Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 44 No. 4 (2023)

Calculation of the Mean and Variance in the Birth-Death Immigration process with Catastrophe for M\M\1 Queueing Model

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Abstract: - This paper, delineates how catastrophe affect the Birth-Death and Immigration Process (BDI). To find the general solution for the population size distribution, the Probability Generating Function (PGF) is used. When the rates of birth, death, immigration, and catastrophe are remain's constant, the explicit solution of mean and variance is also obtained. In addition, the mean and variance are time dependent. The system's behavior, a numerical analysis is presented for investigated.

Keywords: *BDI, PGF, Catastrophe, Mean, Variance.*

1. Introduction

A lot of real-world phenomena, including queues, inventory, evolution, population biology, and epidemiology, can be modeled using birth-death processes. Although assuming that these rates are independent of time makes it easier to analyze the stochastic process that is being modeled, the above rates are frequently time-dependent in practical applications. In this note, a straightforward BDI process is taken into account. A Catastrophe, which happen occasionally and make the population disappear, have an impact on this process. The effects of different types of catastrophes on population processes had been examined by Granita [1], Getz [2], E.G. Kyriakidis et al. [4], Michael et al. [8], and Sindayigaya [3]. Some researchers discuss the recurring and progressive nature of the birth and death processes in random environments [8-10]. The main objective of this paper is to examine the BDI process using a catastrophe parameter and based on the DDE. Using a PGF, the general solution is obtained, which ultimately results in the determination of the mean and variance of the population and, in the end, yields numerical examples.

Assumption and Postulates of the model

If R(t) represents the total number of members at time 't' and P_ℓ (t)= P_ℓ [R(t)=l /R(0)=0].
Birth and Death rates are proportional to population size, as in the straight forward birth and death processes with a birth rate $\lambda > 0$ and a death rate $\mu > 0$ with immigration rate $\nu > 0$ will happen regardless of the size of the population. Additionally, the probability of a catastrophe is affected by the size of the population and will occur at a rate $\nu > 0$.

2. The Transient probabilities:

The common argument demonstrates that $P_{\boldsymbol{\ell}}$(t) satisfies using the forward Kolmogorov equations

$$P'_{\boldsymbol{\ell}}(t) = (\boldsymbol{\ell} + 1)\,\mu\, P_{\boldsymbol{\ell}+1}(t) + ((\boldsymbol{\ell} - 1)\,\lambda + \nu)P_{\boldsymbol{\ell}-1}(\boldsymbol{t}) - (\boldsymbol{\ell}\,(\mu + \lambda) + \nu + \gamma)\,\boldsymbol{P}_{\boldsymbol{\ell}}\,(t) \tag{1}$$

$$P'_0(t) = \mu P_1(t) - (\nu + \gamma)\ P_0(t) \tag{2}$$

Letting

$$G(S,t) = \sum_{\boldsymbol{\ell}=0}^{\infty} P_j(t) S^{\boldsymbol{\ell}} \tag{3}$$

be the probability generating function $\{P_{\ell}(t)\}$.
So $G(S,t)$ satisfies the linear partial differential equation:

$$\frac{\partial G}{\partial t} = (\lambda S - \mu)(S-1)\frac{\partial G}{\partial S} + (\nu(S-1) - \gamma)\, G \tag{4}$$

For the purpose of simplicity, since we are consider a process involving immigration, the initial conditions $P_0(0) = 1$. This gives $G(S, 0) = 1$. The method of solving equation (4) is sketched and, in the broadest sense for constant parameters, yields a generating function for the distribution of population size at each time t. This appears to be a novel outcome of the BDI with Catastrophe in this situation. Finally, the solution of (4) is

$$G(S,t) = \left[\frac{(\mu-\lambda S)+\lambda(S-1)\exp\{(\lambda-\mu)t\}}{\mu-\lambda}\right]^{\frac{-\nu}{\lambda}} \left\{\sum_{\ell=0}^{\infty} m_{\ell}\left[\frac{\mu-\lambda S+\mu(S-1)\exp\{(\lambda-\mu)t\}}{\mu-\lambda S+\lambda(S-1)\exp\{(\lambda-\mu)t\}}\right]^{\ell}\right\}. \tag{5}$$

3. Determination of the Mean and Variance for the BDI

The mean and variance, which can be easily determined, are the two key distributional moments.
Differentiating (3) with respect to S and again with respect to S, then put S=1, we obtain

$$\frac{\partial G}{\partial S}(1,t) = \sum_{\ell=0}^{\infty} \ell P_{\ell}(t) \triangleq E(\ell) \tag{6}$$

$$\frac{\partial^2 G}{\partial S^2}(1,t) = \sum_{\ell=0}^{\infty} \ell(\ell-1)P_{\ell}(t) \triangleq E(\ell(\ell-1)) \tag{7}$$

If the function is a probability distribution, then the first moment is the mean value $\bar{k}(t)$ and second central moment is the variance $c^2(t)$ at time t.
Without initially solving for the distribution, it is straightforward to calculate the mean value and variance from (4). Differentiating (4) with respect to S and (6) with respect to t.
If S=1, then we have

$$\frac{\partial^2 G}{\partial S \partial t}(1,t) = \frac{d\bar{k}(t)}{dt} \tag{8}$$

Similarly, using (7)

$$\frac{\partial^3 G}{\partial S^2 \partial t}(1,t) = \frac{d}{dt}E(\ell(\ell-1)) = \frac{d}{dt}E(\ell^2) - \frac{d}{dt}\bar{k}(t) \tag{9}$$

Putting S=1 and differentiating (4) with respect to S, we get

$$\frac{d\bar{k}(t)}{dt} - (\lambda(t) - \mu(t) - \gamma(t))\bar{k}(t) = \nu(t) \tag{10}$$

$$\frac{d}{dt}E(\ell^2) - (2\lambda(t) - 2\mu(t) - \gamma(t))E(\ell^2) = (2\nu(t) + \lambda(t) + \mu(t))\bar{k}(t) + \nu(t) \tag{11}$$

Hence,

$$c^2(t) = E(\ell^2) - \bar{k}^2(t) \tag{12}$$

Applying the initial conditions to the equations (10) and (11), we have

$$\bar{k}(0) = k_0 \tag{13}$$

$$E(\ell^2)|_{t=0} = c_0^2 + k_0^2 \tag{14}$$

It is easy to solve these equations considering initial conditions.
Let

$$\int_0^t (\lambda(\tau) - \mu(\tau) - \gamma(\tau))d\tau = r(t) \tag{15}$$

The integrating factor for (15) is then $e^{-r(t)}$, so that

$$\bar{k}(t) = e^{r(t)}\left(\int_0^t \nu(\tau)\, e^{-r(t)} d\tau + \rho\right) \tag{16}$$

and the initial conditions gives

$$\rho = 0 \tag{17}$$

If v (t) and γ(t) are chosen as the constants and λ(t) and μ(t) are both constants, we get

$$r(t) = (\lambda - \mu - \gamma)t,$$

So that

$$\bar{k}(\mathrm{t}) = \frac{v}{(\mu+\gamma-\lambda)}(1 - \exp\{(\lambda - \mu - \gamma)t\}) \tag{18}$$

This is an explicit solution for the average time-varying trajectory of BDI process with catastrophe.
In a similar manner, using equation (12), solving (11) subject to (14), and constant parameters, we get

$$c^2(t) = \frac{-v^2(\exp\{(\lambda-\mu-\gamma)t\}-1)^2}{(\lambda-\mu-\gamma)^2} + \frac{2(v\lambda\gamma+v\lambda\mu-v^2\lambda+v^2\mu+v^2\gamma-v\lambda^2)}{(2\lambda-2\mu-\gamma)(\mu+\gamma-\lambda)(\lambda-\mu)}(\exp\{(2\lambda-2\mu-\gamma)t\}-1) \tag{19}$$

Equation (19) is an explicit solution for the time-varying process of dispersion of the BDI process with catastrophe.

4. Numerical Analysis:

Tables: 1 to 16 contains the numerical results, and figures A to D shows the corresponding graphs for each table.

Table 1

λ=1.5, μ=1, $v = 0.1, \gamma = 0.01$

t	0	1	2	3	4	5
$\bar{k}$(t)	0	0.129044	0.339685	0.683517	1.24476	2.16089

Table 2

λ=1.6, μ=1, $v = 0.1, \gamma = 0.01$

t	0	1	2	3	4	5
$\bar{k}$(t)	0	0.136269	0.382097	0.825568	1.62558	3.06881

Table 3

λ=1.7, μ=1, $v = 0.1, \gamma = 0.01$

t	0	1	2	3	4	5
$\bar{k}$(t)	0	0.144017	0.431145	1.0036	2.1449	4.42035

Table 4

λ=1.8, μ=1, $v = 0.1, \gamma = 0.01$

t	0	1	2	3	4	5
$\bar{k}$(t)	0	0.152329	0.487969	1.22752	2.85704	6.44751

Table 5

λ=1.5, μ=1, $v = 0.1, \gamma = 0.42$

t	0	1	2	3	4	5
$\bar{k}$(t)	0	0.104109	0.216889	0.339061	0.47141	0.614781

Table 6

λ=1.5, μ=1, $v = 0.1, \gamma = 0.51$

t	0	1	2	3	4	5
$\bar{k}$(t)	0	0.0995017	0.198013	0.295545	0.392106	0.487706

Table 7

λ=1.5, μ=1, $v = 0.1, \gamma = 0.63$

t	0	1	2	3	4	5
$\bar{k}$(t)	0	0.0937727	0.176114	0.248418	0.311907	0.367657

Table 8

λ=1.5, μ=1, $v = 0.1, \gamma = 0.73$

t	0	1	2	3	4	5
$\bar{k}$(t)	0	0.089332	0.160311	0.216706	0.261513	0.29114

Table 9

λ=1.5, μ=1, $v = 0.01, \gamma = 0.02$

t	0	1	2	3	4	5
$c^2(t)$	0	0.10142	0.37479	1.0997	3.02997	8.1714

Table 10

$\boldsymbol{\lambda}$**=1.6,** $\boldsymbol{\mu}$**=1,** $\boldsymbol{\nu = 0.01, \gamma = 0.02}$

t	0	1	2	3	4	5
$c^2(t)$	0	0.102346	0.434773	1.51552	5.03093	16.4688

Table 11

$\boldsymbol{\lambda}$**=1.7,** $\boldsymbol{\mu}$**=1,** $\boldsymbol{\nu = 0.01, \gamma = 0.02}$

t	0	1	2	3	4	5
$c^2(t)$	0	0.105117	0.522156	2.17837	8.75912	34.9134

Table 12

$\boldsymbol{\lambda}$**=1.8,** $\boldsymbol{\mu}$**=1,** $\boldsymbol{\nu = 0.01, \gamma = 0.02}$

t	0	1	2	3	4	5
$c^2(t)$	0	0.110174	0.644082	3.2342	15.8055	76.8344

Table 13

$\boldsymbol{\lambda}$**=1.5,** $\boldsymbol{\mu}$**=1,** $\boldsymbol{\nu = 0.01, \gamma = 0.02}$

t	0	1	2	3	4	5
$c^2(t)$	0	0.10242	0.37479	1.0997	3.02997	8.1714

Table 14

$\boldsymbol{\lambda}$**=1.5,** $\boldsymbol{\mu}$**=1,** $\boldsymbol{\nu = 0.01, \gamma = 0.03}$

t	0	1	2	3	4	5
$c^2(t)$	0	0.101829	0.369938	1.07643	2.93905	7.85114

Table 15

$\boldsymbol{\lambda}$**=1.5,** $\boldsymbol{\mu}$**=1,** $\boldsymbol{\nu = 0.01, \gamma = 0.04}$

t	0	1	2	3	4	5
$c^2(t)$	0	0.101241	0.365159	1.05372	2.85105	7.54405

Table 16

$\boldsymbol{\lambda}$**=1.5,** $\boldsymbol{\mu}$**=1,** $\boldsymbol{\nu = 0.01, \gamma = 0.05}$

t	0	1	2	3	4	5
$c^2(t)$	0	0.100658	0.360451	1.03153	2.76588	7.24957

Figure (A) depicts a comparison between the mean system size $\bar{k}$(t) versus time t for different arrival rate λ with the identical values for the parameters. Based on the graphical illustration, the mean system size of the queue grows as the arrival's rate grows.

Figure (B) depicts a comparison of the mean system size $\bar{k}$(t) versus time t with varying values of ν with the same values for the parameters. When the value of ν increases, the mean system size of the queue rapidly decreases.

Figure (C) depicts a comparison of the variance system size C^2(t) and time t with different arrival rate λ with the same parameter values. It clearly displays that as variance of number of customers in the system increases when arrival rate increases.

Figure (D) depicts a comparison of the variance system size C^2(t) versus time t with varying values of ν and the identical values for the parameters. When the value of ν increases, the variance system's size of the queue rapidly decreases.

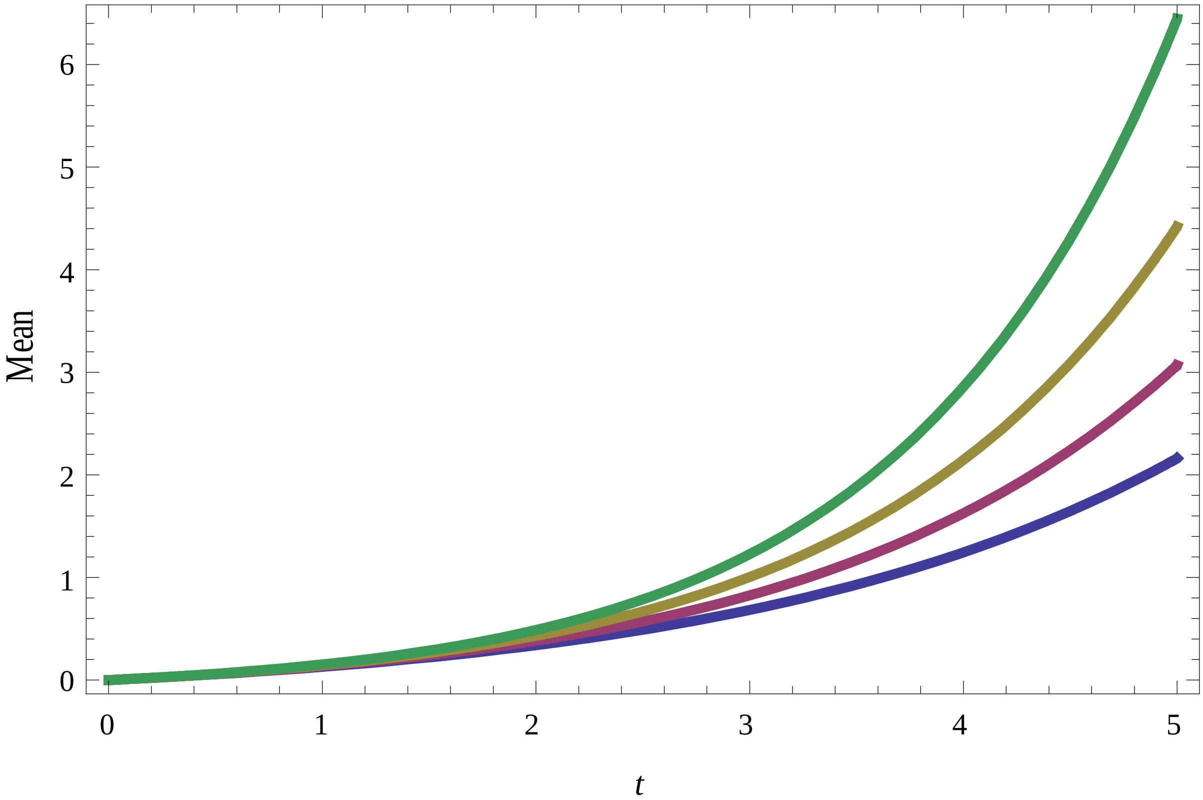

Figure (A)- $\bar{k}(t)$ versus t

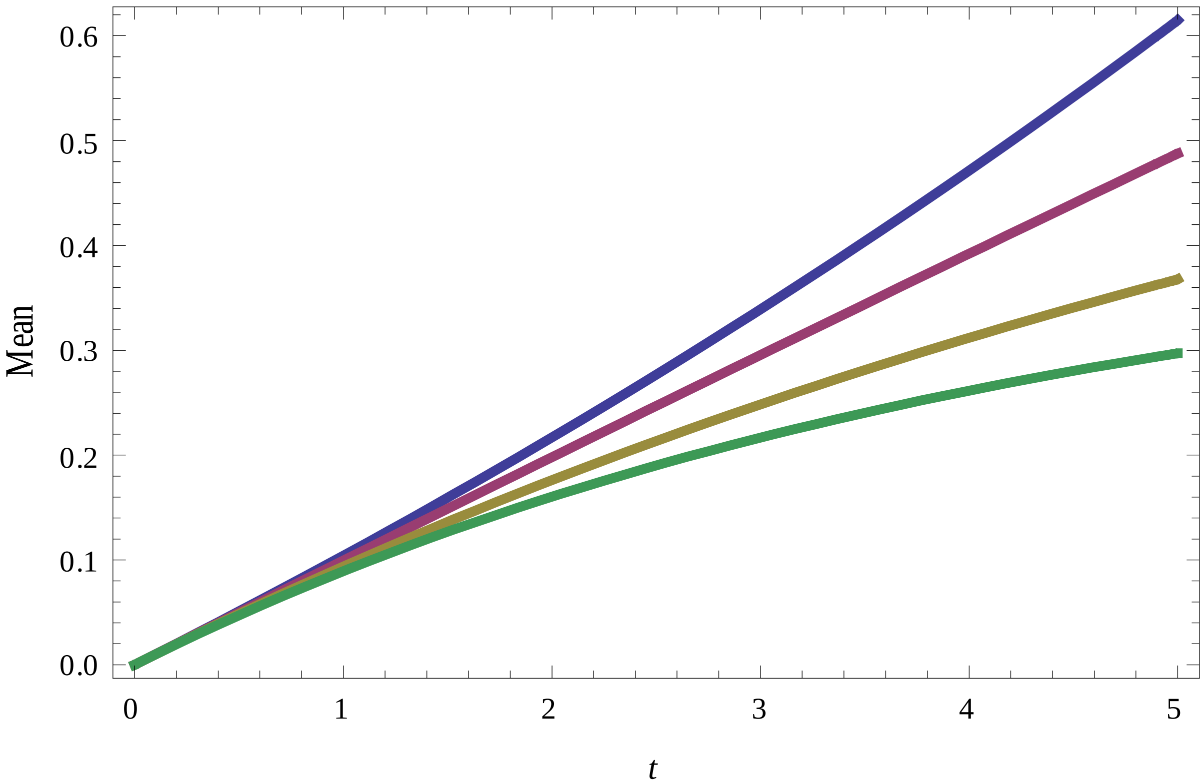

Figure (B) - $\bar{k}(t)$ versus t

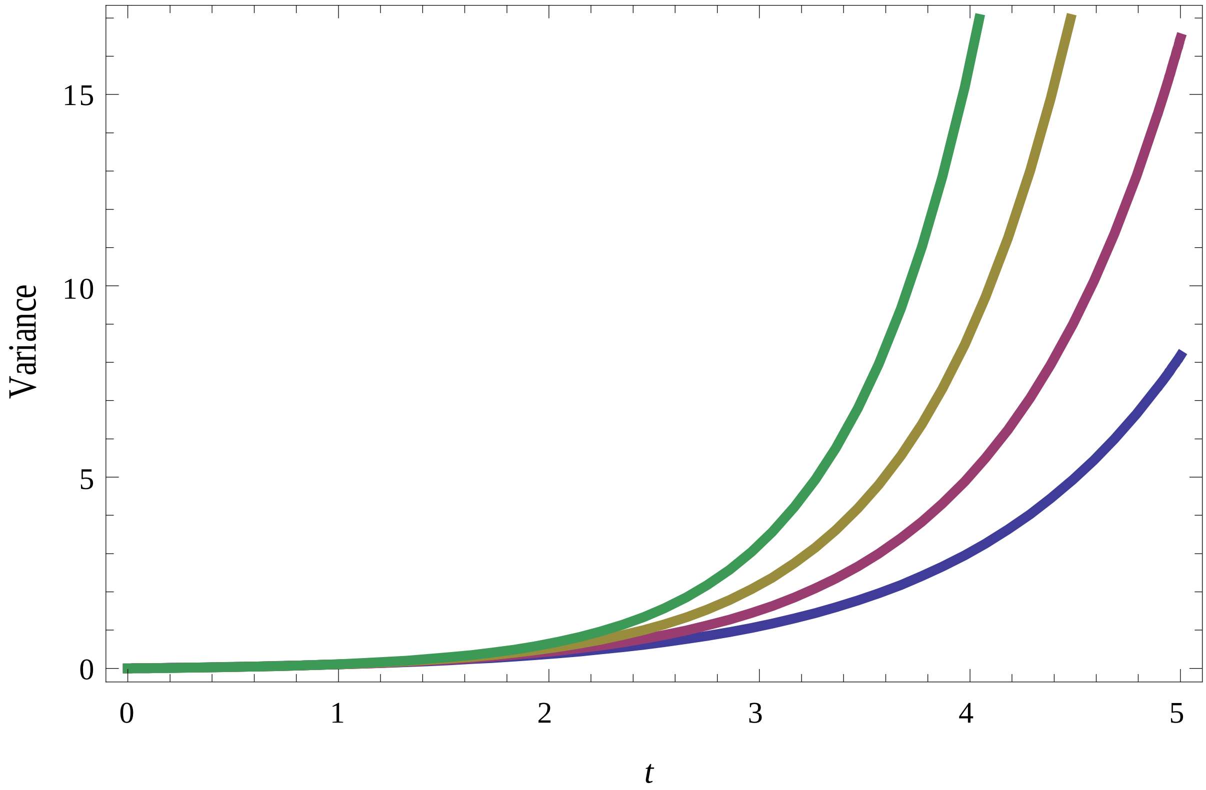

Figure (C)- $c^2(t)$ versus t

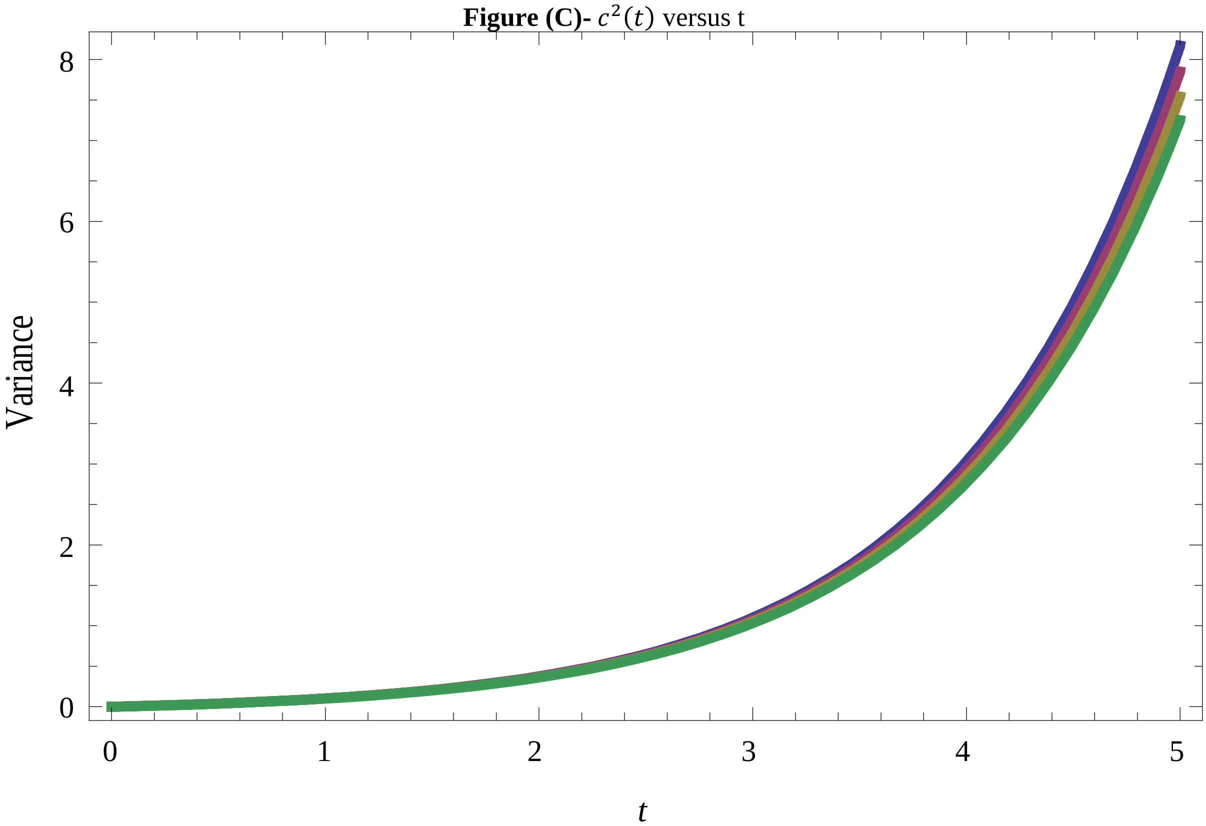

Figure (D)- $c^2(t)$ versus t

Conclusion:

This paper, using the PGF to obtain the general form for the partial differential equation (pde) for the BDI process with catastrophe. For BDI process with catastrophe, the mean and variance functions have been accurately determined. A numerical analysis is also provided to facilitate further investigation of the system's behavior.

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