On Irresolute Generalized $\beta$-$\alpha$-topological Group

1R. Rama vani, 2R.Selvi
1Research Scholar, 2Associate Professor
Department of Mathematics, Sriparasakthi College for Women, Courtallam,Tamilnadu, India
Affiliated to Manonmaniam Sundaranar University,Abishikapatti, Tirunelveli-627012, Tamilnadu, India.

Abstract: In this paper, we introduce and study two types of topological groups namely irresolute generalized $\beta$-$\alpha$-topological group and Irr-$\mathcal{G}$-$\beta$-$\alpha$ topological group. Also, some basic concepts and results related to the above two topological groups are studied.

Keywords: $\mathcal{G}$-$S_\alpha$-open,$\mathcal{G}$-$S_\alpha^\beta$-open,$\mathcal{G}$-$S_\alpha$-Int,$\mathcal{G}$-$S_\alpha$-Cl,$\mathcal{G}$-$\alpha_\beta$-regular,irresolute $\mathcal{G}$-$\beta$-$\alpha$ topological group

AMS Subject Classification: 22-xx,54-xx,22A05,54A05.

1. Introduction.

Topological groups are logically the combination of topological spaces and groups. A topological group is defined as a group binded with a topology such that the binary operations are continuous. A. csaszar [2] introduced the concept of generalized topology in 2002. In 2013 Muard Hussian et.al [13] introduced the concept of generalized topological groups.In 2013 Ibrahim [7] introduced the concept of $\alpha$-open sets and $\alpha_\beta$-open. A.B. Khalaf et.al [8] introduced some basic concept of $\beta$-$\alpha$-topological groups. In 2015 Moiz ud din khan et.al [14] introduced the concept of Irresolute-topological groups.In this paper, we introduced the concept of irresolute generalized $\beta$-$\alpha$-topological group and study the related concepts.

2. Preliminaries

Definition 2.1 [2] Let X be any set and let $\mathcal{G} \subseteq P(X)$ be a subfamily of power set of X.

Then $\mathcal{G}$ is called a generalized topology if $\emptyset \in \mathcal{G}$ and for any index set I , $\bigcup_{i \in I} O_i \in \mathcal{G}$, $O_i \in \mathcal{G}$ , $i \in I$

Definition 2.2 [13] A triple $(G, *, \mathcal{G})$ is said to be an $\mathcal{G}$-topological group if $(G, *)$ is a group, $(G, \mathcal{G})$ is a generalized topological space and,

(i) The multiplication mapping $m: G \times G \rightarrow G$ defined by $m(x, y) = x * y$ , $x, y \in G$ is $\mathcal{G}$-continuous.

(ii) The inverse mapping $i : G \rightarrow G$ defined by $i(x) = x^{-1}$ , $x \in G$ is $\mathcal{G}$ – continuous.

Definition 2.3 [7] Let $(G, \tau)$ be a topological space and $\beta : \alpha O(G, \tau) \rightarrow P(G)$ an operation on $\alpha O(G, \tau)$. A nonempty set A of G is called $\alpha_\beta$-open set if for each point $x \in A$, there exists an $\alpha$-open set U of G containing x such that $U^\beta \subseteq A$.

Definition 2.4 [15] Let $(G, *)$ be a group and $(G, \tau)$ be a topological space. A triple $(G, *, \tau)$ is called a $\beta$-$\alpha$-topological group if the inversion map is $\beta$-$\alpha$-continuous and the multiplication map is jointly $\beta$-$\alpha$-continuous in both variable.

Definition 2.5[14] A topological group $(G, *, \tau)$ is called an irresolute-topological group if for each $x, y \in G$ and each semi-open neighbourhood W of $x * y^{-1}$ in G there exist semi-open neighbourhood U of x and V of y such that $U * V^{-1} \subseteq W$. 

4649
Definition 2.6[11] A topological space \((G, \tau)\) is said to be \(\alpha\beta\)-regular if for each \(x \in G\) and for each \(\alpha\)-open set \(V\) in \(G\) containing \(x\), there exist an \(\alpha\)-open set \(U\) in \(G\) containing \(x\) such that \(U^\beta \subseteq V\).

Definition 2.7[3] A subset \(A\) of a generalized topological space \((G, \mathcal{G})\) is called \(G\)-\(\alpha\beta\)-open if \(A \subseteq G - \text{Int}(G - \text{Cl}(G - \text{Int}(A)))\). The complement of an \(G\)-\(\alpha\beta\)-open set is called \(G\)-\(\alpha\beta\)-closed.

Irresolute \(G\)-\(\beta\alpha\)-topological group

Definition: 3.1 A subset \(A\) of a generalized topology \(X\) is said to be generalized semi-\(\alpha\)-open if there exists a \(G\)-\(\alpha\)-open set \(U\) in \(X\) such that \(U \subset A \subset clU\). The family of all generalized semi-\(\alpha\)-open is denoted by \(G\)-\(S\_\alpha\). The complement of a \(G\)-\(S\_\alpha\)-open set is said to be \(G\)-\(S\_\alpha\)-closed. The union of all \(G\)-\(S\_\alpha\)-open sets contained in \(A\) is called the generalized semi-\(\alpha\)-interior of \(A\) and denoted by \(G\)-\(S\_\alpha\)-Int. The intersection of all \(G\)-\(S\_\alpha\)-closed containing \(A\) is called the generalized semi-\(\alpha\)-closure and denoted by \(G\)-\(S\_\alpha\)-Cl.

Definition: 3.2 Let \((G, \mathcal{G})\) be a generalized topology and \(\beta : G\-S\_\alpha O(G, \mathcal{G}) \to \mathcal{P}(G)\) an operation on \(G\-S\_\alpha O(G, \mathcal{G})\). A subset \(A\) of \(G\) is called \(G\)-\(S\_\alpha\)\(\beta\)-open set if for each point \(x \in A\), there exists a \(G\)-\(S\_\alpha\) open set \(U\) of \(G\) containing \(x\) such that \(U^\beta \subseteq A\). The family of all generalized semi-\(\alpha\)-open is denoted by \(G\)-\(S\_\alpha\)\(\beta\)-open. The complement of a \(G\)-\(S\_\alpha\)\(\beta\)-open set is said to be \(G\)-\(S\_\alpha\)\(\beta\)-closed. The union of all \(G\)-\(S\_\alpha\)\(\beta\)-open sets contained in \(A\) is called the generalized semi-\(\alpha\)-interior of \(A\) and denoted by \(G\)-\(S\_\alpha\)\(\beta\)-Int. The intersection of all \(G\)-\(S\_\alpha\)\(\beta\)-closed containing \(A\) is called the generalized semi-\(\alpha\)-closure and denoted by \(G\)-\(S\_\alpha\)\(\beta\)-Cl.

Definition: 3.3 A \(G\)-\(\beta\alpha\)-topological group is called an irresolute \(G\)-\(\beta\alpha\)-topological group if for each \(x, y \in G\) and each \(G\)-\(S\_\alpha\)-open neighbourhood \(W\) of \(x \ast (y^{-1})\) in \(G\), there exists \(G\)-\(S\_\alpha\)-open neighbourhood \(U\) of \(x\) and \(V\) of \(y\) such that \(U^\beta \ast (V^\beta)^{-1} \subseteq W^\beta\).

Example: 3.4 Let \((Z_3, +_3)\) is a group under addition and \(G = \{Z_3, \emptyset, \{1\}, \{0, 2\}, \{1, 2\}, \{0, 1\}\}\). Then \((Z_3, +_3, G)\) is the irresolute \(G\)-\(\beta\alpha\)-topological group.

Remark 3.5 If \((G, \ast, \mathcal{G})\) is an irresolute \(G\)-\(\beta\alpha\)-topological group such that the family \(G\)-\(S\_\alpha\)-open is a generalized topology on \(G\) with \(G\)-\(S\_\alpha\)(\(G\)) \(\neq\) \(G\), then \((G, \ast, G\-S\_\alpha(G))\) is a \(G\)-\(\beta\alpha\)-topological group.

Theorem: 3.6 If \((G, \ast, \mathcal{G})\) is an irresolute \(G\)-\(\beta\alpha\)-topological group, then the multiplication mapping \(f : G \times G \to G\) and the inverse mapping \(i : G \to G\) are irresolute.

Proof: Let \((x, y) \in G \times G\) and let \(W\) be a \(G\)-\(S\_\alpha\)-open neighbourhood subgroup of \(G\).

Define \(f : G \times G \to G\) by \(f(x, y) = x \ast y\). By definition 3.3, for every \(G\)-\(S\_\alpha\)-open neighbourhood \(W\) of \(x \ast (y^{-1})^{-1}\), there exist \(G\)-\(S\_\alpha\)-open neighbourhood \(U\) of \(x\) and \(V\) of \(y^{-1}\) such that \(U^\beta \ast (V^\beta)^{-1} \subseteq W^\beta\). Here \(V^{-1}\) is a \(G\)-\(S\_\alpha\)-open neighbourhood of \(y\).

Then \(f(U \times V^{-1}) \subseteq f(U^\beta \times (V^\beta)^{-1}) \subseteq U^\beta \ast (V^\beta)^{-1} \subseteq W^\beta\) and \((x, y) \in U \times V\) in \(G\)-\(S\_\alpha\)-open, which implies \(f\) is irresolute on \(G \times G\). Let \(x \in G\) be a point and let \(W\) be a \(G\)-\(S\_\alpha\)-open neighbourhood of \(G\). Define \(i : G \to G\) by \(i(x) = x^{-1}\). Here \(U\) be a \(G\)-\(S\_\alpha\)-open neighbourhood with \(x \in U\) and \(i(U^\beta) = (U^\beta)^{-1} \subseteq W^\beta\).

Hence \(i\) is irresolute.

Definition: 3.7 A \(G\)-\(\beta\alpha\)-topological group \((G, \ast, \mathcal{G})\) is said to be an \(Irr\)-\(G\)-\(\beta\alpha\)-topological group if both the multiplication mapping \(m : G \times G \to G\) and the inverse mapping \(i : G \to G\) are irresolute.

Example: 3.8 Let \((Z_3, +_3)\) is a group under addition and \(G = \{Z_3, \emptyset, \{0\}, \{0, 2\}, \{1, 2\}, \{1, 0\}\}\) be the generalized topology on \(Z_3\).

Then \((Z_3, +_3, G)\) is the \(Irr\)-\(G\)-\(\beta\alpha\)-topological group and \(G\)-\(\beta\alpha\)-topological group.
Theorem: 3.9 Let $G$ be an $\text{Irr-}G-\beta-\alpha$-topological group and $\mathcal{B}_e$ be the collection of all $G-\alpha$-open neighbourhood of identity $e$ of $G$. Then for every $O \in \mathcal{B}_e$, there is an element $U \in \mathcal{B}_e$ such that $(U^\beta)^{-1} \subseteq O^\beta$.

Proof: Let $G$ be an $\text{Irr-}G-\beta-\alpha$-topological group. Since the inverse mapping $\iota : G \rightarrow G$ is irresolute, for every $O \in \mathcal{B}_e$ with $a^{-1} \in O$, there is $U \in \mathcal{B}_e$ such that $a \in U$ and $(U^\beta)^{-1} \subseteq O^\beta$.

Theorem: 3.10 Every $G-\alpha$ open subgroup $H$ of an irresolute $G-\beta-\alpha$ topological group $(G, \ast, \mathcal{G})$ is also an irresolute $G-\beta-\alpha$ topological group.

Proof: Let $H$ be a $G$-open subgroup of $G$. We prove that for each $x, y \in H$, by definition 3.3, for each $G-\alpha$-open neighbourhood $x \ast y^{-1} \in W \subset H$, there exists $G-\alpha$-open neighbourhood $x \in U \subset H$ and $y \in V \subset H$ such that $U^\beta \ast (V^\beta)^{-1} \subseteq W^\beta$.

Since $H$ is $G$-open and $W$ is $G-\alpha$-open in $G$, the set $W \cap H = W$ is a $G-\alpha$-open in $G$ and contains $x \ast y^{-1}$. Apply now the fact that $G$ is an irresolute $G-\beta-\alpha$-topological group to find $G-\alpha$-open neighbourhood $A$ of $x$ and $B$ of $y$ such that $A^\beta \ast (B^\beta)^{-1} \subseteq W^\beta$. The sets $U = A \cap H$ and $V = B \cap H$ are $G-\alpha$-open subsets of $H$ which contain $x$ and $y$. Therefore $U^\beta \ast (V^\beta)^{-1} \subseteq A^\beta \ast (B^\beta)^{-1} \subseteq W^\beta$. Hence $H$ is an irresolute $G-\beta-\alpha$ topological group.

Theorem: 3.11 A nonempty subgroup $H$ of an irresolute $G-\beta-\alpha$ topological group $G$ is $G-\alpha$-open if and only if its $G-\alpha$-Int is nonempty.

Proof: Let $H$ be a subgroup of $G$ and $x \in G-\alpha$-Int($H$).

There exists a $G-\alpha$-open set $V$ such that $x \in V \subset H$, which implies $x \ast V \subset H$.

For every $y \in H$, we have $y \ast V = y \ast x^{-1} \ast x \ast V \subset y \ast x^{-1} \ast H = H$. Since $y \ast V$ is $G-\alpha$-open and the union of $G-\alpha$-open sets is $G-\alpha$-open, $H = \bigcup \{y \ast V : y \in H\}$ is $G-\alpha$-open.

Conversely, Let $H = \bigcup \{U \ast a : a \in H\}$ is $G-\alpha$-open. Let $x = U \ast a$ for some $u \in U$.

Take $U = U \ast a \ast a^{-1} = x \ast a^{-1}$, which implies $u = x \ast a^{-1}$ for some $u \in U$. There exists a $G-\alpha$-open set $U$ containing $x$ such that $x \in U \subset H$, which implies $x \in G-\alpha$-Int($H$).

Theorem: 3.12 Let $G$ be an irresolute $G-\beta-\alpha$ topological group, $A \subset G$ and $\mathcal{B}_e$ the system of all $G-\alpha$-open neighbourhoods of $e$. Then $G-\alpha$ Cl($A$) $\subseteq AO^\beta$.

Proof: Let $G$ be an irresolute $G-\beta-\alpha$-topological group and $A$ be a subset of $G$. Let $G-\alpha$-open neighbourhood $U$ containing $e$ such that $(U^\beta)^{-1} \subseteq O^\beta$. Take $x \in G-\alpha$ Cl($A$). Then $xU^\beta$ is a $G-\alpha$-open set containing $x$.

Now $a \in A \cap xU^\beta$, which implies $a = xb$ for some $b \in U^\beta$, then $x = ab^{-1} \in A(U^\beta)^{-1} \subseteq AO^\beta$. Hence $G-\alpha$ Cl($A$) $\subseteq AO^\beta$.

Theorem: 3.13 Let $(G, \ast, \mathcal{G})$ is an irresolute $G-\beta-\alpha$-topological group and $A, B \subseteq G$. If $A$ is arbitrary and $B$ is $G-\alpha$-open, then $A \ast B$ is $G-\alpha$-open.

Proof: Let $(G, \ast, \mathcal{G})$ be an irresolute $G-\beta-\alpha$-topological group and $B$ be $G-\alpha$-open, then $G-\alpha$ Int($B$) = $B$. Consider $a \in A$, then $a \ast B = a \ast G-\alpha$ Int($B$) = $G-\alpha$ Int($a \ast B$).

Hence, $A \ast B = A \ast G-\alpha$ Int($B$) = $\bigcup_{a \in A} a \ast G-\alpha$ Int($B$) = $\bigcup_{a \in A} G-\alpha$ Int($a \ast B$).

Then $A \ast B$ is the union of $G-\alpha$-open sets. Hence $A \ast B$ is $G-\alpha$-open.

Theorem: 3.14 Let $(G, \ast, \mathcal{G})$ be an irresolute $G-\beta-\alpha$-topological group on $G$, then the following statements are true:
(i) If H is a subgroup of G, then it is $G$-$S_{\alpha}$-closed in G

(ii) If a subgroup H of G contains a non empty $G$-$\alpha_B$-open set, then H is $G$-$S_{\alpha}$-open in G.

**Proof.**

(i) Let H be a subgroup of G. Then, by theorem 3.13 every left coset $x \ast H$ is $G$-$S_{\alpha}$-open. Since the union of $G$-$S_{\alpha}$-open sets is $G$-$S_{\alpha}$-open, $Y=\bigcup_{x\in G/H} x \ast H$ is $G$-$S_{\alpha}$-open.

Hence $H=G \setminus Y$ is $G$-$S_{\alpha}$-closed.

(ii) Let H be a subgroup of G and let B be a non empty $G$-$\alpha_B$-open subset of G such that $B \subseteq H$. Then $h \ast B$ is $G$-$S_{\alpha}$-open in G for any $h \in H$. Since H is a subgroup of G, $h \ast B \subseteq H$ for all $h \in H$. Since the union of $G$-$S_{\alpha}$-open sets is $G$-$S_{\alpha}$-open, $H=\bigcup_{x \in H} h \ast B$ is also $G$-$S_{\alpha}$-open.

**Theorem: 3.15** Let U be a symmetric $G$-$S_{\alpha}$-open neighbourhood of the identity e in an irresolute $G$-$\beta$-$\alpha$-topological group G. Then the set $L=\bigcup_{n \in \mathbb{N}} U^n$ is a $G$-$S_{\alpha}$-open and $G$-$S_{\alpha}$-closed subgroup of G.

**Proof:** Let U be a symmetric $G$-$S_{\alpha}$-open neighbourhood of the identity e. Let L be a subgroup of G. Let $x \in U^p$, $y \in U^q$ be elements of L. Then $x \ast y \in U^{p+q} \subseteq L$. Now if $x \in L$, say $x \in U^k$, then $x^{-1} \in (U^{-1})^k = U^k \subseteq L$. Since G is $G$-$S_{\alpha}$-open, $U^n$ is a $G$-$S_{\alpha}$-open for each $n \in \mathbb{N}$. Here L is also $G$-$S_{\alpha}$-open in G. Then by theorem 3.14(i), L is $G$-$S_{\alpha}$-closed.

**Quotients on irresolute generalized $\beta$-$\alpha$-topological group**

**Definition 3.16.** Let $(G, \ast, G)$ be an irresolute $G$-$\beta$-$\alpha$-topological group and S be a normal subgroup of a group G and the mapping $\pi : G \rightarrow G/S$ be defined by $\pi(g) = g \ast S$, for each $g \in G$. In the set $G/S$, we define a family $G'$ and $G$-$S_{\alpha}$ $O(G/S, G')$ of subset as follows:

$G' = \{ O \subseteq G/S : \pi^{-1}(O) \in G \}$ and $G$-$S_{\alpha}$ $O(G/S, G') = \{ O \subseteq G/S : \pi^{-1}(O) \in G$-$S_{\alpha}$ $O(G, G') \}$.

From the operation $\beta$ which is defined on $G$-$S_{\alpha}$ $O(G/S, G')$, we defined the operation $\beta_{G/S}$: $G$-$S_{\alpha}$ $O(G/S, G') \rightarrow \mathcal{P}(G/S)$ as follows:

$(\pi(U))_{\beta_{G/S}} = \pi(U^P)$ for every $U \in G$-$S_{\alpha}$ $O(G, G')$ and $\pi(U) \in G$-$S_{\alpha}$ $O(G/S, G')$.

**Example:** Let $(Z_3, +_3)$ is a group under addition and $G = \{ Z_3, \emptyset, [0,2], [1,2], [0], [0,1] \}$ is an irresolute $G$-$\beta$-$\alpha$-topological group and $\{ Z_3, \emptyset, [1,2], [0] \}$ is $G$-$S_{\alpha}$-open and for each $A \in G$-$S_{\alpha}$ $O(Z_3, G')$ we define $\beta$ on $G$-$S_{\alpha}$ $O(Z_3, G')$ by

$$A^\beta = \begin{cases} A & \text{if } A \text{ is singleton set,} \\ \{ Z_3 \} & \text{otherwise} \end{cases}$$

Let $S = \{ 0,3 \}$, so $Z_3/S = \{ S, 1+S, 2+S \}$ Then $G' = \{ \emptyset, Z_3/S, [S], [1+S, 2+S], [1+S, 2+S] \}$

$G$-$S_{\alpha}$ $O(G/S, G') = \{ \emptyset, Z_3/S, [S], [1+S, 2+S] \}$.

Then $\{ S \}_{\beta_{Z_3/S}} = \pi([0,3])_{\beta_{Z_3/S}} = \pi((0,3)^\beta) = \pi(Z_3) = Z_3/S$

$\{ 1+S, 2+S \}_{\beta_{Z_3/S}} = \pi((1, 2, 0, 3))_{\beta_{Z_3/S}} = \pi((1, 2, 0, 3)^\beta) = \pi(Z_3) = Z_3/S$.

Therefore, $(Z_3/S, +_3, G')$ is an irresolute $G$-$\beta_{Z_3/S}$-$\alpha$-topological group.

**Definition 3.18** A $G$-topology $(G, G)$ is said to be $G$-$S_{\alpha\beta}$-regular if for each $x \in G$ and for each $G$-$S_{\alpha}$-open set V in G containing x, there exist an $G$-$S_{\alpha}$-open set U in G containing x such that $U^\beta \subseteq V$. 

4652
Theorem 3.19 Let $(G, *, g)$ be an irresolute $G$-regular group, $(G, g)$ be $G$-$g$-$\alpha$-topological group and $S$ be a normal subgroup of $G$. $\pi$ be the natural mapping of $G$ onto $G/S$ and let $U$ and $V$ be a $G$-$S_{\alpha}$-open subset of $G$ such that $e \in U, e \in V$ and $V^{-1} \ast \bar{V} \subseteq U$. Then $G - S_{\alpha}g_{G/S}$ is $\pi(V) \subseteq \pi(U)$.

Proof. Let any $x \in G$ and $\pi(x) \in G - S_{\alpha}g_{G/S}$ be defined by $\pi(V) = V \ast x$ is $G$-$S_{\alpha}g_{G/S}$-open set. Let $V \ast x$ be an $G$-$S_{\alpha}$-open set containing $x$ and the mapping $\pi : G \rightarrow G/S$ be defined by $\pi(V) = V \ast x$ is $G$-$S_{\alpha}g_{G/S}$-open, then $\pi(V \ast x)$ is $G$-$S_{\alpha}g_{G/S}$-open set containing $\pi(x)$. Here $\pi(V \ast x) \cap \pi(V) = \emptyset$, we have $\pi(a \ast x) = \pi(b)$ for some $a \in V$ and $b \in V$, i.e. $a \ast x = b \ast h$, for some $h \in S$, which implies $x = (a^{-1} \ast b) \ast h \in U \ast S$, since $a^{-1} \ast b \in V^{-1} \ast \bar{V} \subseteq U$. Therefore, $\pi(x) \in \pi(U \ast S) = \pi(U)$. Hence $G - S_{\alpha}g_{G/S}$ is $\pi(V) \subseteq \pi(U)$.

Theorem 3.20 Let $(G, *, g)$ be an irresolute $G$-$\beta$-regular group, $(G, g)$ be $G$-$g$-$\alpha$-topological group and $S$ be a normal subgroup of $G$. Then the quotient space $G/S$ is $G$-$g_{G/S}$-$\beta$-regular.

Proof. The mapping $\pi : G \rightarrow G/S$ be defined by $\pi(V) = V \ast x$. Let $W$ be a $G$-$S_{\alpha}$-open set of $\pi(e)$ in $G/S$, where $e$ is the neutral element of $G$. By the continuity of $\pi$, we can find an $G$-$S_{\alpha}$-open set $U$ of $e$ in $G$ such that $\pi(U) \subseteq W$. Since $G$ is an irresolute $G$-$\beta$-topological group, we can choose an $G$-$\beta$-open set $V$ of $e$ such that $V^{-1} \ast \bar{V} \subseteq U$. Then, by theorem 3.19, $\pi(V) \subseteq \pi(U) \subseteq W$. Since $\pi(V)$ is a $G$-$g_{G/S}$-open neighbourhood of $\pi(e)$, the $G$-$g_{G/S}$-$\beta$-regularity of $G/S$ at the point $\pi(e)$ is satisfied. Hence the space $G/S$ is $G$-$g_{G/S}$-$\beta$-regular.

Example 3.21 Let $(Z_2, +/3)$ be a group under addition and $\mathfrak{g} = \{ Z_2, \emptyset, \{0,2\}, \{1,2\}, \{0,1\} \}$ is an irresolute $G$-$\beta$-topological group and for each $A \in G - S_0 \mathfrak{g}(Z_2, \mathfrak{g})$ we define $\beta$ on $G - S_0 \mathfrak{g}(Z_2, \mathfrak{g})$ by $A^\beta = Cl(A)$. Let $S = \{0,3\}$, so $Z_2/S = \{1,2\}$ and $\mathfrak{g}^* = \{0, Z_2/S, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}$. Then $G - S_0 \mathfrak{g}(G/S, \mathfrak{g}^*) = \{0, Z_2/S\}$.

Theorem 3.22 Let $(G, *, g)$ be an irresolute $G$-$\beta$-topological group and let $S$ be normal subgroup of $G$. If $(G, g)$ is $G$-$g_{\beta}$-regular, then $(G/S, *, g^*)$ is an irresolute $G$-$\beta_{G/S}$-$\alpha$-topological group.

Proof. Let $U$ be an irresolute $G$-$\beta_{G/S}$-$\alpha$-topological group and $G$-$\beta$-regular. First we prove that $\pi(U) \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$ for every $U \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$. By the definition of $G - S_0 \mathfrak{g}(G/S, g^*)$, $\pi(U) \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$ at $\pi^{-1}(\pi(U)) \subseteq G - S_0 \mathfrak{g}(G, g)$. Let the mapping $\pi : G \rightarrow G/S$ be defined by $\pi(g) = g \ast S$. For each $g \in G$, then the mapping is $G$-homomorphism, we have $\pi^{-1}(\pi(g)) = g \ast S$, for each $g \in G$. Therefore $\pi^{-1}(\pi(U)) = U \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$.

Since $(G, g)$ is $G$-$g_{\beta}$-regular, $U \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$. Since $\pi(U) \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$ for every $U \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$, $\pi^{-1}(\pi(U)) \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$.

Next we prove that the $(G/S, *, g^*)$ is an irresolute $G$-$\beta_{G/S}$-$\alpha$-topological group.

Let $\mathfrak{g} : G - S_0 \mathfrak{g}(G/S, g^*)$ and $a, b \in G/S$ such that $a \ast b \in O$. Let $x, y \in G$ and the mapping $\pi : G \rightarrow G/S$ be defined by $\pi(x) = a$ and $\pi(y) = b$. Since $\pi$ is a $G$-homomorphism, $\pi(x \ast y) = \pi(x) \ast \pi(y) = a \ast b \in O$, which implies $x \ast y \in \pi^{-1}(0) \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$.

By definition of 3.3, for each $G$-$S_{\alpha}$-open neighbourhood $W$ of $x \ast y^{-1} \in \pi^{-1}(0) \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$, there exists $G$-$S_{\alpha}$-open neighbourhood $U$ of $x$ and $V$ of $y$ such that $U^\beta \ast (V^\beta)^{-1} \subseteq \pi^{-1}(0)^^\beta$. Again since $\pi$ is a $G$-homomorphism, we have $\pi(U^\beta \ast (V^\beta)^{-1}) = \pi(U^\beta) \ast \pi(V^\beta)^{-1}$. Here $U^\beta \ast (V^\beta)^{-1} \subseteq \pi^{-1}(0)^\beta$.

Thus, $\pi(U^\beta) \ast \pi(V^\beta)^{-1} \subseteq \pi((\pi^{-1}(0))^\beta) = O_{G/S}$.

Therefore, $\pi(U^\beta) \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$ and $\pi(V^\beta)^{-1} \subseteq G - S_0 \mathfrak{g}(G/S, g^*)$. Since $a = \pi(x) \subseteq \pi(U)$ and $b = \pi(y) \subseteq \pi(V)$, $(G/S, *, g^*)$ is an irresolute $G$-$\beta_{G/S}$-$\alpha$-topological group.
References


