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Group Difference Cordial Labeling in Graphs

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Abstract

Let $G = (VG), E(G))$ be a graph . Let Γ be a group .For $u \in \Gamma$, let $o(u)$ denotes the order of u in Γ. Let $f :$ $V(G) \to \Gamma$ be a function . For each edge uv assign the label $|o(f(u)) - o(f(v))|$. Let $v_f(i)$ denotes the number of vertices of G having label i under f. Also e_f (1), e_f (0) respectively denote the number of edges labeled with 1and not with 1.Now f is called a group difference cordial labeling if $|v_f(i) - v_f(j)| \le 1$ for every $i, j \in \Gamma, i \ne j$ and $|e_f(1) - e_f(0)| \le 1$. A graph which admits a group difference cordial labeling($GDCL$)is called group difference cordial graph . In this paper we fix the group Γ as the group $\{1, -1, \text{i}, -\text{i}\}$which is the group of fourth roots of unity, that is cyclic with generators i and -i. We proved that path graph, bistar graph, and Crown graph are group differenceCordial graph We further characterized Ladder graph, and star graph is a group difference cordial graph. **AMS subject classification**: 05C78

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1 Introduction

Graphs considered here are finite, undirected and simple. An assignment of integers to the vertices, edges or both in a graph is known as labeling and it depends on a few factors. Cahit et.al introduced the concept of cordial labeling [3].Ponraj et al. introduced a new labeling called difference cordial labeling[6].Athisayanathan et al. introduced the concept of group A cordial labeling[1].Labelled graphs are valuable models for a variety of applications including constraint programming across finite domains, circuit design, addressing in communication networks and astronomy.

Definition 1.1. [3] Let $f : V(G) \to \{0,1\}$ be any function . For each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1. Also the number of edges labelled 0 and the number of edges labeled 1 differ by at most 1.

In[6] , Ponraj et al. introduced a new labeling called difference cordial labeling .

Definition 1.2. [6] Let G be a (p, q) graph . Let f $: V(G) \to \{1,2 \dots p\}$ be a bijection . For each edge uv, assign the label $|f(u) - f(v)|$. f is called a difference cordial labeling if f is $1 - 1$ and $|e_f(0) - e_f(1)| = 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges with 1 and not labeled with 1 respectively . A graph with a difference cordial labeling is called a difference cordial graph.

Athisayanathan et al. [1] introduced the concept of group A cordial labeling.

Definition 1.3. [1] Let A be a group. We denote the order of an element $a \epsilon A$ by $o(a)$. Let $f: V(G) \to A$ be a function. For each edge uv assign the label 1 if $\left(o(f(u)), o(f(v))\right) = 1$ or 0 otherwise. f is called a group A Cordial labeling if $|v_f(a) - v_f(b)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(x)$ and $e_f(n)$ respectively denote the number of vertices labeled with an element x and number of edges labeled with $n (n = 0,1)$ A graph which admits a group A Cordial labeling is called a group A cordial graph.

Motivated by these, we define group difference cordial labeling of graphs. Terms not defined here are used in the sense of Harary [4] and Gallian [3]. For any real number x, we denote $\lfloor x \rfloor$, the greatest integer smaller than or equal to x and by $\lceil x \rceil$, we mean the smallest integer greater than or equal to x.

A path is an alternating sequence of $v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n$. A path on n vertices is denoted by P_n. A bipartite graph is a graph whose vertex set $V(G)$ can be partitioned in to two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. If every vertex of V_1 is adjacent with every vertex of V_2, then G is a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$ then the complete bipartite graph is denoted by $K_{m,n}$. $K_{1,n}$ is called a star graph. The Bistar $B_{m,n}$ is the graph obtained by making adjacent the two central vertices of $K_{1,m}$ and $K_{1,n}$. The graph $L_n = P_n \times P_2$ is called a ladder. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i th vertex of G_1 with an edge to every vertex in the ith copy of G_2. The graph $C_n \odot K_1$ is called a crown.

2. Group Difference cordial Graphs

Definition 2.1. Let $G = (V(G), E(G))$ be a graph. Let Γ be a group. For $u \epsilon \Gamma$, let $o(u)$ denote the order of u in Γ. Let $f: V(G) \to \Gamma$ be a function. For each edge uv assign the label $|o(f(u)) - o(fv))|$. Let $v_f(i)$ denote the number of vertices of G having label i under f. Also $e_f(1), e_f(0)$ respectively denote the number of edges labeled with 1 and not with 1. Now f is called a group difference cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ for every $i, j \in \Gamma, i \neq j$ and $|e_f(1) - e_f(0)| \leq 1$. A graph which admits a group difference cordial labeling is called group difference cordial graph.

In this paper we take the group Γ as the group $\{1, -1, i, -i\}$ which is the group of fourth roots of unity, that is cyclic with generators i and $-$ i.

Theorem 2.2. The Path P_n is a group difference cordial graph for all 'n'.

Proof : Let $G = P_n$ have n vertices and f be the group difference cordial labeling of G.

Let $V(G) = \{u_1, u_2 \dots u_n\}$. Clearly P_n is a group difference cordial graph for $n \leq 3$.

Assume $n \geq 4$ and define $f : V(G) \rightarrow \{1, -1, \mathrm{i}, -\mathrm{i}\}$ as follows

Case (i) : $n \equiv 0 \pmod 4$, Let $n = 4k, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if\ 1 \leq i \leq k \\ \mathrm{i} & if\ k+1 \leq i \leq 2k \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if\ 1 \leq i \leq k \\ -\mathrm{i} & if\ k+1 \leq i \leq 2k \end{cases}$$

Clearly $V_f(1) = k, V_f(-1) = k, V_f(\mathrm{i}) = V_f(-\mathrm{i}) = k$. As $2k$ consecutive vertices are labelled alternatively, with 1 and -1.we get $e_f(1) = 2k - 1$ and $e_f(0) = 2k$.Therefore, f is a group difference cordial labeling of G .

Case (ii): $n \equiv 1 \pmod 4$, Let $n = 4k + 1, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if\ 1 \leq i \leq k+1 \\ \mathrm{i} & if\ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if\ 1 \leq i \leq k \\ -\mathrm{i} & if\ k+1 \leq i \leq 2k \end{cases}$$

Clearly $V_f(1) = k + 1$, $V_f(-1) = V_f(\mathrm{i}) = V_f(-\mathrm{i}) = k$. Also $e_f(1) = 2k = e_f(0)$. Therefore, f is a group difference cordial labeling of G.

Case (iii): $n \equiv 2 \pmod 4$, Let $n = 4k + 2, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if\ 1 \leq i \leq k+1 \\ \mathrm{i} & if\ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if\ 1 \leq i \leq k+1 \\ -\mathrm{i} & if\ k+2 \leq i \leq 2k+1 \end{cases}$$

Clearly $V_f(1) = k + 1 = V_f(-1)$, $V_f(\mathrm{i}) = V_f(-\mathrm{i}) = k$. Also $e_f(1) = 2k + 1$ and $e_f(0) = 2k$.

Therefore, f is a group difference cordial labeling of G .

Case (iv): $n \equiv 3 \pmod 4$,Let $n = 4k + 3, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if\ 1 \leq i \leq k+1 \\ \mathrm{i} & if\ k+2 \leq i \leq 2k+2 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if\ 1 \leq i \leq k+1 \\ -\mathrm{i} & if\ k+2 \leq i \leq 2k+1 \end{cases}$$

Clearly $V_f(1) = V_f(-1) = V_f(\mathrm{i}) = k + 1$, $V_f(-\mathrm{i}) = k$. Also $e_f(1) = 2k + 1 = e_f(0)$

Therefore, f is a group difference cordial labeling of G.

Theorem 2.3. The Ladder L_n is a group difference cordial graph if and only if n is odd, $n \geq 3$.

Proof: Assume $G = L_n$ is a group difference cordial graph and f is a group difference cordial labeling of G.

Claim: n is odd

Suppose if n is even, $n = 2k$, for $k \geq 1$ then by definition L_n has $4k$ vertices and $6k - 2$ edges. So $V_f(1) = V_f(-1) = V_f(\mathrm{i}) = V_f(-\mathrm{i}) = k$ for any group difference cordial labeling f.To get an edge $e = u_i u_{i+1}$ with label 1 we must have the labeling as $f(u_i) = 1$ or

$f(u_{i+1}) = -1$ and vice versa for $2k$ vertices. Therefore, the maximum number of edges that could be labelled with 1 are $3k-2$

So the number of edges that are labelled other than 1 is $3k$ (ie), $e_f(1) = 3k - 2$ and $e_f(0) = 3k$. which is a contradiction.

Conversely, assume n is odd, that is $n = 2k+1, k \geq 1$.Therefore $G = L_n$ has $4k+2$ vertices and $6k+1$ edges.Let $V(G) = \{u_1, u_2 \ldots u_n\}$.Define $f : V(G) \to \{1, -1, i, -i\}$ as follows

$$f(u_{2i-1}) = \begin{cases} 1 & if \ 1 \leq i \leq k+1 \\ i & if \ k+2 \leq i \leq 2k+1 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \ 1 \leq i \leq k+1 \\ -i & if \ k+2 \leq i \leq 2k+1 \end{cases}$$

Clearly $V_f(1) = k+1 = V_f(-1), V_f(i) = V_f(-i) = k$.Also $e_f(1) = 3k+1$ and $e_f(0) = 3k$.

Therefore, f is a group difference cordial labeling of G

Theorem 2.4. The Crown graph $C_n \odot k_1$ is a group difference cordial graph for every n, $n \geq 3$.

Proof: Let $G = C_n \odot k_1$ have $2n$ vertices and f be the group difference cordial labeling of G.

Let $V(G) = \{u_1, u_2 \ldots u_{2n}\}$. Define $f : V(G) \to \{1, -1, i, -i\}$ as follows

Case (i) : $n \equiv 0 \pmod 4$, Let $n = 4k, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \ 1 \leq i \leq 2k \\ i & if \ 2k+1 \leq i \leq 4k \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \ 1 \leq i \leq 2k \\ -i & if \ 2k+1 \leq i \leq 4k \end{cases}$$

Clearly $V_f(1) = 2k, V_f(-1) = V_f(i) = V_f(-i) = 2k$. Also $e_f(1) = 4k = e_f(0)$. Therefore, f is a group difference cordial labeling of G.

Case (ii): $n \equiv 1 \pmod 4$, Let $n = 4k+1, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \ 1 \leq i \leq 2k+1 \\ i & if \ 2k+2 \leq i \leq 4k+1 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \ 1 \leq i \leq 2k+1 \\ -i & if \ 2k+2 \leq i \leq 4k+1 \end{cases}$$

Clearly $V_f(1) = 2k+1 = V_f(-1)$, $V_f(i) = V_f(-i) = 2k$. Also $e_f(1) = 4k+1 = e_f(0)$.

Therefore, f is a group difference cordial labeling of G.

Case (iii): $n \equiv 2 \pmod 4$, Let $n = 4k+2, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \ 1 \leq i \leq 2k+1 \\ i & if \ 2k+2 \leq i \leq 4k+2 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \ 1 \leq i \leq 2k+1 \\ -i & if \ 2k+2 \leq i \leq 4k+1 \end{cases}$$

Clearly $V_f(1) = V_f(-1) = V_f(i) = V_f(-i) = 2k+1$. Also $e_f(1) = 4k+2 = e_f(0)$

Therefore, f is a group difference cordial labeling of G.

Case (iv): $n \equiv 3 \pmod 4$, Let $n = 4k+3, k \geq 1$.

$$f(u_{2i-1}) = \begin{cases} 1 & if \ 1 \leq i \leq 2k+2 \\ i & if \ 2k+3 \leq i \leq 4k+3 \end{cases}$$

$$f(u_{2i}) = \begin{cases} -1 & if \ 1 \leq i \leq 2k+2 \\ -i & if \ 2k+3 \leq i \leq 4k+3 \end{cases}$$

Clearly $V_f(1) = 2k+2 = V_f(-1), V_f(i) = V_f(-i) = 2k+1$.Also $e_f(1) = 4k+3 = e_f(0)$

Therefore, f is a group difference cordial labeling of G.

Theorem 2.5. The S_n is a group difference cordial graph if and only if $n \leq 6$.

Proof: Let $G = S_n$ have n vertices and f be the group difference cordial labeling of G.

Let $V(G) = \{u_1, u_2 \dots u_n\}$.

Suppose $n \leq 6$.The group difference cordial labeling of S_n is given in the following table

n	u_1	u_2	u_3	u_4	u_5	u_6
1	1					
2	1	−1				
3	1	−1	i			
4	1	−1	i	−i		
5	1	−1	i	−i	−1	
6	1	−1	i	−i	−1	1

Conversely suppose S_n is a group difference cordial graph .To prove $n \leq 6$.

Suppose $n > 6$.

Case (i): $n \equiv 0 \pmod 4$,

Let $n = 4k, k \geq 2$. So $V_f(1) = V_f(-1) = V_f(i) = V_f(-i) = k$ for any group difference cordial labeling. Let u_1 be the apex vertex that is assigned label 1 for all graph. Since-1 is assigned to k vertices,we get k edges with label 1 and remaining $3k - 1$ edges without label 1.(ie)$e_f(1) = k$ and $e_f(0) = 3k - 1$ which is a contradiction for $k \geq 2$.

Case (ii): $n \equiv 1 \pmod 4$

Let $n = 4k + 1, k \geq 2$. So $V_f(1) = V_f(-1) = V_f(i) = k, V_f(-i) = k + 1$ for any group difference cordial labeling. Here $k + 1$ vertices are assigned label -1 and so we get $k + 1$ edges with labeling 1 and remaining $3k - 1$ edges are labelled with labels other than 1. (ie)$e_f(1) = k + 1$ and $e_f(0) = 3k - 1$ which is a contradiction for $k \geq 2$.

Case (iii): $n \equiv 2 \pmod 4$

Let $n = 4k + 2, k \geq 2$. So $V_f(1) = V_f(-1) = k + 1, V_f(i) = V_f(-i) = k$ for any group difference cordial labeling. Here $k + 1$ vertices are assigned label-1 and so we get $k + 1$ edges with labeling 1 and remaining $3k$ edges as without 1 .(ie) $e_f(1) = k + 1$ and $e_f(0) = 3k$. which is a contradiction for $k \geq 2$.

Case (iv): $n \equiv 3 \pmod 4$

Let $n = 4k + 3, k \geq 1$. So $V_f(1) = V_f(-1) = V_f(i) = k + 1, V_f(-i) = k$ for any group difference cordial labeling. Here $k + 1$ vertices are assigned label -1 and so we get $k + 1$

edges with labeling 1 and remaining $3k+1$ edges as without 1 .(ie) $e_f(1) = k+1$ and $e_f(0) = 3k+1$ which is a contradiction for $k \geq 1$. Hence $G = S_n$ is a group difference cordial graph for $n \leq 6$.

Theorem2. 6. The Bistar $B_{n,n}$ is a group difference cordial graph for all 'n'.

Proof: Let $G = B_{n,n}$ have $2n+2$ vertices, f be the group difference cordial labeling of G.Let $V(G) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$. and $E(G) = \{uv, uu_i, vv_i : 1 \leq i \leq n\}$.

Clearly $B_{n,n}$ is a group difference cordial graph for $n \leq 3$.Assume $n \geq 4$ and fix u as 1 *and* v as -1 . Define $f : V(G) \to \{1, -1, \text{i}, -\text{i}\}$ *as* follows.

Case (i): $n \equiv 0 \pmod 4$, Let $n = 4k, k \geq 1$.

$$f(u_i) = \begin{cases} -1 & if \quad 1 \leq i \leq 2k \\ -\text{i} & if \quad 2k+1 \leq i \leq 4k \end{cases}$$

$$f(v_i) = \begin{cases} 1 & if \quad 1 \leq i \leq 2k \\ \text{i} & if \quad 2k+1 \leq i \leq 4k \end{cases}$$

Clearly $V_f(1) = 2k+1 = V_f(-1)$, $V_f(\text{i}) = V_f(-\text{i}) = 2k$.Also $e_f(1) = 4k+1$ and $e_f(0) = 4k$. Therefore, f is a group difference cordial labeling of G.

Case (ii): $n \equiv 1 \pmod 4$, Let $n = 4k+1, k \geq 1$.

$$f(u_i) = \begin{cases} -1 & if \quad 1 \leq i \leq 2k \\ -\text{i} & if \quad 2k+1 \leq i \leq 4k+1 \end{cases}$$

$$f(v_i) = \begin{cases} 1 & if \quad 1 \leq i \leq 2k \\ \text{i} & if \quad 2k+1 \leq i \leq 4k+1 \end{cases}$$

Clearly $V_f(1) = V_f(-1) = V_f(\text{i}) = V_f(-\text{i}) = 2k+1$. Also $e_f(1) = 4k+1$ and $e_f(0) = 4k+2$. Therefore f is a group difference cordial labeling of G.

Case (iii): $n \equiv 2 \pmod 4$, Let $n = 4k+2, k \geq 1$.

$$f(u_i) = \begin{cases} -1 & if \quad 1 \leq i \leq 2k+1 \\ -\text{i} & if \quad 2k+2 \leq i \leq 4k+2 \end{cases}$$

$$f(v_i) = \begin{cases} 1 & if \quad 1 \leq i \leq 2k+1 \\ \text{i} & if \quad 2k+2 \leq i \leq 4k+2 \end{cases}$$

Clearly $V_f(1) = 2k+2 = V_f(-1)$, $V_f(\text{i}) = V_f(-\text{i}) = 2k+1$.Also $e_f(1) = 4k+3$ and $e_f(0) = 4k+2$. Therefore, f is a group difference cordial labeling of G.

Case (iv): $n \equiv 3 \pmod 4$, Let $n = 4k+3, k \geq 1$.

$$f(u_i) = \begin{cases} -1 & if \quad 1 \leq i \leq 2k+1 \\ -\text{i} & if \quad 2k+2 \leq i \leq 4k+3 \end{cases}$$

$$f(v_i) = \begin{cases} 1 & if \quad 1 \leq i \leq 2k+1 \\ \text{i} & if \quad 2k+2 \leq i \leq 4k+3 \end{cases}$$

Clearly $V_f(1) = 2k+2 = V_f(-1)$, $V_f(\text{i}) = 2k+2 = V_f(-\text{i})$.Also $e_f(1) = 4k+3$ *and* $e_f(0) = 4k+4$. Therefore, f is a group difference cordial labeling of G.

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