Trajectory Controllability of Artificial Satellite Under the Effect of Oblateness of Earth

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Abstract

This article explores the trajectory controllability of artificial satellites, taking into account the gravitational force of Earth and the force resulting from Earth's oblateness. Controllability in this context is investigated by transforming the equations of motion into a cylindrical coordinate system and leveraging the concept of Lipschitz continuity within the nonlinear terms, coupled with the application of functional analysis.

Keywords- The motion of Satellite, Oblate Earth, Trajectory controllability, Nonlinear functional analysis,

AMS Subject Classification- 3B05, 34H05, 93C10, 93C15

1 Introduction

Artificial satellites have assumed a vital role in modern society, serving essential functions in navigation, communication, and Earth's environmental monitoring [1, 2]. Extensive research efforts have been dedicated to comprehending the intricate dynamics governing the motion of these satellites. Previous studies have employed a diverse range of methodologies, including analytical, semi-analytical, and numerical approaches.

For instance, King-Hele [5] made significant strides in solving the two-body problem for satellites, incorporating Earth's oblateness into an analytical framework. Raj [3] introduced regularized equations of motion using KS transformations [4], accounting for atmospheric drag effects. Sehnal [6] delved into the perturbations caused by the upper terrestrial atmosphere, while Knowles *et.al.* [7]unearthed the influence of geomagnetic storms on upper atmosphere density, thus affecting satellite orbits.

Yan and Kapila [8] took a novel approach by developing dynamical equations of satellite motion within a spherical rotating frame, exploring conditions for maintaining a fixed osculating plane of motion. Khalil [9] contributed with analytical solutions incorporating atmospheric drag and Earth's oblateness up to the 4th order zonal harmonic, using Hamiltonian mechanics. Bezdvěk and Vokrouhlický [10] presented a semi-analytic theory accommodating the oblateness of Earth up to the 9th order zonal harmonic, factoring in atmospheric drag with the TD88 empirical model, and validating their predictions with real-world satellite data.

Furthermore, researchers like Hassan *et. al.* [11], Chen and Jing [12], and Lee *et. al.* [14] explored perturbed satellite motion under the influence of Earth's oblateness and atmospheric drag. The study of satellite formation flight with aerodynamic forces was conducted by Reid and Misra [13], while Xu and Chen [15] derived analytical solutions based on Keplerian angular elements, accounting for atmospheric drag effects. *et. al.* [16]. scrutinized the impact of Earth's oblateness and atmospheric drag on the Cosmos1484 satellite's orbit. Using Lie transformations, Delhaise[17] derived an analytical solution for the satellite's motion by considering gravity and air drag Sharma *et.al.* [18, 19]investigated satellite motion, considering variables like initial velocities, Earth's oblateness, atmospheric drag, and even predicting satellite re-entry times. Realizing the importance of maintaining satellites in their designated orbits for prolonged periods, researchers like Hajovsky [20] employed atmospheric drag as a control mechanism. B. Palancz [2, 21] pursued trajectory control through pole placement strategies.

Recent work by Lamba [22] addressed controllability, observability, and stability issues related to artificial satellites using state space methods. However, it predominantly employed a two-dimensional model, yielding four equations in polar form.

In the present study, we extend this research by considering the 3D motion of satellites, specifically examining the influence of the J_2 zonal harmonic in a Cartesian coordinate system. Our focus centers on understanding the controllability of artificial satellite trajectories in this comprehensive context.

2 Prelieminaries

Definition 2.1. *The evolution system is said to be* ***completely controllable on the interval*** $\mathbb{I}$ *if for any* $x_0, x_1 \in \mathbb{X}$ *there exists a controller* $w(t)$ *in control space* $\mathbb{U}$ *such that the state of system steers initial state* x_0 *at* $t = 0$ *to desire final state* x_1 *at* $t = T_0$*.*

Definition 2.2. *The evolution system is* ***totally controllable on the interval*** $\mathbb{I}$ *if it is completely controllable over all its subintervals* $[t_k, t_{k+1}]$*.*

Let $\mathcal{C}_{\mathcal{T}}$ be the set of all functions $y(\cdot)$ defined over the interval $\mathbb{I}$ satisfying the initial state $y(0) = x_0$ and final state $y(T_0) = x_1$. This set $\mathcal{C}_{\mathcal{T}}$ is called a set of all feasible trajectories. The controller obtained from the concept of complete and total controllability for the linear system will be optimal but for the semilinear or nonlinear system may not be optimal. To overcome this situation one has to design a trajectory having optimum energy or cost and define a controller in such a way that the state of the system steers along this trajectory. Finding the controller that steers the system on the prescribed optimal trajectory from an initial state to the desired final state is called TC.

Definition 2.3 (TC)**.** *The evolution system is said to be* ***trajectory controllable (T- Controllable)*** *if for any trajectory* $y \in \mathcal{C}_{\mathcal{T}}$*, there exist* L^2 *control function* $w \in \mathbb{U}$ *such that the state of the system* $x(t)$ *satisfy* $x(t) = y(t)$ *almost everywhere over the interval* $\mathbb{I}$*.*

In TC one has to find the controller such that the system should steer from an arbitrary initial state to a desired final state along a prescribed path or trajectory. Therefore, TC is strongest among all the forms of controllability.

3 Motion of Artificial Satellite

The equations of motion of satellite under the effect of the oblateness of the earth is given by

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{a}_O \tag{3.1}$$

where, $\mu = GM$, G is the gravitational constant and M is the mass of the earth, and $\vec{a}_O$ is the acceleration due to the oblateness of the earth, considering zonal harmonic J_2.
The equations of motion in cylindrical coordinate systems are represented by Humi.

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\mu r\left[\frac{1}{(r^2+z^2)^{\frac{3}{2}}} + \frac{3R^2J_2(r^2-4z^2)}{2(r^2+z^2)^{\frac{7}{2}}}\right] \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0 \\ \ddot{z} &= -\mu z\left[\frac{1}{(r^2+z^2)^{\frac{3}{2}}} + \frac{3R^2J_(3r^2-2z^2)}{2(r^2+z^2)^{\frac{7}{2}}}\right]. \end{aligned} \tag{3.2}$$

Under the effect of zonal harmonic J_2, the satellite will deviate from its desired orbit, hence its motion becomes uncontrollable. Eventually, it will hit on Earth. Therefore, to make a satellite in orbit one has to plant a trajectory controller in the system. The figure shows the trajectory of the motion of an artificial satellite under the effect of the oblateness of the Earth. By considering

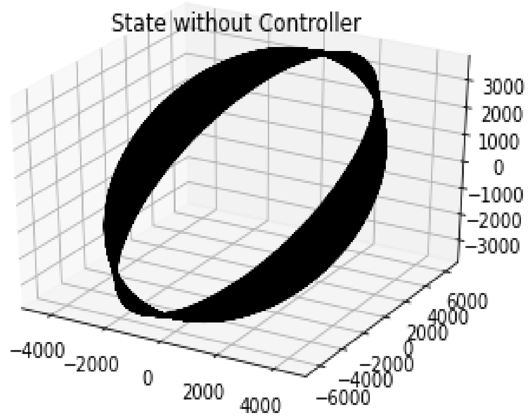

Figure 1: System without control
Data: $\bar{r_0} = (0, -5888.9727, -3400)$,$\bar{v_0} = (7, 0, 0)$, $R = 6378.1363$, $a = |\bar{r_0}|$, $\mu = G * M$, $J_2 = 1082.63 \times 10^{-6}$, Time Span: 540000 sec.

$$\begin{aligned} X_1 &= r - \sigma, \\ X_2 &= \dot{r}, \\ X_3 &= \sigma\left(\theta - \omega t\right), \\ X_4 &= \sigma\left(\dot{\theta} - \omega\right), \\ X_5 &= z, \\ X_6 &= \dot{z}. \end{aligned} \tag{3.3}$$

The system becomes:

$$
\begin{aligned}
\frac{dX_1}{dt} &= X_2, \\
\frac{dX_2}{dt} &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu\,(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + u_1\,(t)\,, \\
\frac{dX_3}{dt} &= X_4, \\
\frac{dX_4}{dt} &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)}, \\
\frac{dX_5}{dt} &= X_6, \\
\frac{dX_6}{dt} &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3\,(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\}.
\end{aligned}
\tag{3.4}
$$

Plugging trajectory controller the system becomes:

$$
\begin{aligned}
\frac{dX_1}{dt} &= X_2 + W_1(t), \\
\frac{dX_2}{dt} &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2 - \mu\,(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + W_2(t), \\
\frac{dX_3}{dt} &= X_4 + W_3(t), \\
\frac{dX_4}{dt} &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)} + W_4(t), \\
\frac{dX_5}{dt} &= X_6 + W_5(t), \\
\frac{dX_6}{dt} &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}} + \frac{3R^2J_2\left[3\,(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\} + W_6(t).
\end{aligned}
\tag{3.5}
$$

4 Generaliztion of the Problem

The form of the equation (3.5) is

$$
\begin{aligned}
\frac{dx}{dt} &= f(t, x(t)) + w(t) \\
x(0) &= x_0
\end{aligned}
\tag{4.1}
$$

over the interval $\mathbb{I}$. Here, for each t, the state of the system $x(t)$ lies in the Banach space $\mathbb{R}^n$, $f : \mathbb{I}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfying the conditions of the hypothesis and $w(\cdot)$ is trajectory controller for the system (4.1). Let $\mathcal{C}(\mathbb{I}, \mathbb{R}^n)$ be the Banach space of continuous functions from $\mathbb{I}$ into $\mathbb{R}^n$ equipped with the norm $\|x\| = \sup\limits_{t\in\mathbb{I}} \|x(t)\|$.

The function $x(t)$ is the mild solution of the evolution system (4.1) over the interval $\mathbb{I}$ if, $x(t)$ satisfies the integral equation

$$x(t) = x_0 + \int_0^t [f(s, x(s)) + w(s)]ds \tag{4.2}$$

Theorem 4.1. *If the nonlinear function $f : \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable with respect to the first argument and continuous with respect to the second argument. Moreover, there exist a constant $l : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|f(t, x_1) - f(t, x_2)\| \leq l\|x_1 - x_2\|$$

for all $x_1, x_2 \in B_r(\mathbb{R}^n)$, $\forall r < r_0$ for some r_0 and $t \in \mathbb{I}$ Satisfies then the system (4.1) is trajectory controllable.

Proof. Let $u(t)$ be any trajectory from $\mathcal{C}_{\mathcal{T}}$ and define feed-back control of the system as:

$$w(t) = \frac{du(t)}{dt} - f(t, u(t)) \tag{4.3}$$

Putting feedback control $w(t)$,

$$\frac{dx}{dt} = f(t, x(t)) + \frac{du(t)}{dt} - f(t, u(t))$$

Taking $z(t) = x(t) - u(t)$,

$$\begin{aligned} \frac{dz}{dt} &= f(t, x(t)) - f(t, u(t)) \\ z(0) &= 0, \end{aligned} \tag{4.4}$$

and mild solution of the equation (4.4) is

$$z(t) = \int_0^t f(s, x(s)) - f(s, u(s))ds$$

Since,

$$\begin{aligned} ||z(t)|| &\leq \int_0^t ||f(s, x(s)) - f(s, u(s))||ds \\ &\leq l \int_0^t ||x(s) - u(s)|| (\text{Applying hypotheses of the theorem}) \\ &\leq lT_0||z(t)||, \end{aligned}$$

and the equation $(I + lT_0)||z(t)|| \leq 0$ has unique trivial solution $||z(t)|| = 0$ for all t. Thus, $z(t) = 0$ almost everywhere. Hence the system is trajectory-controllable. $\square$

5 Simulation of the Problem

In the system (3.5) the nonlinear function can be written as

$$\begin{aligned}
f_1 &= X_2,\\
f_2 &= (X_1+\sigma)\left(\frac{X_4}{\sigma}+\omega\right)^2-\mu\,(X_1+\sigma)\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}+\frac{3R^2J_2\left[(X_1+\sigma)^2-4X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\},\\
f_3 &= X_4,\\
f_4 &= -\frac{2X_5\sigma\left(\frac{X_4}{\sigma}+\omega\right)}{(X_1+\sigma)},\\
f_5 &= X_6,\\
f_6 &= -\mu X_5\left\{\frac{1}{\left[(X_1+\sigma)^2+X_5^2\right]^{3/2}}+\frac{3R^2J_2\left[3\,(X_1+\sigma)^2-2X_5^2\right]}{2\left[(X_1+\sigma)^2+X_5\right]^{7/2}}\right\},
\end{aligned}$$

This function $f=[f_1,f_2,f_3,f_4,f_5,f_6]$ is continuous with respect to t and there exist a constant $l>0$ such that $||f(t,x)-f(t,y)||<l||x-y||$. This is possible by choosing the center of the earth as the frame of reference. Therefore, the system (3.5) is trajectory controllable, and the trajectory controller for the system is defined as

$$w(t)=\frac{du(t)}{dt}-f(t,u(t))$$

for the prescribed trajectory $u(t)$. For the trajectory $\bar{u}=\left(7\sqrt{\frac{a^3}{\mu}}sin(\sqrt{\frac{\mu}{a^3}}t),7cos(\sqrt{\frac{\mu}{a^3}}t),-\sqrt{\frac{a^3}{\mu}}tsin(\sqrt{\frac{\mu}{a^3}}t)\right)$

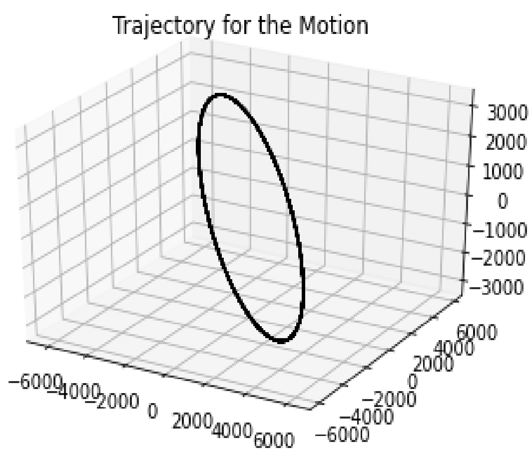

Figure 2: Trajectory

6 Conclusion

the paper contributes to the understanding of satellite motion and offers a mathematical framework for trajectory controllability, which is essential for maintaining satellites in their designated orbits. This research has implications for satellite control strategies, ensuring their stable and precise movement in space.

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