Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 44 No. 04 (2023)

Estimating Approximate Solutions of Non-Linear Caputo Fractional Differential Equations with Forcing Functions Using Picard's Iteration Method

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Abstract: The objective of this research is to extend the applicability of Picard's iterative existence and uniqueness theorem in solving non-linear Caputo fractional differential equations of higher order that involve a forcing function satisfying the usual Lipchitz's condition. To demonstrate the effectiveness of our approach, we have presented numerical examples of order α (where $1 < \alpha < 2$ and $2 < \alpha < 3$) along with the application of the fractional damped duffing oscillator. The solutions obtained through Picard's iterative method are accompanied by graphical representations and some solutions compared with ADM solutions using graphs.

Keywords*: Non-linear Factional Differential Equations, Approximate solution, Picard's Iterative method, Forcing Function, Fractional Duffing oscillator, MATLAB.*

1. Introduction

Differential Equations are a valuable tool in solving real-world problems involving both linear and non-linear systems. Depending on the nature of the phenomenon, we may need to employ either linear or non-linear differential equations. However, the majority of scientific problems and phenomena occur in a non-linear form. While only a small subset of these problems can be solved analytically, many can be addressed through approximate analytical methods or numerical techniques, such as linearization, decomposition, or homotopy methods [14].

As science and technology continue to advance, many phenomena cannot be adequately described by classical differential equations. For instance, various physical processes exhibit memory and hereditary properties that are not captured by local differential operators. Therefore, new tools, such as fractional differential equations with nonlocal operators, are necessary to represent these nonlocal phenomena accurately. However, most fractional differential equations do not possess exact solutions, and hence analytical and numerical methods have become increasingly important for solving them [4]. In recent years, several efficient methods have been developed for solving fractional differential equations, including the Adomian decomposition method[12], variational iteration method, homotopy perturbation method[5] [7], Haar wavelet operational method[3], neural networks[15]. For some nonlinear fractional differential equations, an exact analytical solution is not possible, and numerical methods like Picard's method [8], Taylor series method [10], and Lagrange's polynomial method [1] are employed to obtain approximate solutions. Numerical methods typically produce more accurate results than approximate analytic methods.

Recently, numerous techniques have emerged to generate numerical solutions for various types of fractional differential equations. For example, Jafari &Daftardar-Gejjib proposed a method for solving nonlinear fractional differential equations using Adomain Decomposition Method [9]. In this approach, an approximate solution for a non-linear fractional differential equation is obtained and compared with its exact solution.

Athassawat Kammanee has developed a method for generating numerical solutions for fractional differential equations with variable coefficients using Taylor basis functions [10]. This method is computationally straightforward and offers a structured approach to obtaining numerical approximations.

Aisha F. Fareed, Mourad S. Semary, and Hany N. Hassan have developed an approximate solution for fractional order Riccati equations using controlled Picard's method with Atangana-Baleanu fractional derivative [2]. This method is versatile in that it can be applied to a wide range of integer and fractional order differential equations, including non-linear ones. The approach employs an additional auxiliary parameter that enhances convergence, making it a highly effective method.

In their paper, Rainey Lyons and et.al. Present an extension of Picard's Iterative Existence and Uniqueness Theorem to Caputo fractional differential equations with non-homogeneous terms that meet the typical Lipschitz's condition [13].

Ejikeme, C. L.and et.al. Proposed a solution to the nonlinear Duffing oscillator with fractional derivatives using the Homotopy Analysis Method (HAM). Their research provides an analytical solution to the fractional-order Duffing oscillator with graphical representation [6].

This research presents a solution to a non-linear fractional differential equation with a non-homogeneous term that includes a forcing function using Picard's iterative method. The method is demonstrated through several numerical examples, and the solution to a non-linear fractional Duffing oscillator is also presented graphically.

2. Preliminaries

This section aims to establish a clear understanding of the fractional calculus used in the paper, as there are several definitions of fractional derivatives and integrals. The Caputo derivative and Riemann-Liouville integral are both commonly used, but the paper specifically focuses on the Riemann-Liouville fractional integral [11].

Definition 1 The Caputo's fractional derivative of order α is defined as

$$D^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{u^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds\,,\ 0\le n-1<\alpha\le n,\quad n\in N$$

where α is the order of the derivative and n is the smallest integer which is greater than α.

Definition 2 The Riemann-Liouville fractional integral operator of order α, I^{α} is given by

$$I^{\alpha}u(t)=\begin{cases}\dfrac{1}{\Gamma\alpha}\displaystyle\int_0^t(t-s)^{\alpha-1}u(s)ds\,, & \alpha>0\\ u(t), & \alpha=0\end{cases}$$

An important property of the Riemann-Liouville fractional integral that it is a linear operator

$$I^{\alpha}\big(\lambda f(t)+g(t)\big)=\ \lambda I^{\alpha}f(t)+I^{\alpha}g(t)$$

where λ is a constant. Moreover, a well-known Riemann-Liouville fractional integral formula regards its effects on powers of the integrand

$$I^{\alpha}t^{n}=\frac{\Gamma n+1}{\Gamma n+1+\alpha}t^{n+\alpha}\,,for\ \ n>-1$$

Furthermore, Caputo's fractional derivative of order α is

$$D^{\alpha}u(t)=I^{n-\alpha}\left(\frac{d^{n}u(t)}{dt^{n}}\right)$$

where $\alpha\in\boldsymbol{R}$and $n-1<\alpha\le n$ with$n\in\boldsymbol{N}$. The relation between the Caputo fractional derivative and Riemann-Liouville integral operator is that they are almost inverses, except for integration constants,

$$D^{\alpha}(I^{\alpha}u(t))=u(t)$$

$$I^{\alpha}(D^{\alpha}u(t))=u(t)-\sum_{k=0}^{n-1}u^{k}(0)\frac{t^{k}}{k!}\qquad \text{for } t>0$$

$I^{\alpha}(e^{kt}) = \frac{e^{kt}}{k^{\alpha}}$, Where k is a constant.

3. **Existence of Picard's Iteration Method for System of FDE:**

In this research, Picard's method was used to solve a system of fractional differential equations with initial conditions. The method was applied to a non-homogeneous term of the FDEs, with an attempt to satisfy the Lipschitz condition. It should be noted that in a previous study [13], Picard's method was developed for Caputo fractional differential equations with initial conditions, but for the case where the nonlinear term satisfies a time-independent Lipschitz condition.

Consider the Caputo fractional initial value problem

$$\begin{cases} D^{\alpha} y = f(x,y) \\ y(x_0) = y_0 \end{cases} \quad (1)$$

where $0 < \alpha < 1$, and $f(x,y) \in C\big[[x_0, x_0 + x] \times R, R\big]$. The integral representation of (1) is given *by*

$$y(x) = \frac{1}{\Gamma\alpha}\int_0^t (x-t)^{\alpha-1} f(t, y(t))dt \quad (2)$$

For example,

$$D^{\frac{3}{2}} y(x) = f(x, y, y') \qquad y(x_0) = y_0, y'(x_0) = y'_0 \quad (3)$$

We take $y' = y_1$, then $D^{\frac{1}{2}}(y_1) = f(x, y, y')$

So, we have a system of two equation, one is of fractional order α (0< α< 1) and other is of order one,

$$y' = y_1 \quad (3.1)$$

$$D^{\frac{1}{2}}(y_1) = f(x, y, y') \quad (3.2)$$

Lipschitz condition: A vector valued function $\bar{f}$ defined for $(x, \bar{y})$ in some set S.

We say that $\bar{f}$ satisfies Lipchitz condition on S, if there exist a constant k > 0 such that

$\left|\bar{f}(x,\bar{y}) - \bar{f}(x,\bar{z})\right| \le k|\bar{y} - \bar{z}| \forall (x,\bar{y}), (x,\bar{z}) \in$ S

The constant k is called Lipschitz constant for $\bar{f}$ on S.

Let us consider a nonlinear non homogeneous equation.

Consider the system of two equations

$$y_1{}' = ay_1 + by_2$$
$$D^{\alpha}(y_2) = cy_1 + dy_2$$

Applying Lipschitz condition,

$$\left|\bar{f}(x,\bar{y}) - \bar{f}(x,\bar{z})\right| \le k|(ay_1 + by_2, cy_1 + dy_2) - (az_1 + bz_2, cz_1 + dz_2)|$$
$$\le k|(ay_1 - az_1 + by_2 - bz_2, cy_1 - cz_1 + cy_2 - dz_2|$$
$$\le k[|a(y_1 - z_1)| + |b(y_2 - z_2)| + |c(y_1 - z_1)| + |d(y_2 - z_2)|$$

$\le k|(a + c)(y_1 - z_1)|+|(b + d)(y_2 - z_2)|$

$\le k(|a| + |c|), (|b| + |d|\ \ )|y_1, y_2 - z_1, z_2|$

$\le k(|a| + |c|), (|b| + |d|\ \ )|\bar{y} - \bar{z}|$

Lipschitz constant k: $\bar{f}(x,\bar{y}) = (ay_1 + by_2, cy_1 + dy_2)$

$\frac{\partial \bar{f}}{\partial y_1} = (a,c), \left|\frac{\partial \bar{f}}{\partial y_1}\right| = |(a,c)| = |a| + |c|$

$\frac{\partial \bar{f}}{\partial y} = (b,d), \left|\frac{\partial \bar{f}}{\partial y}\right| = |(b,d| = |b| + |d|$

Lipschitz constant $k\ is$ $\max(|a| + |c|, |b| + |d|)$

Therefore $\bar{f}$ satisfies Lipschitz condition with constant k

Numerical Solutions:

Example 1: Consider Non linear fractional differential equation.

$$D^{\frac{5}{2}} y = \frac{1}{4} y^4 + x, \text{ y (0) = 0}, y'(0) = 0, y''(0) = 0 \quad (4)$$

Equation can be written as

$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{d^2y}{dx^2}\right) = \frac{1}{4}y^4 + x \quad => \frac{d^{\frac{1}{2}}z}{dx^{\frac{1}{2}}} = \frac{1}{4}y^4 + x(4.1)$

$\frac{d^2y}{dx^2} = z => \frac{dy}{dx} = t \quad (4.2)$

$\frac{dt}{dx} = z(4.3)$

On solving (4.1), (4.2) and (4.3) using Picard's Iterative method, we get y

$y(x) = \frac{14!}{4\Gamma(\frac{9}{2})^4\Gamma\frac{35}{2}} x^{\frac{33}{2}} + \frac{x^{\frac{7}{2}}}{\Gamma\frac{9}{2}}........$

On solving (4.1), (4.2) and (4.3) using ADM method, we get y [17]

$y(x) = \frac{16x^{\frac{7}{2}}}{105\sqrt{\pi}} + \frac{274877906944\ x^{\frac{33}{2}}}{2009196669692953125\ \pi^{\frac{3}{2}}} + ...$

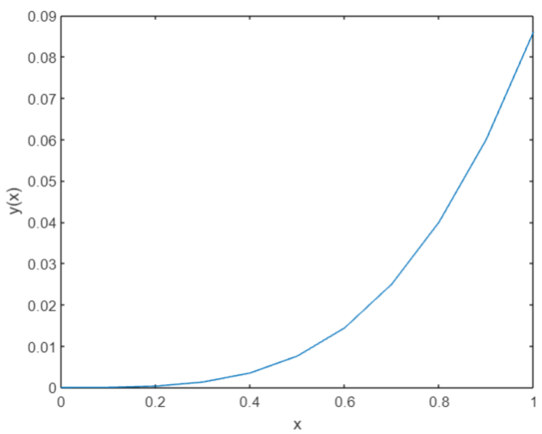

Figure 1(A): shows graphical representation of solution IVP (4) using Picard's Iterative method using MATLAB

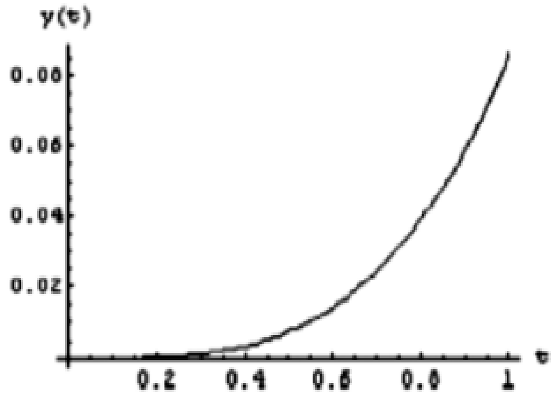

Figure 1(B): shows graphical representation of solution IVP (4) using ADM method[17]

Figures 1(A) and 1(B) show comparison between ADM and Picard's Method for solution ofNon-Linear Non homogeneous FDE with fractional order $\alpha(2 < \alpha < 3)$ and forcing term x. These graphs conclude that error of Picard's solution (Numerical) by comparing with ADM Solution (Semi analytic) is negligible.

Example2: $D^{\frac{3}{2}}y = \frac{1}{2}y^2 + x^2, y(0) = 0,\ y'(0) = 1$ (5)

Equation can be written as

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{dy}{dx}\right) = \frac{1}{2}y^2 + x^2$$

$\frac{dy}{dx} = z$ (5.1)

$\frac{d^{1/2}z}{dx^{1/2}} = \frac{1}{2}y^2 + x^2$ (5.2)

On solving (5.1) and (5.2) using Picard's Iterative method, we get y

$$y(x) = \frac{2.7!}{\frac{17}{2}\Gamma(\frac{9}{2})^2\Gamma\frac{17}{2}} x^{\frac{17}{2}} + \frac{2x^{\frac{7}{2}}}{\frac{7}{2}\Gamma\frac{7}{2}}.........$$

Using ADM method equation (5) gives solution as follows [17]

$$y(x) = \frac{32x^{\frac{7}{2}}}{105\sqrt{\pi}} + \frac{1048576\, x^{\frac{17}{2}}}{1206079875\,\sqrt{\pi}} + ...$$

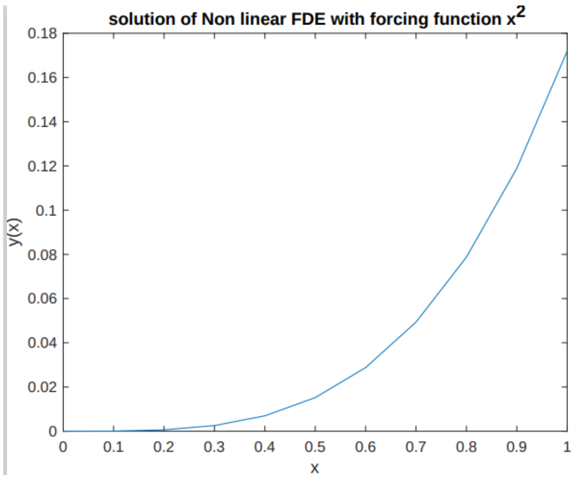

Figure 2(A): shows graphical representation of solution of IVP (5) using Picard's Iterative method using MATLAB

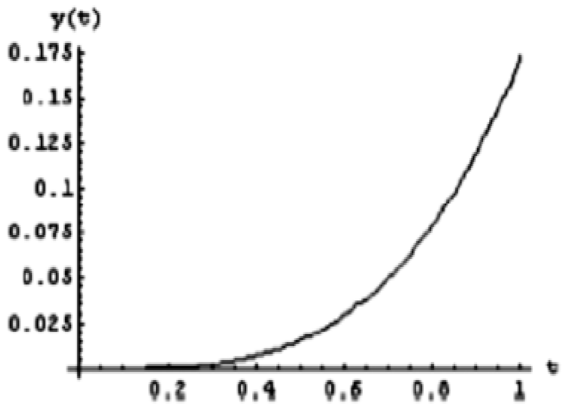

Figure 2(B): shows graphical representation of solution of IVP (5) using ADM method[17]

Figures 2(A) and 2(B) show comparison between ADM and Picard's Method for solution of Non-Linear Non homogeneous FDE with order $\alpha(1 < \alpha < 2)$ and forcing term x^2. These graphs conclude that error of Picard's solution (Numerical) by comparing with ADM Solution (Semi analytic) is also negligible.

Example 3: Consider Multi term Non linear Fractional differential equation.

$$D^{\frac{3}{2}}y(x) + Dy(x) = 2y^2 + x, \qquad y(0) = 0,\ y'(0) = 1 \tag{6}$$

Equation can be written as

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} = 2y^2 + x$$

$$\frac{dy}{dx} = z \tag{6.1}$$

$$\frac{d^{1/2}z}{dx^{1/2}} + z = 2y^2 + x \tag{6.2}$$

On solving (6.1) and (6.2) simultaneous equations using Picard's Iterative method, we get y

$$y(x) = \frac{2}{\Gamma\left(\frac{7}{2}\right)^2 \Gamma\frac{15}{2}} x^{\frac{13}{2}} + \frac{x^{\frac{5}{2}}}{\Gamma\frac{7}{2}} - \frac{x^3}{6} + \frac{x^{\frac{7}{2}}}{\Gamma\frac{9}{2}} \dots ....$$

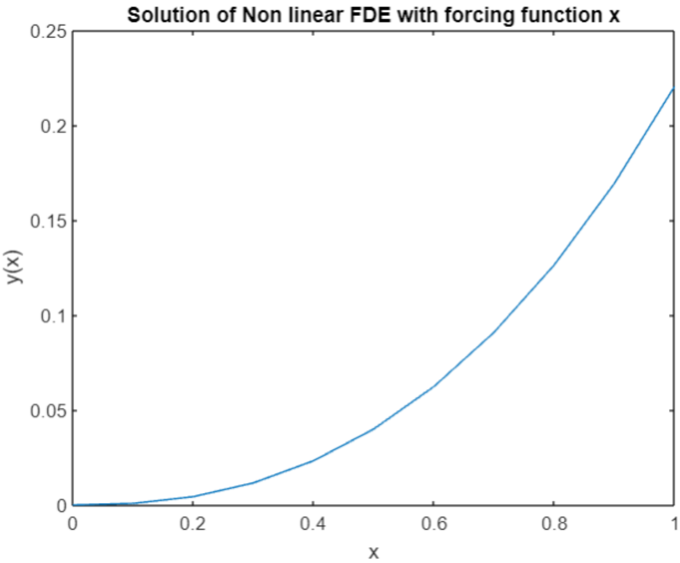

Figure 3: Shows graphical representation of approximate solution IVP (6) using Picard's Iterative method using MATLAB.

Figure 3 show approximate solution of Non linear Non homogeneous multi term FDE of order $\alpha(1 < \alpha < 2)$. In ADM method, ADM polynomial is difficult to calculate for multi term FDE, so Picard's Method gives approximate numerical solution and easy to implement.

Application: Consider Fractional Nonlinear duffing Oscillator

$$D^{\frac{3}{2}}u(t) + \delta D^{\frac{1}{2}}u(t) + \rho u + \mu u^3 = \lambda sin\omega t \quad (7)$$

$$u(0) = \frac{2}{\Gamma\frac{1}{2}}, D^{\frac{1}{2}}u(0) = 1, \lambda = 20, \rho = 1, \mu = 2, \omega = \frac{\pi}{2}, \delta = 1$$

Equation can be written as

$$\frac{d^{\frac{1}{2}}u}{dt^{\frac{1}{2}}} = z \quad (7.1)$$

$$\frac{dz}{dt} = 20sin\omega t - u - 2u^3 - z \quad (7.2)$$

After solving (7.1) and (7.2) using Picard's Iterative method, we get

$$u(t) = \frac{2}{\Gamma\frac{1}{2}} + \frac{t^{\frac{1}{2}}}{\Gamma\frac{3}{2}} + \frac{20t^{\frac{1}{2}}}{\omega^{\frac{1}{2}}\Gamma\frac{3}{2}} - \frac{20w^{\frac{3}{2}}t^{\frac{5}{2}}}{\Gamma\frac{7}{2}} + \frac{20\omega^{\frac{7}{2}}t^{\frac{9}{2}}}{\Gamma\frac{11}{2}} - \frac{t^{\frac{3}{2}}}{\Gamma\frac{5}{2}}\left(1+\frac{2}{\Gamma\frac{1}{2}}+\frac{16}{\Gamma\left(\frac{1}{2}\right)^3}\right) + \ldots.$$

Table 1: Numerical values shows oscillating behavior of displacement u(t) at different values of time(t).

t	u(t)	t	u(t)	t	u(t)	t	u(t)	t	u(t)
0	1.13	0.6	8.12	1.2	-1.48	1.8	-11.20	2.4	0.59
0.1	5.80	0.7	7.25	1.3	-3.73	1.9	-12.00	2.5	8.06
0.2	7.41	0.8	6.04	1.4	-5.97	2.0	-12.96	2.6	17.69
0.3	8.31	0.9	4.50	1.5	-8.21	2.1	-11.46	2.7	24.87
0.4	8.68	1.0	2.70	1.6	-9.32	2.2	-8.93	2.8	38.9
0.5	8.60	1.1	0.69	1.7	-10.81	2.3	-4.98	2.9	50.12

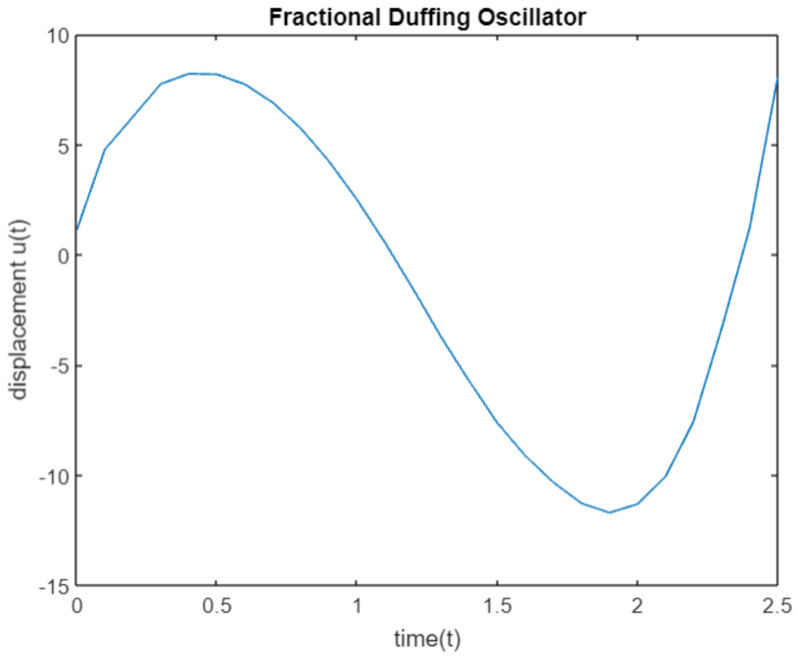

Figure 4: Shows Graph of fractional duffing oscillator [6] for h = 0.1, $\omega = \frac{\pi}{2}, \lambda = 20 , \rho = 1 , \mu = 2 , \delta = 1$ using MATLAB.

Table 1 shows values of displacement against time in Fractional duffing oscillator. Values show oscillating behavior with time. And figure 4 also presents oscillating behavior in graph .so Picard's Method works efficiently for this type of application of Non-linear higher order FDE with forcing term.

Conclusion:

This research proposes a method to solve non-linear fractional differential equations of order α (1<α<2) and (2<α<3) which includes a forcing function as a non-homogeneous term, using Picard's Iteration Method. The presented technique is easy to implement and applicable to non-linear problems of fractional order. Several numerical problems are solved using the proposed method, and graphical representations are obtained for each numerical example and some solutions compared with solution of ADM Method using graphs. The approach is simpler than other semi-analytic methods, as there is no need to calculate Adomian polynomials or general Lagrange multipliers. Additionally, an application of the fractional non-linear differential equation with a forcing function, namely the fractional non-linear Duffing oscillator, is presented and solved using the proposed method. This work can be extended to solve non-linear systems of FDEs for related problems.

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