolation theorem.

Keywords:*Lieb's Inequality, Fourier transform, Gabor transform,Shearlet transform,Wavelet transform, Continuous modulated shearlet transform.*

1. Introduction

Lieb's uncertainty inequality, also known as the Lieb's bound or the Lieb inequality, is a fundamental result in quantum mechanics that establishes a fundamental limit on the uncertainty of certain pairs of observables. It was first derived by Elliott Lieb in 1973.

In quantum mechanics, the uncertainty principle, as formulated by Werner Heisenberg, states that the more precisely we try to measure certain pairs of complementary observables, such as position and momentum, the more uncertain their values become. However, Lieb's uncertainty inequality goes beyond thestandard Heisenberg uncertainty principle by providing a quantitative bound on the uncertainty relation between certain observables.

Gröchenig in 1998 proved the following version of Lieb's inequality for short-time Fourier transform on locally compact abelian groups (see [**3**, Theorem 6.3.1]).

Theorem 1.1: For$f, \psi \in L^2(G)$,where$\psi$is a window function and$2 \leq q \leq \infty$,

$$\left\|G_\psi f\right\|_{L^q(G\times\hat{G})} \leq \|\psi\|_{L^2(G)}\|f\|_{L^2(G)}.$$

Lieb's inequality has also been established for continuous quaternion wavelet transform in [**7**] and for continuous spherical Gabor transform in [**2**].

In section 2, we recall continuous modulated Shearlet transform and some of its results. Section 3 contains the formulation of main results of the paper including a version of Lieb's inequality.

2. Continuous Modulated Shearlet Transform

Consider G to be a second countable, unimodular locally compact group of type I.Let μ_G be the left Haar measure on G and $\mu_{\hat{G}}$ be the Plancherel measure on the dual space $\hat{G}$. For $f \in L^1(G)$, the Fourier transform $\hat{f}$ is defined as the operator

$$\hat{f}(\pi) = \int_G f(x)\,\pi(x)^*\,d\mu_G(x).$$

The continuous modulated shearlet transform has been introduced in [**1**]. We briefly recall the notations. Let H be a second countable, locally compact Abelian group with Haar measure μ_H. The group of automorphisms of H be denoted by $\mathrm{Aut}(H)$. Let μ_L be the left Haar measure on a locally compact group L.Suppose that $\lambda : L \to \mathrm{Aut}(H)$ be a homomorphism $l \mapsto \lambda_l$ satisfying the property that the mapping from $L \times H$ onto H given by$(l,h) \mapsto \lambda_l(h)$ iscontinuous. The group $D = L \times_\lambda H$, which is the semi-direct product of L and H, is a locally compact group with the group operation

$$(l,h)(l',h') = (ll', h\lambda_l(h')).$$

By [**4**, (15.29)], the left Haar measure on $D$is given by

$$d\mu_D(l,h) = \delta_\lambda(l)\,d\mu_L(l)\,d\mu_H(h),$$

Here, δ_λ is a positive-continuous homomorphism on Lsatisfying

$$d\mu_H(h) = \delta_\lambda(l)\,d\mu_H\big(\lambda_l(h)\big).$$

Also, the left Haar measure on the locally compact group $\mathcal{S} = D \times G$is given by

$$d\mu_{\mathcal{S}}(l,h,x) = \delta_\lambda(l)\,d\mu_L(l)\,d\mu_H(h)\,d\mu_G(x).$$

For each $\psi \in L^2(H \times G)$and $(l,h,x) \in \mathcal{S}$, we define$\mathcal{U}^{\psi}_{(l,h,x)} : H \times G \to \mathbb{C}$ by

$$\mathcal{U}^{\psi}_{(l,h,x)}(k,y) = \delta_\lambda^{1/2}(l)\,\psi(\lambda_{l^{-1}}(h^{-1}k), x^{-1}y)$$

for all $(k,y) \in H \times G$.

From [**1**, Proposition 2.2], it is clear that$\mathcal{U}^{\psi}_{(l,h,x)} \in L^2(H \times G)$ and

$$\left\|\mathcal{U}^{\psi}_{(l,h,x)}\right\|_{L^2(H\times G)} = \|\psi\|_{L^2(H\times G)}. \tag{2.1}$$

Definition 2.1: A function $\psi \in L^2(H \times G)$ is called*admissible* if

$$C_\psi = \int_{L\times G} \left|\mathcal{F}_H\tilde{\psi}(\eta \circ \lambda_l, x)\right|^2 d\mu_{L\times G}(l,x) < \infty,$$

which is independent of almost every $\eta \in \widehat{H}$. Here $\mathcal{F}_H$ denotes the Fourier transform on H and

$$\tilde{\psi}(k,y) = \overline{\psi(k^{-1}, y^{-1})}.$$

Let $C_c(H \times G)$ denote the set of all continuous, complex-valued functions on $H \times G$having compact supports.

Definition 2.2: Let $f \in C_c(H \times G)$ andsuppose$\psi \in L^2(H \times G)$ be admissible. Then, the measurable field of operators on $\mathcal{S} \times \widehat{G}$defined by

$$\mathcal{MS}_\psi f(l,h,x,\pi) = \int_{H\times G} f(k,y)\overline{\mathcal{U}^{\psi}_{(l,h,x)}(k,y)}\,\pi(y)^*\,d\mu_{H\times G}(k,y)$$

is called *continuous modulated shearlet transform*(CMST) of $f$with respect to ψ.

By [**1**, Proposition 2.11], we have the following:

Proposition 2.3: Let $\psi \in L^2(H \times G)$ be an admissible function. Then, the linear operator

$$\mathcal{MS}_\psi : C_c(H \times G) \to \mathcal{H}^2\big(\mathcal{S} \times \widehat{G}\big)$$

given by $f \mapsto \mathcal{MS}_\psi f$ satisfies

$$\left\|\mathcal{MS}_{\psi} f\right\|_{\mathcal{H}^2(\mathcal{S}\times\hat{G})} = C_{\psi}^{1/2}\|f\|_{L^2(H\times G)}.$$

The above equality shows that $\mathcal{MS}_{\psi} : C_c(H\times G) \to \mathcal{H}^2(\mathcal{S}\times\hat{G})$ defined by $f \mapsto \mathcal{MS}_{\psi} f$ is a multiple of an isometry. So, we can extend $\mathcal{MS}_{\psi}$uniquely to a bounded linear operator from $L^2(H\times G)$ into a closed subspace N of $\mathcal{H}^2(\mathcal{S}\times\hat{G})$ which we still denote by $\mathcal{MS}_{\psi}$ and this extension satisfies

$$\left\|\mathcal{MS}_{\psi} f\right\|_{\mathcal{H}^2(\mathcal{S}\times\hat{G})} = C_{\psi}^{1/2}\|f\|_{L^2(H\times G)},$$

for each $f \in L^2(H\times G)$.

Throughout this paper, we consider G to be an Abelian group. In that case $\mathcal{MS}_{\psi} f \in L^2(\mathcal{S}\times\hat{G})$ and it satisfies

$$\left\|\mathcal{MS}_{\psi} f\right\|_{L^2(\mathcal{S}\times\hat{G})} = C_{\psi}^{1/2}\|f\|_{L^2(H\times G)}. \qquad (2.2)$$

Gabor transform, wavelet transform and shearlet transform may be obtained from CMST, for details see [**1**, Section 4].

3. Main Results

Before proving the main results, we shall first state Riesz-Thorin interpolation theorem. For more details, one may refer to [**6**, Page 52].

Theorem 3.1 (Riesz-Thorin interpolation theorem): Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and T be a bounded linear operator from $L^{p_0}(X,A,\mu)$to $L^{q_0}(Y,B,\nu)$ with norm M_0andfrom $L^{p_1}(X,A,\mu)$to $L^{q_1}(Y,B,\nu)$ with norm M_1. Then T is bounded operator from $L^{p_\theta}(X,A,\mu)$to $L^{q_\theta}(Y,B,\nu)$ with norm M_θ such that

$$M_\theta \le M_0^{1-\theta} M_1^{\theta}$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ \ \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},\ \ \theta \in (0,1).$$

We shall now prove the first main result of the paper.

Theorem 3.2: Let $f \in L^2(H\times G)$ and $\psi \in L^2(H\times G)$ be admissible function. For $2 \le q < \infty$,

$$\left\|\mathcal{MS}_{\psi} f\right\|_{L^q(\mathcal{S}\times\hat{G})} \le C_{\psi}^{\frac{1}{q}} \|\psi\|_{L^2(H\times G)}^{1-\frac{2}{q}} \|f\|_{L^2(H\times G)}.$$

Proof: $\mathcal{MS}_{\psi}$ is bounded from $L^2(H\times G)$ to $L^2(\mathcal{S}\times\hat{G})$ such that

$$\left\|\mathcal{MS}_{\psi} f\right\|_{L^2(\mathcal{S}\times\hat{G})} = C_{\psi}^{1/2}\|f\|_{L^2(H\times G)}$$

and

$$\left\|\mathcal{MS}_{\psi} f(l,h,x,\gamma)\right\| \le \|\psi\|_{L^2(H\times G)}\|f\|_{L^2(H\times G)}$$

for each $(l,h,x,\gamma) \in \mathcal{S}\times\hat{G}$.

It implies that $\mathcal{MS}_{\psi}$ is bounded from $L^2(H\times G)$ to $L^\infty(\mathcal{S}\times\hat{G})$ such that

$$\left\|\mathcal{MS}_{\psi} f\right\|_{L^\infty(\mathcal{S}\times\hat{G})} \le \|\psi\|_{L^2(H\times G)}\|f\|_{L^2(H\times G)}.$$

Applying Theorem 3.1 for $p_0 = 2$, $q_0 = 2$, $p_1 = 2$, $q_1 = \infty$, $p_\theta = 2$ and $q_\theta = q$, we obtain $\theta = 1 - \frac{2}{q}$ and $\mathcal{MS}_{\psi}$ as a bounded operator from $L^2(H\times G)$ to $L^q(\mathcal{S}\times\hat{G})$ such that

$$\left\|\mathcal{MS}_{\psi} f\right\|_{L^q(\mathcal{S}\times\hat{G})} \leq C_{\psi}^{\frac{1}{q}} \|\psi\|_{L^2(H\times G)}^{1-\frac{2}{q}} \|f\|_{L^2(H\times G)}.$$,

Corollary 3.3: Let $f,\psi \in L^2(H)$ with ψ an admissible function. For $2 \leq q \leq \infty$,

$$\left\|\mathcal{W}_{\psi} f\right\|_{L^q(L\times_\lambda H)} \leq C_{\psi}^{\frac{1}{q}} \|\psi\|_{L^2(H)}^{1-\frac{2}{q}} \|f\|_{L^2(H)}.$$

Theorem 3.4 (Lieb's Inequality): Let $f_1, f_2, \psi_1, \psi_2 \in L^2(H\times G)$ with ψ_1, ψ_2 as admissible functions. For $1 \leq p \leq \infty$, the function

$$(l,h,x,\gamma) \mapsto \mathcal{MS}_{\psi_1} f_1(l,h,x,\gamma)\, \mathcal{MS}_{\psi_2} f_2(l,h,x,\gamma)$$

belongs to $L^p\left(\mathcal{S}\times\hat{G}\right)$ and

$$\left\|\mathcal{MS}_{\psi_1} f_1 \ \mathcal{MS}_{\psi_2} f_2\right\|_{L^p(\mathcal{S}\times\hat{G})} \leq C_{\psi_1}^{\frac{1}{2p}} C_{\psi_2}^{\frac{1}{2p}} \|\psi_1\|_{L^2(H\times G)}^{1-\frac{1}{p}} \|\psi_2\|_{L^2(H\times G)}^{1-\frac{1}{p}} \|f_1\|_{L^2(H\times G)} \|f_2\|_{L^2(H\times G)}. \quad (3.1)$$

Proof: Using Cauchy-Schwarz inequality and equation (2.2), we have

$$\int_{\mathcal{S}\times\hat{G}} \left|\left(\mathcal{MS}_{\psi_1} f_1 \ \mathcal{MS}_{\psi_2} f_2\right)(l,h,x,\gamma)\right| d\sigma(l,h,x,\gamma)$$

$$= \int_{\mathcal{S}\times\hat{G}} \left|\mathcal{MS}_{\psi_1} f_1(l,h,x,\gamma)\ \mathcal{MS}_{\psi_2} f_2(l,h,x,\gamma)\right| d\sigma(l,h,x,\gamma)$$

$$\leq \left(\int_{\mathcal{S}\times\hat{G}} \left|\mathcal{MS}_{\psi_1} f_1(l,h,x,\gamma)\right|^2 d\sigma(l,h,x,\gamma)\right)^{1/2} \left(\int_{\mathcal{S}\times\hat{G}} \left|\mathcal{MS}_{\psi_2} f_2(l,h,x,\gamma)\right|^2 d\sigma(l,h,x,\gamma)\right)^{1/2}$$

$$= \left\|\mathcal{MS}_{\psi_1} f_1\right\|_{L^2(\mathcal{S}\times\hat{G})} \left\|\mathcal{MS}_{\psi_2} f_2\right\|_{L^2(\mathcal{S}\times\hat{G})}$$

$$= C_{\psi_1}^{1/2} C_{\psi_2}^{1/2} \|f_1\|_{L^2(H\times G)} \|f_2\|_{L^2(H\times G)}.$$

(3.2)

Therefore $\mathcal{MS}_{\psi_1} f_1 \ \mathcal{MS}_{\psi_2} f_2 \in L^1\left(\mathcal{S}\times\hat{G}\right)$ and

$$\left\|\mathcal{MS}_{\psi_1} f_1 \ \mathcal{MS}_{\psi_2} f_2\right\|_{L^1(\mathcal{S}\times\hat{G})} \leq C_{\psi_1}^{\frac{1}{2}} C_{\psi_2}^{\frac{1}{2}} \|f_1\|_{L^2(H\times G)} \|f_2\|_{L^2(H\times G)}.$$

Again using Cauchy-Schwarz inequality and equation (2.1), we have

$$\left|\mathcal{MS}_{\psi_1} f_1(l,h,x,\gamma)\right|$$

$$\leq \int_{H\times G} \left|f_1(k,y)\overline{\mathcal{U}_{(l,h,x)}^{\psi_1}(k,y)}\,\gamma(y^{-1})\right| d\mu_{H\times G}(k,y)$$

$$\leq \left(\int_{H\times G} |f_1(k,y)|^2\, d\mu_{H\times G}(k,y)\right)^{1/2} \left(\int_{H\times G} \left|\mathcal{U}_{(l,h,x)}^{\psi_1}(k,y)\right|^2 d\mu_{H\times G}(k,y)\right)^{1/2}$$

$$= \|f_1\|_{L^2(H\times G)} \left\|\mathcal{U}_{(l,h,x)}^{\psi_1}\right\|_{L^2(H\times G)}$$

$$= \|f_1\|_{L^2(H\times G)} \|\psi_1\|_{L^2(H\times G)}.$$

Similarly, $\left|\mathcal{MS}_{\psi_2} f_2(l,h,x,\gamma)\right| \leq \|f_2\|_{L^2(H\times G)} \|\psi_2\|_{L^2(H\times G)}$. So

$$\left|\left(\mathcal{MS}_{\psi_1} f_1 \ \mathcal{MS}_{\psi_2} f_2\right)(l,h,x,\gamma)\right| \leq \|f_1\|_{L^2(H\times G)} \|\psi_1\|_{L^2(H\times G)} \|f_2\|_{L^2(H\times G)} \|\psi_2\|_{L^2(H\times G)}.$$

Therefore $\mathcal{MS}_{\psi_1} f_1 \ \mathcal{MS}_{\psi_2} f_2 \in L^\infty\left(\mathcal{S}\times\hat{G}\right)$ and

$$\left\|\mathcal{MS}_{\psi_1}f_1\ \mathcal{MS}_{\psi_2}f_2\right\|_{L^\infty(\mathcal{S}\times\hat{G})} \leq \|f_1\|_{L^2(H\times G)}\|\psi_1\|_{L^2(H\times G)}\|f_2\|_{L^2(H\times G)}\|\psi_2\|_{L^2(H\times G)}. \quad (3.3)$$

Thus (3.1) holds for $p=\infty$. Now for $1\leq p<\infty$, we can write using (3.2) and (3.3)

$$\int_{\mathcal{S}\times\hat{G}}\left|\left(\mathcal{MS}_{\psi_1}f_1\ \mathcal{MS}_{\psi_2}f_2\right)(l,h,x,\gamma)\right|^p\, d\sigma(l,h,x,\gamma)$$

$$=\int_{\mathcal{S}\times\hat{G}}\left|\left(\mathcal{MS}_{\psi_1}f_1\ \mathcal{MS}_{\psi_2}f_2\right)(l,h,x,\gamma)\right|^{p-1}\left|\left(\mathcal{MS}_{\psi_1}f_1\ \mathcal{MS}_{\psi_2}f_2\right)(l,h,x,\gamma)\right|\, d\sigma(l,h,x,\gamma)$$

$$\leq\left\|\mathcal{MS}_{\psi_1}f_1\ \mathcal{MS}_{\psi_2}f_2\right\|^{p-1}_{L^\infty(\mathcal{S}\times\hat{G})}\int_{\mathcal{S}\times\hat{G}}\left|\left(\mathcal{MS}_{\psi_1}f_1\ \mathcal{MS}_{\psi_2}f_2\right)(l,h,x,\gamma)\right|\, d\sigma(l,h,x,\gamma)$$

$$=C_{\psi_1}^{1/2}C_{\psi_2}^{1/2}\|\psi_1\|^{p-1}_{L^2(H\times G)}\|\psi_2\|^{p-1}_{L^2(H\times G)}\|f_1\|^{p}_{L^2(H\times G)}\|f_2\|^{p}_{L^2(H\times G)}.$$

So $\mathcal{MS}_{\psi_1}f_1\ \mathcal{MS}_{\psi_2}f_2\in L^p\left(\mathcal{S}\times\hat{G}\right)$ and

$$\left\|\mathcal{MS}_{\psi_1}f_1\ \mathcal{MS}_{\psi_2}f_2\right\|_{L^p(\mathcal{S}\times\hat{G})}\leq C_{\psi_1}^{\frac{1}{2p}}C_{\psi_2}^{\frac{1}{2p}}\|\psi_1\|^{\frac{p-1}{p}}_{L^2(H\times G)}\|\psi_2\|^{\frac{p-1}{p}}_{L^2(H\times G)}\|f_1\|_{L^2(H\times G)}\|f_2\|_{L^2(H\times G)}.$$,

Remark 3.5: The above theorem provides an alternative proof for Theorem 3.2 as follows:

Considering $f_1=f_2=f$ and $\psi_1=\psi_2=\psi$ in Theorem 3.4, we have

$$\left(\int_{\mathcal{S}\times\hat{G}}\left|\mathcal{MS}_\psi f(l,h,x,\gamma)\right|^{2p}\, d\sigma(l,h,x,\gamma)\right)^{1/p}\leq C_\psi^{\frac{1}{p}}\|\psi\|^{\frac{2(p-1)}{p}}_{L^2(H\times G)}\|f\|^2_{L^2(H\times G)}.$$

For $2\leq q\leq\infty$, we substitute $p=\frac{q}{2}$ with $1\leq p\leq\infty$ to obtain

$$\left(\int_{\mathcal{S}\times\hat{G}}\left|\mathcal{MS}_\psi f(l,h,x,\gamma)\right|^{q}\, d\sigma(l,h,x,\gamma)\right)^{2/q}\leq C_\psi^{\frac{2}{q}}\|\psi\|^{\frac{2(q-2)}{q}}_{L^2(H\times G)}\|f\|^2_{L^2(H\times G)}.$$

Hence

$$\left\|\mathcal{MS}_\psi f\right\|_{L^q(\mathcal{S}\times\hat{G})}\leq C_\psi^{\frac{1}{q}}\|\psi\|^{1-\frac{2}{q}}_{L^2(H\times G)}\|f\|_{L^2(H\times G)}.$$,

Remark 3.6: Using Theorem 3.2, one may deduce Lieb's inequality for Gabor transform, wavelet transform and shearlet transform on locallay compact Abelian groups.

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