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Analysis of Single Server Queueing System with Encouraged Arrival rate of Customers

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Abstract:- In this study, we explore a queuing system with a single server that facilitates the arrival of Encouraged consumers while limiting server control. For first-time users, the server in this configuration provides three distinct service types (one by one service, bulk service with an accessible collection, and non-accessible collection). To calculate the predicted number of customers waiting in queue, the likelihood of a steady state, and a system of difference differential equation, the Laplace transform is used. The tabular format is used to display the numerical findings.

***Keywords**: batch service with accessible batch and non-accessible batch, encouraged customers' arrival, Laplace transforms, difference differential equation, steady state probability*

1. Introduction

A.Sridhar and S.Senthilkumar (2022) [6] talk about three distinct service categories that may be handled by single server queue. Queueing system analysis with several vacations and motivated customers (2022). They looked at a queueing model with a limited server capacity and encouraged customers on numerous vacation days. In this work, we analyze the effects of encouraging client arrival in a queueing system with a single batch of services, where some customers have access to those services while others do not, and where the mean service rate for those with access and those without access also varies. It is assumed that customers come as per the Poisson process with $\lambda\,(1+\varsigma)$ parameter, where ς is the variable that corresponds to the desired arrival rate. The server may process requests individually, in batches, with or without the ability to see the groups as they are processed. In the system, if the number of users is equal to or less than control limit,c_1., then the FCFS rule applies, and server provides a single user at a time, with service time distributed exponentially having mean service rateμ_S. If the queue length is greater than,c_1., server serves the entire queue in a single batch, and new customers are ready to participate the batch until its size decreases below,c_2., at which point service time is exponentially distributed with a μ_A mean service rate . However, if the queue length is equal to or greater than,c_2., then server serves the entire queue in a single batch without letting new customers join it with mean service rateμ_N.

The server performs an inspection of the system after each service completion epoch and classifies the system size (β) into 3 groups:(1) $0 \le \beta \le c_1$, (2)$c_1 + 1 \le \beta \le c_2 - 1$ and (3)$\beta \ge c_2$.

An accessible batch is one in which the server processes requests from a batch of users in sequence and accepts additional users into that batch while it is processing those requests, up until the point where either the batch size reaches the maximum allowed value or the service ends, whichever comes first (AB). Finally, Non-accessible batches are those in which the server has already taken all the service units and does not permit new users to join (NAB) with condition$\beta \ge c_2$,

The steady state probabilities of the system size were determined by Baburaj and Manoharan (1999) [1] for both single as well as bulk queuing systems. To access the batches when the service is in process. Sivasamy (1990) [2] considers the idea of accessibility. In this study, we analyze the dynamic behavior of a single queueing system providing both accessible and non-accessible batches of services at three distinct service rates and encouraging consumers to arrive at any time. In Section 2, we do an analysis of the model, the predicted queue length is derived in Section 3. Estimating the Busy Period is done in Section 4. Tables with numerical findings were shown in Section 5. After this, in Section 6, the conclusion is drawn.

2. Analysis of the Model

Let us consider $P(0,\beta,t)$, $\beta = 0,1,2,\dots,c_1$ indicates that there are β customers at time t in the system and server is providing a single service (or is idle when β =0), $P(1,\beta,t), \beta = c_1+1, c_1+2, \dots, c_2-1$ assess the probability further that server is occupied processing AB while serving β users and $P(2,\beta,t), \beta \geq 0$ is that β customers are waiting to be served (excluding those now being served) while the server is occupied with a NAB at time t. Here, we have a convenient way to express the system's state space: $S = S_1 \cup S_2 \cup S_3$, where $S_1 = \{(0,\beta), \beta = 0,1,2,\dots,c_1\}$, $S_2 = \{(1,\beta), \beta = c_1+1, c_1+2,\dots,c_2-1\}$ and $S_3 = \{(2,\beta), \beta \geq 0\}$.

$\boldsymbol{P(i,j,t)}$,$\boldsymbol{P(i,j,t)}$ **and** $\boldsymbol{P(i,j,t)}$ conform to the set of difference differential equations shown below.

$$\frac{d}{dt}P(0,0,t) = -\lambda(1+\varsigma)P(0,0,t) + \mu_S P(0,1,t) + \mu_A \sum_{\beta=c_1+1}^{c_2-1} P(1,\beta,t) + \mu_N P(2,0,t) \quad (1)$$

$$\frac{d}{dt}P(0,\beta,t) = -\left((\lambda(1+\varsigma)) + \mu_S\right)P(0,\beta,t) + \lambda(1+\varsigma)P(0,\beta-1,t) + +\mu_S P(0,\beta+1,t)$$

$$+\mu_N P(2,\beta,t), 1 \leq \beta \leq c_1 - 1 \quad (2)$$

$$\frac{d}{dt}P(0,c_1,t) = -\left((\lambda(1+\varsigma)) + \mu_S\right)P(0,c_1,t) + \lambda(1+\varsigma)P(0,c_1-1,t) + \mu_N P(2,c_1,t) \quad (3)$$

$$\frac{d}{dt}P(1,c_1+1,t) = -\left((\lambda(1+\varsigma)) + \mu_A\right)P(1,c_1+1,t) + \lambda(1+\varsigma)P(0,c_1,t) + \mu_N P(2,c_1+1,t) \quad (4)$$

$$\frac{d}{dt}P(1,\beta,t) = -\left((\lambda(1+\varsigma)) + \mu_A\right)P(1,\beta,t) + \lambda(1+\varsigma)P(1,\beta-1,t) + \mu_N P(2,\beta,t),$$

$$c_1 + 2 \leq \beta \leq c_2 - 1 \quad (5)$$

$$\frac{d}{dt}P(2,0,t) = -\left((\lambda(1+\varsigma)) + \mu_N\right)P(2,0,t) + \lambda(1+\varsigma)P(1,c_2-1,t) + \mu_N \sum_{\beta=c_2}^{\infty} P(2,\beta,t) \quad (6)$$

$$\frac{d}{dt}P(2,\beta,t) = -\left((\lambda(1+\varsigma)) + \mu_N\right)P(2,\beta,t) + \lambda(1+\varsigma)P(2,\beta-1,t), \beta \geq 1. \quad (7)$$

Let $P(i,\beta,s), i = 0,1,2$ be the Laplace transforms $\mathrm{P}(i,\beta,t), i = 0,1,2$ respectively. By applying the Laplace Transform with the initial condition $\mathrm{P}(0,0,t) = 1$ and using Laplace finial value theorem, $\mathrm{P}(i,j) = \lim_{t\to\infty} \mathrm{P}(i,j,t) = \lim_{s\to 0} s\mathrm{P}(i,j,s)$ to equations (1) through (7) and then supposing the steady state criteria are met, we get the following set of transition probabilities.

$$\lambda(1+\varsigma))\,P(0,0) = \mu_S P(0,1) + \mu_A \sum_{\beta=c_1+1}^{c_2-1} P(1,\beta) + \mu_N P(2,0) \quad (8)$$

$$(\lambda(1+\varsigma) + \mu_S)\,P(0,\beta) = \lambda(1+\varsigma)P(0,\beta-1) + +\mu_S P(0,\beta+1) + \mu_N P(2,\beta)$$

$$1 \leq \beta \leq c_1 - 1 \quad (9)$$

$$(\lambda(1+\varsigma) + \mu_S)\,P(0,c_1) = \lambda(1+\varsigma)P(0,c_1-1) + \mu_N P(2,c_1) \quad (10)$$

$$(\lambda(1+\varsigma) + \mu_A)\,P(1,c_1+1) = \lambda(1+\varsigma)P(0,c_1) + \mu_N P(2,c_1+1) \quad (11)$$

$$(\lambda(1+\varsigma) + \mu_A)P(1,\beta) = \lambda(1+\varsigma)P(1,\beta-1) + \mu_N P(2,\beta), c_1 + 2 \leq \beta \leq c_2 - 1 \quad (12)$$

$(\lambda(1+\varsigma)+\mu_N)P(2,0,)=\lambda(1+\varsigma)P(1,c_2-1)+\mu_N\sum_{\beta=c_2}^{\infty}P(2,\beta)$ (13)

$(\lambda(1+\varsigma)+\mu_N)P(2,\beta)=\lambda(1+\varsigma)P(2,\beta-1)$, $\beta\geq 1.$ (14)

Resolve (14) repeatedly we find,

$P(2,\beta)=P(2,0)\sigma_1^{\beta}$, Where $\sigma_1=\frac{\lambda(1+\varsigma)}{(\lambda(1+\varsigma)+\mu_N)}$

By solving (9), which is the difference equation in, $P(0,\beta)$, we get

$$P(0,\beta)=F.\phi^{\beta}-P(2,0)\frac{\mu_N\sigma_1^{\beta}}{W(z)},1\leq\beta\leq c_1-1$$

Where F is a constant, $W(z)=\mu_S z^2-(\lambda(1+\varsigma)+\mu_S)z+\lambda(1+\varsigma)$ and $\phi=\frac{\lambda(1+\varsigma)}{\mu_S}$.

By putting this into (10), we obtain

$P(0,c_1)=F\,\sigma_2\phi^{c_1-1}-P(2,0)[\frac{\mu_N\sigma_2\sigma_1^{c_1-1}}{W(z)}-\sigma_3\sigma_1^{c_1}]$,

Where $\sigma_2=\frac{\lambda(1+\varsigma)}{\lambda(1+\varsigma)+\mu_S}$ and $\sigma_3=\frac{\mu_N}{\lambda(1+\varsigma)+\mu_S}$.

From (11)

$P(1,c_1+1)=F\sigma_2\sigma_4\phi^{c_1-1}-P(2,0,).U_1$

Where $\sigma_4=\frac{\lambda(1+\varsigma)}{\lambda(1+\varsigma)+\mu_A}$, $\sigma_5=\frac{\mu_N}{\lambda(1+\varsigma)+\mu_A}$ and $U_1=\frac{\mu_N\sigma_2\sigma_4\sigma_1^{c_1-1}}{W(z)}$-$(\sigma_1\sigma_4\sigma_5+\sigma_3\sigma_4)\sigma_1^{c_1}$

From (12)

$$P(1,\beta)=F\sigma_2\sigma_4^{\beta-c_1}\phi^{c_1-1}-P(2,0)\{\,[U_1\sigma_4^{\beta-c_1-1}-(\beta-c_1-1)\sigma_5\sigma_1^{\beta}\},\ c_1+2\leq\beta\leq c_2-1$$

From (13)

$$P(2,0)=F.U_2$$

Where $\sigma_6=\frac{\mu_N}{\lambda(1+\varsigma)+\mu_N}$ and $U_2=\frac{\sigma_1\sigma_2\phi^{c_1-1}\sigma_4^{c_2-c_1-1}}{1+U_1\sigma_4^{c_2-c_1-1}\sigma_1-(c_2-c_1-1)\sigma_5\sigma_1^{c_2}-\frac{\sigma_1^{c_2}\sigma_6}{(1-\sigma_1)}}$

From (8)

$$P(0,0)=F[1-\frac{U_2\mu_S\mu_N\sigma_1}{W(z)\lambda(1+\varsigma)}+\frac{\mu_A}{\lambda(1+\varsigma)}U_3+\frac{\mu_N}{\lambda(1+\varsigma)}U_2]$$

Where $U_3=\sigma_2\sigma_4\phi^{c_1-1}-U_1U_2+\phi^{c_1-1}\sigma_2\left(\frac{\sigma_4{}^2-\sigma_4{}^{c_2-c_1}}{1-\sigma_4}\right)-U_2\left[U_1\frac{\sigma_4-\sigma_4{}^{c_2-c_1-1}}{1-\sigma_4}\right]+T_1$

$$-(c_1+1)\frac{\sigma_1{}^{c_1+2}-\sigma_1{}^{c_2}}{1-\sigma_1}$$

and $T_1=\sum_{\beta=c_1+1}^{c_2-1}\beta.\sigma_1^{\beta}=(1-\sigma_1)^{-2}(\sigma_1^{c_1+2}-\sigma_1^{c_2})+(1-\sigma_1)^{-1}(c_1\sigma_1^{c_1+1}-(c_2-1)\sigma_1^{c_2})$.

Consequently, we derive Laplace transformation for the transition probabilities, such as

$P(0,0)=F[1-\frac{U_2\mu_S\mu_N\sigma_1}{W(z)\lambda(1+\varsigma)}+\frac{\mu_A}{\lambda(1+\varsigma)}U_3+\frac{\mu_N}{\lambda(1+\varsigma)}U_2]$ (15)

$P(0,\beta)=F[\phi^{\beta}-U_2\frac{\mu_N\sigma_1^{\beta}}{W(z)}],1\leq\beta\leq c_1-1$ (16)

$P(0,c_1)=F[\,\sigma_2\phi^{c_1-1}-U_2[\frac{\mu_N\sigma_2\sigma_1^{c_1-1}}{W(z)}-\sigma_3\sigma_1^{c_1}]\,]$ (17)

$P(1, c_1+1) = F[б_2 б_4 ɇ^{c_1-1} - U_1 U_2]$ (18)

$$P(1,\beta) = F[б_2 б_4^{\beta - c_1} ɇ^{c_1-1} - U_2\{\ U_1 б_4^{\beta - c_1 - 1} - (\beta - c_1 - 1) б_5 б_1^{\beta}\}],\ \ c_1 + 2 \le \beta \le c_2 - 1$$

(19)

$P(2,\beta) = F U_2 б_1^{\beta}, \beta \ge 0$ (20)

Then using the normalizing condition

$$\sum_{\beta=0}^{c_1} P(0,\beta) + \sum_{\beta=c_1+1}^{c_2-1} P(1,\beta) + \sum_{\beta \ge 0} P(2,\beta) = 1$$

We get

$$F = \{1 - \frac{U_2 \mu_S \mu_N б_1}{W(z)\lambda\,(1+ς)} + \frac{\mu_A}{\lambda\,(1+ς)} U_3 + \frac{\mu_N}{\lambda\,(1+ς)} U_2 + \frac{ɇ - ɇ^{c_1}}{1-ɇ} - U_2 \frac{\mu_N (б_1 - б_1^{c_1})}{(1-б_1)}$$

$$+ɇ^{c_1-1} б_2 - U_2[\frac{\mu_N б_1^{c_1-1} б_2}{W(z)} - б_3 б_1^{c_1}] + U_3 + \frac{U_2}{1-б_1}\}^{-1}\ \ (21)$$

3. Expected Queue Length

If there are m customers, and, $(1 \le \beta \le c_1)$.., then the queue size will be $\beta - 1$ and the server will be providing just one service, with the probability being, $P_L(0,\beta)$. The server is handling AB requests if the queue length is equal to or more than c_1 *but less then* c_2 ($c_1 + 1 \le \beta \le c_2 - 1$), and the availability is $P_L(1,\beta)$. The server is handling NAB requests if the queue size is equal to or more than, $c_2(\beta \ge c_2)$, so the probability is, $P_L(2,\beta)$. In other words, clients who are waiting for NAB service must wait till it is finished.

. $Ɨ_{\mathbb{Q}} = \sum_{\beta=2}^{c_1} (\beta - 1) P(0,\beta) + \sum_{\beta \ge 1} \beta(2,\beta)$

Making use of the equations (16), (17),(20) and (21), we discover,

$Ɨ_{\mathbb{Q}} =$

$F\left[ɇ^2[(1-ɇ)^{-2}(1-ɇ^{c_1-1}) - (c_1-1)ɇ^{c_1-2}(1-ɇ)^{-1}] + (c_1-1)ɇ^{c_1-1}б_2 + U_2[\frac{б_1}{(1-б_1)^2} + (c_1 1)\left[\frac{\mu_N б_1^{c_1-1} б_2}{W(z)} -\right.\right.$

б3б1c1− μNб12Wz1−б1−21−б1c1−1− c1−1б1c1−21−б1−1].(22)

4. Expected Busy Period

This configuration only allows the server to rest when there are no clients using the service. Here ʙ (busy period) and ɪ (idle period) alternates and form a busy cycle. Let $\mathcal{Y}(\mathrm{t})$ denote the server's state and $\mathcal{X}(\mathrm{t})$ signify the system's state at t time. In this model the server becomes busy when a single unit arrives.

Hence $Ȩ\,[ɪ\,] = \frac{1}{\lambda\,(1+ς)}$.

From the theory of renewal process $P(0,0) = \lim_{t\to 0} P\,\{\mathcal{Y}(\mathrm{t}) = 0, \mathcal{X}(\mathrm{t}) = 0\}$

$$= \frac{Ȩ[ɪ]}{Ȩ[ɪ] + Ȩ[ʙ]}$$

Therefore expected busy period $Ȩ[ʙ] = \frac{1-P(0,0)}{\lambda\,(1+ς)P(0,0)}$

$$= \frac{1 - F[1 - \frac{U_2\mu_S\mu_N б_1}{W(z)\lambda\,(1+ς)} + \frac{\mu_A}{\lambda\,(1+ς)} U_3 + \frac{\mu_N}{\lambda\,(1+ς)} U_2]}{\lambda\,(1+ς) F[1 - \frac{U_2\mu_S\mu_N б_1}{W(z)\lambda\,(1+ς)} + \frac{\mu_A}{\lambda\,(1+ς)} U_3 + \frac{\mu_N}{\lambda\,(1+ς)} U_2]}$$

5. Numerical Illustrations

Take the values of the variables as $\lambda = 10, \varsigma = 2$, $\mu_S = 10, \mu_A = 8$, $\mu_N = 7$, $c_1 = 5$ and $c_2 = 20$ for numeric operations.

The numerical findings for evaluating steady-state probability using equations (15) to (20) are shown in Table 1.

Table 1. Numerical results of steady state probabilities

β	$P(\mathbf{0},\beta)$	β	$P(\mathbf{1},\beta)$	β	$P(\mathbf{2},\beta)$
0	0.1560	5	0.3083	0	3.9823e-05
1	0.0152	6	0.2438	1	3.2289e-05
2	0.0457	7	0.1921	2	2.6180e-05
3	0.1370	8	0.1517	3	2.9657e-05
4	0.4110	9	0.1198	4	1.7211e-05
		10	0.0945	5	1.3955e-05
		11	0.0746	6	1.1315e-05
		12	0.0589	7	9.1741e-06
		13	0.0465	8	7.4385e-06
		14	0.0367	9	6.0312e-06
		15	0.0290	10	4.8902e-06
		16	0.0229	11	3.9650e-06
		17	0.0181	12	3.2149e-06
		18	0.0143	13	2.6067e-06
		19	0.0113	14	2.1135e-06
				15	1.7136e-06
				…	…
				…	…
				…	…

Therefore, expected queue length computed for various c_1 and c_2 values by using equation (22) are as follows.

Table 2

$c_1 \downarrow / c_2 \rightarrow$	14	15	16	17	18	19
6	3.6103	3.6885	3.7424	3.7746	3.7878	3.7839
7	4.2272	4.3649	4.4766	4.5651	4.6331	4.6833
8	4.8160	5.0155	5.1839	5.3237	5.4381	5.5303
9	5.3268	5.5972	5.8305	6.0284	6.1939	6.3306
10	5.7290	6.0800	6.3890	6.6559	6.8827	7.0729

Here it can be noted that expected queue length increases when both c_1 and c_2 increases.

Therefore expected queue length computed for various ς , μ_S, μ_A and μ_N values by using equation (22). The result on both ς and μ_S expounds in table 3, the output on

both ς and μ_A clarifies in table 4, the outturn on both ς and μ_N exucidates in table 5.

Table 3Effects on ς and μ_S

$\lambda = 8$, $\mu_A = 8$, $\mu_N = 7$,$c_1 = 50$ and $c_2 = 100$

$\varsigma \downarrow / \mu_S \rightarrow$	10	11	12	13	14	15
1	44.5473	16.1514	0.4372	0.0114	0.0004	0.00002
2	46.8310	46.6519	46.4694	46.2776	46.0679	45.8267
3	47.4642	47.3358	47.2090	47.0827	46.9555	46.8257
4	47.8088	47.7111	47.6147	47.5192	47.4243	47.3294
5	47.9853	47.9092	47.8340	47.7595	47.6855	47.6117

Here it can be noted that the expected queue length decreases when μ_S increases but queue length increases when **ς**increases.

Table 4 Effects on ς and μ_A

$\lambda = 8,$, $\mu_S = 10 \mu_N = 7, c_1 = 50$, and $c_2 = 100$

$\varsigma \downarrow / \mu_A \rightarrow$	8	9	10	11	12	13
2	46.8310	47.1373	47.4292	47.7077	47.9738	48.2282
3	47.4642	47.7814	48.0870	48.3820	48.6669	48.9424
4	47.8088	48.1190	48.4169	48.7055	48.9858	49.2586
5	47.9853	48.2994	48.5891	48.8662	49.1347	49.3965
6	48.0130	48.3700	48.6679	48.9396	49.1976	49.4473

Here it can be noted that the expected queue length increases when both **ς**and μ_A increases.

Table 5 Effects of on ς and μ_N

$\lambda = 8$, $\mu_S = 10, \mu_A = 8, c_1 = 50$, and $c_2 = 100$

$\varsigma \downarrow / \mu_N \rightarrow$	7	8	9	10	11	12
1	44.5473	44.0076	43.4989	43.0161	42.5552	42.1130
2	46.8310	46.3767	45.9578	45.5705	45.2112	44.8770
3	47.4642	47.0342	46.6323	46.2559	45.9026	45.5704
4	47.8088	47.4176	47.0484	46.6993	46.3687	46.0553
5	47.9853	47.6321	47.2962	46.9763	46.6714	46.3804

Here it can be noted that the expected queue length increases when **ς**increases but queue length decreases μ_N increases.

Table6 Queue length with respect to service rates

S.No.	Queue Length($l_{\mathbb{Q}}$)	Service rates condition
1	47.5549	$\mu_S = \mu_A = \mu_N$
2	47.9416	$\mu_S > \mu_A > \mu_N$
3	47.4287	$\mu_S < \mu_A < \mu_N$
4	47.8827	$\mu_S = \mu_A > \mu_N$
5	47.6013	$\mu_S > \mu_A = \mu_N$
6	47.2773	$\mu_S = \mu_N > \mu_A$
7	47.6376	$\mu_S < \mu_A = \mu_N$

From the table number 6, we conclude that the queue length is decreases when single service rate and non-accessible batch service rate are euqual each other but both are greater than accessible batch service rate.

6. Conclusions

Further we can find customer spend time in the system and queue length, busy period of the server of the encouraged arrival of customersς, service rates μ_S, μ_A and μ_N, control limits c_1 and c_2. While the server is active, we can also notice that the customer spends considerable time waiting in the system's queues. Some numerical results demonstrating system actions for charging clients at various rates for the same service are shown.

References

[1] Baburaj,C. and Manoharan,M., A Single and Batch Service M/M/1 Queue, Vol.35, pp.39-44, 1999.
[2] Sivasamy,R., A bulk Service queue with accessible and non-accessible batches Opsearch, Vol.27,No.1,pp.46-54,1990.
[3] A. Sridhar and R. Allah Pitchai , A Single and Batch Service Queueing System with Additional Service Station , International Journal of Advanced Computer and Mathematical Sciences ISSN 2230-9624. Vol4, Issue3, 2013, pp199-209.
[4] Sridhar and R. Allah Pitchai, Two Server Queueing System with Single and Batch Service , (2014),International Journal of Applied Operational Research,Vol. 4, No. 2, pp. 15-26, Spring 2014 .
[5] Sridhar and R. Allah Pitchai ,Analysis Of a Markovian Queue with Two Heterogeneous Servers and Working Vacation , International Journal of Applied Operational Research, 5(4),Vol. 5, No. 4, pp. 1-15, Autumn 2015.
[6] S.Senthilkumar and A.Sridhar , Single Server Queueing System with Three Types of Services, Mathematical Statistician and Engineering Applications ISSN:2094-0343 Vol.71No.3s2(2022) pp.1046-1055.
[7] S.Malik and R.Gupta , Analysis of Finite Capacity Queuing System with Multiple Vacations and Encouraged Customers, International Journal of Scientific Research in Mathematics and Statistical Sciences, Vol-9, Issue-2,pp.17-22, April(2022).
[8] Baburaj,C. and Manoharan,M., On A Markovian Single and Batch Service Queue with Accessibility, International and Management Science, Vol.10,Number 3,pp-17-23,1999.