Calculation of the Mean and Variance in the Birth-Death Immigration processes for M/M/1 Queueing Model

H. Saddam Hussain¹, A. Sridhar²
¹PG and Research Department of Mathematics, C. Abdul Hakeem College (Autonomous), Melvisharam-632509, Tamil Nadu, India
²PG and Research Department of Mathematics, Government Thirumagal Mill’s College, Gudiyattam-632602, Tamil Nadu, India

Abstract: This paper delineates how catastrophe affect the Birth-Death and Immigration Process (BDI). To find the general solution for the population size distribution, the Probability Generating Function (PGF) is used. When the rates of birth, death, immigration, and catastrophe are remain's constant, the accurate solution of mean and variance is also obtained. In addition, the mean and variance are time dependent. The system's behavior, a numerical example is presented for investigated.

Keywords: BDI, PGF, Catastrophe, Mean, Variance

1. Introduction

A lot of real-world phenomena, including queues, inventory, evolution, population biology, and epidemiology, can be modelled using birth-death processes. Although assuming that these rates are independent of time makes it easier to analyze the stochastic process that is being modelled, the above rates are frequently time-dependent in practical applications. In this note, a straightforward BDI process is taken into account. Total catastrophes, which happen occasionally and make the population disappear, have an impact on this process. The effects of different types of catastrophes on population processes had been examined by Granita [1], Getz [2], E.G. Kyriakidis et al. [4], Michael et al. [8], and Sindayigaya [3]. Some researchers discuss the recurring and progressive nature of the birth and death processes in random environments [8-10]. The main objective of this paper is to examine the BDI process using a catastrophe parameter and based on the DDE. Using a PGF, the general solution is obtained, which ultimately results in the calculating the population's mean and variance and, in the end, yields numerical examples.

2. The Transient probabilities:

The common argument demonstrates that \( P_\ell(t) \) satisfies using the forward Kolmogorov equations

\[
P_\ell'(t) = (\ell + 1) \mu P_{\ell+1}(t) + ((\ell - 1) \lambda + \nu) P_{\ell-1}(t) - (\ell (\mu + \lambda) + \nu + \gamma) P_\ell(t)
\]

(1)

\[
P_0'(t) = \mu P_1(t) - (\nu + \gamma) P_0(t)
\]

(2)

Letting

\[
G(S, t) = \sum_{\ell=0}^{\infty} P_\ell(t) S^\ell
\]

(3)

be the probability generating function \( \{ P_\ell(t) \} \).

So \( G(S, t) \) satisfies the linear partial differential equation:

\[
\frac{\partial G}{\partial t} = (\lambda S - \mu)(S-1)\frac{\partial G}{\partial S} + (\nu(S - 1) - \gamma) G
\]

(4)
For the purpose of simplicity, since we are thinking about a process involving immigration, the initial conditions \( P(0) = 1 \). This gives \( G(S, 0) = 1 \). The method of solving equation (4) is sketched and, in the broadest sense for constant parameters, yields a generating function for the distribution of population size at each time \( t \). This appears to be a novel outcome of the BDI with Catastrophe in this situation. Finally, the solution of (4) is

\[
G(S, t) = \left[ \frac{(\mu - \lambda S) + (\lambda S - 1) \exp[(\lambda - \mu) t]}{\mu - \lambda} \right] \sum_{n=0}^{\infty} m_n \left[ \frac{(\mu - \lambda S + \mu (\lambda S - 1) \exp[(\lambda - \mu) t]}{\mu - \lambda} \right] \] (5)

### 3. Calculating the Mean and Variance for the BDI

The mean and variance, which can be easily determined, are the two key distributional moments. Differentiating (3) with respect to \( S \) and again with respect to \( S \), then put \( S=1 \), we obtain

\[
\frac{\partial G}{\partial S} (1, t) = \sum_{n=0}^{\infty} \ell P_n(t) \pm E(\ell) (6)
\]

\[
\frac{\partial^2 G}{\partial S^2} (1, t) = \sum_{n=0}^{\infty} (\ell - 1) P_n(t) \pm E(\ell (\ell - 1)) (7)
\]

If the function is a probability distribution, then the first moment is the mean value \( \bar{k}(t) \) and second central moment is the variance \( c^2(t) \) at time \( t \).

Without initially solving for the distribution, it is straightforward to calculate the mean value and variance from (4). Differentiating (4) with respect to \( S \) and (6) with respect to \( t \), if \( S=1 \), then we have

\[
\frac{\partial^2 G}{\partial S \partial t} (1, t) = \frac{d \bar{k}(t)}{dt} \] (8)

Similarly using (7)

\[
\frac{\partial^3 G}{\partial S^2 \partial t} (1, t) = \frac{d}{dt} E(\ell (\ell - 1)) = \frac{d}{dt} E(\ell^2) - \frac{d}{dt} \bar{k}(t) \] (9)

Putting \( S=1 \) and differentiating (4) with respect to \( S \), we get

\[
\frac{d \bar{k}(t)}{dt} - (\lambda(t) - \mu(t) - \gamma(t)) \bar{k}(t) = \nu(t) (10)
\]

\[
\frac{d}{dt} E(\ell^2) - (2\lambda(t) - 2\mu(t) - \gamma(t)) E(\ell^2) = (2\nu(t) + \lambda(t) + \mu(t)) \bar{k}(t) + \nu(t) (11)
\]

Hence,

\[
c^2(t) = E(\ell^2) - \bar{k}^2(t) \] (12)

Applying the initial conditions to the equations (10) and (11), we have

\[
\bar{k}(0) = k_0 (13)
\]

\[
E(\ell^2)_{t=0} = c_0^2 + k_0^2 \] (14)

It is easy to solve these equations considering initial conditions.

Let

\[
\int_{\lambda(\tau)}^{\mu(\tau) - \gamma(\tau)} d\tau = r(t) \] (15)

The integrating factor for (15) is then \( e^{-r(t)} \), so that

\[
\bar{k}(t) = e^{r(t)} \left( \int_{0}^{t} \nu(\tau) e^{-r(\tau)} d\tau + \rho \right) \] (16)
and the initial conditions gives

$$\rho = 0 \quad (17)$$

To evaluate (16) the controls $v(t)$ form must be specified, and because of this flexibility, we are free to choose the controls that work best for us. If $v(t)$ and $y(t)$ are chosen as the constants and $\lambda(t)$ and $\mu(t)$ are both constants, we get

$$r(t) = (\lambda - \mu - y)t,$$

So that

$$\bar{k}(t) = \frac{v}{(\mu + y - \lambda)}(1 - \exp((\lambda - \mu - y)t))(18)$$

This is an explicit solution for the average time-varying trajectory of BDI processes with catastrophes. In a similar manner, using equation (12), solving (11) subject to (14), and constant parameters, we get

$$c^2(t) = -v^2(\exp((\lambda - \mu - y)t) - 1)^2 + \frac{2(v^2 + v^2 + v^2 + v^2 + v^2 + v^2)}{(2\lambda - 2\mu - y)(\mu + y - \lambda)}(\exp((2\lambda - 2\mu - y)t) - 1) \quad (19)$$

Equation (19) is an explicit solution for the time-varying process of dispersion of the BDI process with catastrophe.

4. Numerical Analysis:

Tables 1 to 16 contains the numerical results, and figures A to D show the corresponding graphs for each table.

**Table 1**

| $\lambda=1.5, \mu=1, \nu = 0.1, y = 0.01$ |
|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
| \( \bar{k}(t) \) | 0.129044 | 0.339685 | 0.683517 | 1.24476 | 2.16089 |

**Table 2**

| $\lambda=1.6, \mu=1, \nu = 0.1, y = 0.01$ |
|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
| \( \bar{k}(t) \) | 0.136269 | 0.382097 | 0.825568 | 1.62558 | 3.06881 |

**Table 3**

| $\lambda=1.7, \mu=1, \nu = 0.1, y = 0.01$ |
|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
| \( \bar{k}(t) \) | 0.144017 | 0.431145 | 1.0036 | 2.1449 | 4.42035 |

**Table 4**

| $\lambda=1.8, \mu=1, \nu = 0.1, y = 0.01$ |
|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
| \( \bar{k}(t) \) | 0.152329 | 0.487969 | 1.22752 | 2.85704 | 6.44751 |

**Table 5**

| $\lambda=1.5, \mu=1, \nu = 0.1, y = 0.42$ |
|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
| \( \bar{k}(t) \) | 0.104109 | 0.216889 | 0.339061 | 0.47141 | 0.614781 |

**Table 6**

| $\lambda=1.5, \mu=1, \nu = 0.1, y = 0.51$ |
|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
| \( \bar{k}(t) \) | 0.0995017 | 0.198013 | 0.295545 | 0.392106 | 0.487706 |

**Table 7**

| $\lambda=1.5, \mu=1, \nu = 0.1, y = 0.63$ |
|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
| \( \bar{k}(t) \) | 0.0937727 | 0.176114 | 0.248418 | 0.311907 | 0.367657 |
\[ \lambda = 1.5, \mu = 1, \nu = 0.1, \gamma = 0.73 \]

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k(t) )</td>
<td>0.089332</td>
<td>0.160311</td>
<td>0.216706</td>
<td>0.261513</td>
<td>0.29114</td>
<td></td>
</tr>
</tbody>
</table>

Table 9

\[ \lambda = 1.5, \mu = 1, \nu = 0.01, \gamma = 0.02 \]

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c^2(t) )</td>
<td>0.10142</td>
<td>0.37479</td>
<td>1.0997</td>
<td>3.02997</td>
<td>8.1714</td>
<td></td>
</tr>
</tbody>
</table>

Table 10

\[ \lambda = 1.6, \mu = 1, \nu = 0.01, \gamma = 0.02 \]

<table>
<thead>
<tr>
<th>t</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c^2(t) )</td>
<td>0.102346</td>
<td>0.434773</td>
<td>1.51552</td>
<td>5.03093</td>
<td>16.4688</td>
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Table 11

\[ \lambda = 1.7, \mu = 1, \nu = 0.01, \gamma = 0.02 \]

<table>
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<tr>
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<th>4</th>
<th>5</th>
</tr>
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<tbody>
<tr>
<td>( c^2(t) )</td>
<td>0.105117</td>
<td>0.522156</td>
<td>2.17837</td>
<td>8.75912</td>
<td>34.9134</td>
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Table 12

\[ \lambda = 1.8, \mu = 1, \nu = 0.01, \gamma = 0.02 \]

<table>
<thead>
<tr>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c^2(t) )</td>
<td>0.110174</td>
<td>0.644082</td>
<td>3.2342</td>
<td>15.8055</td>
<td>76.8344</td>
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</tbody>
</table>

Table 13

\[ \lambda = 1.5, \mu = 1, \nu = 0.01, \gamma = 0.02 \]

<table>
<thead>
<tr>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c^2(t) )</td>
<td>0.10242</td>
<td>0.37479</td>
<td>1.0997</td>
<td>3.02997</td>
<td>8.1714</td>
<td></td>
</tr>
</tbody>
</table>

Table 14

\[ \lambda = 1.5, \mu = 1, \nu = 0.01, \gamma = 0.02 \]

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</thead>
<tbody>
<tr>
<td>( c^2(t) )</td>
<td>0.101829</td>
<td>0.369938</td>
<td>1.07643</td>
<td>2.93905</td>
<td>7.85114</td>
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</tbody>
</table>

Table 15

\[ \lambda = 1.5, \mu = 1, \nu = 0.01, \gamma = 0.02 \]

<table>
<thead>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c^2(t) )</td>
<td>0.101241</td>
<td>0.365159</td>
<td>1.05372</td>
<td>2.85105</td>
<td>7.54405</td>
<td></td>
</tr>
</tbody>
</table>

Table 16
Figure (A) $\tilde{k}(t)$ versus $t$

Figure (B) $\tilde{k}(t)$ versus $t$
Figure (C) $c^2(t)$ versus $t$

Figure (D) $c^2(t)$ versus $t$
Conclusion:

This paper, using the PGF to obtain the general form for the partial differential equation (PDE) for the BDI processes with catastrophe. For BDI processes with catastrophe, the mean and variance functions have been accurately determined. A numerical illustrations is also provided to facilitate further investigation of the system's behavior.

References

[8] Michael L.Green, The Immigration-Emigration with Catastrophe Model,California State Polytechnic University, Pomona, CA91768.