Level Set of Direct Product of Intuitionistic Fuzzy BG-ideals in BG-algebra


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Abstract: In this paper, we investigate some properties of level set of direct product of intuitionistic fuzzy BG-ideals in BG-algebra.

Keywords: BG-algebra, fuzzy BG-ideal, intuitionistic fuzzy BG-ideal, direct product of intuitionistic fuzzy BG-ideals, level set.

1. Introduction:


2. Preliminaries

Definition: 2.1

A BG-algebra is a non empty set X with a constant 0 and a binary operation “∗” satisfying the following axioms:

(i) \( x \ast x = 0 \)

(ii) \( x \ast 0 = x \)

(iii) \( (x \ast y) \ast (0 \ast y) = x \forall x, y \in X \).

For brevity we also call X BG-algebra. A binary relation ‘≤’ on X can be defined by \( x \leq y \) if and only if \( x \ast y = 0 \).

A non-empty set S of a BG-algebra X is called a BG-subalgebra of X if \( x \ast yeS \forall x, yeS \).


Definition: 2.2
A fuzzy set \( \mu \) in \( X \) is called a fuzzy BG-ideal of \( X \) if it satisfies the following condition:

(i) \( \mu(0) \geq \mu(x) \)
(ii) \( \mu(x) \geq \min\{\mu(x \ast y), \mu(y)\} \forall x, y \in X \).

Definition: 2.3
If \( A \times B = (\mu_{AXB}, \gamma_{AXB}) \) is an intuitionistic fuzzy sets of BG-Algebra \( X \times Y \) is said to be a intuitionistic fuzzy BG-ideal of \( X \times Y \) if it satisfies the following axioms

(i) \( \mu_{AXB}(0,0) \geq \mu_{AXB}(x_1, y_1) \)
(ii) \( \mu_{AXB}(x_1, y_1) \geq \min\{\mu_{AXB}((x_1, y_1) \ast (x_2, y_2)), \mu_{AXB}((x_2, y_2))\} \)
(iii) \( \mu_{AXB}(x_1, y_1) \ast (x_2, y_2) \geq \min\{\mu_{AXB}((x_1, y_1)), \mu_{AXB}(x_2, y_2)\} \)
(iv) \( \gamma_{AXB}(0,0) \leq \gamma_{AXB}(x_1, y_1) \)
(v) \( \gamma_{AXB}(x_1, y_1) \leq \max\{\gamma_{AXB}((x_1, y_1) \ast (x_2, y_2)), \gamma_{AXB}(x_2, y_2)\} \)
(vi) \( \gamma_{AXB}((x_1, y_1) \ast (x_2, y_2)) \leq \max\{\gamma_{AXB}((x_1, y_1)), \gamma_{AXB}(x_2, y_2)\} \forall x_1, x_2, y_1, y_2 \in X \).

Definition: 2.4
Let \( A = (\mu_A, \gamma_A) \) and \( B = (\mu_B, \gamma_B) \) be intuitionistic fuzzy sets in \( X \) and \( Y \) respectively. Then the direct product of intuitionistic fuzzy sets \( A \) and \( B \) is defined by \( A \times B = (\mu_{AXB}, \gamma_{AXB}) \) where \( \mu_{AXB} : X \times Y \rightarrow [0,1] \) is given by

\[
\mu_{AXB}(x,y) = \min\{\mu_A(x), \mu_B(y)\} \quad \text{and} \quad \gamma_{AXB} : X \times Y \rightarrow [0,1] \text{ is given by}
\]

\[
\gamma_{AXB}(x,y) = \max\{\gamma_A(x), \gamma_B(y)\} \text{ for all } (x,y) \in X \times Y.
\]

3. Level Set of Direct Product of Intuitionistic Fuzzy BG-Ideals
Let \( A \times B = (\mu_{AXB}, \gamma_{AXB}) \) be a intuitionistic BG-ideal of a BG-algebra \( X \times Y \) and \( \alpha, \beta \in [0,1] \) then

\[
\alpha - \text{level cut of } \mu \quad \text{and} \quad \beta - \text{level cut of } \gamma
\]

of \( A \times B \) is as follows

\[
\mu_{AXB,\alpha} = \{(x,y) \in X \times Y / \mu_{AXB}(x,y) \geq \alpha\} \quad \text{and} \quad \gamma_{AXB,\beta} = \{(x,y) \in X \times Y / \gamma_{AXB}(x,y) \leq \beta\}
\]

Theorem 3.1
If \( A \times B = (\mu_{AXB}, \gamma_{AXB}) \) is a intuitionistic fuzzy BG-ideal of \( X \times Y \), then \( \mu_{AXB,\alpha} \) and \( \gamma_{AXB,\beta} \) are BG-ideal of \( X \times Y \) for any \( \alpha, \beta \in [0,1] \).

Solution:
Let \( A \times B = (\mu_{AXB}, \gamma_{AXB}) \) be a intuitionistic fuzzy BG-ideal of \( X \times Y \) and 
Let \( \alpha \in [0,1] \).

Then we have

(i) \( \mu_{AXB}(0,0) \geq \mu_{AXB}(x,y) \forall (x,y) \in X \times Y \)

By definition, \( \mu_{AXB}(x,y) \geq \alpha \forall (x,y) \in \mu_{AXB,\alpha} \)

So \( \mu_{AXB}(0,0) \geq \alpha \)

Therefore \( (0,0) \in \mu_{AXB,\alpha} \).

(ii) Let \( (x_1,y_1), (x_2,y_2) \in X \times Y \) be such that \( (x_1,y_1) \ast (x_2,y_2) \in \mu_{AXB,\alpha} \)

and \( (x_2,y_2) \in \mu_{AXB,\alpha} \).

Then \( \mu_{AXB}((x_1,y_1) \ast (x_2,y_2)) \geq \alpha \)

\( \mu_{AXB}((x_2,y_2)) \geq \alpha \)

Since \( \mu_{AXB} \) is a intuitionistic fuzzy BG-ideal of \( X \times Y \) it follows that

\( \mu_{AXB}((x_1,y_1) \ast (x_2,y_2)) \geq \min \{\mu_{AXB}(x_1,y_1), \mu_{AXB}(x_2,y_2)\} \)

\( \geq \min \{\alpha, \alpha\} \)

\( \geq \alpha \)

Therefore \( \mu_{AXB}((x_1,y_1) \ast (x_2,y_2)) \geq \alpha \).
Hence $(x_1, y_1) * (x_2, y_2) \in X \times Y$

Therefore $\mu_{AXB,a}$ is a intuitionistic fuzzy BG-ideal in BG-algebra.

(iii) Clearly, $\mu_{AXB}$ is a intuitionistic fuzzy BG-ideal of $X \times Y$ it follows that

$\mu_{AXB}((x, y)) \leq \min\{\mu_{AXB}(x, y_1) \times (x_2, y_2)), \mu_{AXB}(x_2, y_2)\}$

$\geq \min\{\alpha, \alpha\}$

$\geq \alpha$

Hence $\mu_{AXB,a}$ is a intuitionistic fuzzy BG-ideal in BG-algebra.

Similarly,

let $\beta \in [0, 1]

Also we have (iv) $\gamma_{AXB}(0, 0) \leq \gamma_{AXB}(x, y) \in X \times Y$

By definition, $\gamma_{AXB}(x, y) \leq \alpha \ \forall (x, y) \in \mu_{AXB,a}$

So $\gamma_{AXB}(0, 0) \leq \alpha$

Therefore $(0, 0) \in \gamma_{AXB,b}$

(v) Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ be such that $(x_1, y_1) \times (x_2, y_2) \in \gamma_{AXB,b}$

and $(x_2, y_2) \in \gamma_{AXB,b}$

Then $\gamma_{AXB}((x_1, y_1) \times (x_2, y_2)) \leq \beta$

$\gamma_{AXB}(x_2, y_2) \leq \beta$

Since $\gamma_{AXB}$ is a intuitionistic fuzzy BG-ideal of $X \times Y$ it follows that

$\gamma_{AXB}((x_1, y_1) \times (x_2, y_2)) \leq \max\{\gamma_{AXB}(x_1, y_1), \gamma_{AXB}(x_2, y_2)\}$

$\leq \max\{\beta, \beta\}$

$\leq \beta$

Therefore $\gamma_{AXB}((x_1, y_1) \times (x_2, y_2)) \leq \beta$

Hence $(x_1, y_1) \times (x_2, y_2) \in X \times Y$

Therefore $\gamma_{AXB,b}$ is a intuitionistic fuzzy BG-ideal in BG-algebra.

(iii) Clearly, $\gamma_{AXB}$ is a intuitionistic fuzzy BG-ideal of $X \times Y$ it follows that

$\gamma_{AXB}((x, y)) \leq \max\{\gamma_{AXB}(x, y_1) \times (x_2, y_2)), \gamma_{AXB}(x_2, y_2)\}$

$\leq \max\{\beta, \beta\}$

$\leq \beta$

Therefore $\gamma_{AXB,b}$ is a intuitionistic fuzzy BG-ideal in BG-algebra.

Hence $\mu_{AXB,a}$ and $\gamma_{AXB,b}$ are intuitionistic fuzzy BG-ideals in BG-algebra.

**Theorem 3.2**

An intuitionistic fuzzy set $A \times B = (\mu_{AXB}, \gamma_{AXB})$ is a intuitionistic fuzzy BG-ideal of $X \times Y$ iff for all $\alpha, \beta \in [0, 1], \mu_{AXB,a}$ and $\gamma_{AXB,b}$ are either empty or BG-ideals of $X \times Y$.

Solution:

Assume that $\mu_{AXB,a}$ and $\gamma_{AXB,b}$ are either empty or BG-ideals of $X \times Y$ for $\alpha, \beta \in [0, 1]$

For any $(x, y) \in X \times Y$

(i) Let $\mu_{AXB}(x, y) = \alpha$ and $\gamma_{AXB}(x, y) = \beta$

Then $(x, y) \in \mu_{AXB,a}$ and $\gamma_{AXB,b}$, so $\mu_{AXB,a} \neq \emptyset \neq \gamma_{AXB,b}$

Since $\mu_{AXB,a}$ and $\gamma_{AXB,b}$ are BG-ideals of $X \times Y$

Therefore $(0, 0) \in \mu_{AXB,a}$ and $\gamma_{AXB,b}$

Hence $\mu_{AXB}(0, 0) \geq \alpha$

Also $\gamma_{AXB}(0, 0) \leq \beta$

$\gamma_{AXB}(x, y) \in X \times Y$

Hence condition (i) satisfy

(ii) If there exist $(x_1, y_1), (x_2, y_2) \in X \times Y$ be such that

$\mu_{AXB}((x_1, y_1)) < \min\{\mu_{AXB}(x_1, y_1) \times (x_2, y_2)), \mu_{AXB}(x_2, y_2)\}$

$\gamma_{AXB}((x_1, y_1)) < \min\{\gamma_{AXB}(x_1, y_1) \times (x_2, y_2)), \gamma_{AXB}(x_2, y_2)\}$
Then by taking
\[ \alpha_0 = \frac{1}{2} (\mu_{AXB}(x_1, y_1)) + \min \{ \mu_{AXB}((x_1, y_1) \ast (x_2, y_2)), \mu_{AXB}(x_2, y_2) \} \]
We have \( \mu_{AXB}(x_1, y_1) < \alpha_0 < \min \{ \mu_{AXB}((x_1, y_1) \ast (x_2, y_2)), \mu_{AXB}(x_2, y_2) \} \)
Hence \((x_1, y_1) \not\in \mu_{AXB, \alpha_0} \)
\( (x_1, y_1) \ast (x_2, y_2) \in \mu_{AXB, \alpha_0} \) and \((x_2, y_2) \in \mu_{AXB, \alpha_0} \)
That is, \( \mu_{AXB, \alpha_0} \) is not a BG-ideals of \( X \times Y \).
Which is a contradiction
Therefore
\[ \mu_{AXB}(x_1, y_1) \geq \min \{ \mu_{AXB}(x_1, y_1) \ast (x_2, y_2), \mu_{AXB}(x_2, y_2) \} \forall (x_1, y_1), (x_2, y_2) \in X \times Y \]
Similiarly,
By taking
\[ \beta_0 = \frac{1}{2} (\gamma_{AXB}(x_1, y_1)) + \max \{ \gamma_{AXB}(x_1, y_1) \ast (x_2, y_2)), \gamma_{AXB}(x_2, y_2) \} \]
We have \( \gamma_{AXB}(x_1, y_1) > \beta_0 > \max \{ \gamma_{AXB}(x_1, y_1) \ast (x_2, y_2)), \gamma_{AXB}(x_2, y_2) \} \)
Hence \((x_1, y_1) \not\in \gamma_{AXB, \beta_0} \)
\( (x_1, y_1) \ast (x_2, y_2) \in \gamma_{AXB, \beta_0} \) and \((x_2, y_2) \in \gamma_{AXB, \beta_0} \)
That is, \( \gamma_{AXB, \beta_0} \) is not a BG-ideals of \( X \times Y \).
Which is a contradiction
Therefore
\[ \gamma_{AXB}(x_1, y_1) \leq \max \{ \gamma_{AXB}(x_1, y_1) \ast (x_2, y_2)), \gamma_{AXB}(x_2, y_2) \} \forall (x_1, y_1), (x_2, y_2) \in X \times Y. \]
(iii) Clearly
\[ \mu_{AXB}(x_1, y_1) \ast (x_2, y_2) \geq \min \{ \mu_{AXB}(x_1, y_1), \mu_{AXB}(x_2, y_2) \} \forall (x_1, y_1), (x_2, y_2) \in X \times Y \]
Similiarly,
\[ \gamma_{AXB}(x_1, y_1) \ast (x_2, y_2) \leq \max \{ \gamma_{AXB}(x_1, y_1), \gamma_{AXB}(x_2, y_2) \} \]
\forall (x_1, y_1), (x_2, y_2) \in X \times Y. \]
Conversely,
Assume \( A \times B = (\mu_{AXB}, \gamma_{AXB}) \) is a intuitionistic fuzzy BG-ideal of \( X \times Y \)
To prove: \( \mu_{AXB, \alpha} \) and \( \gamma_{AXB, \beta} \) are either empty or BG-ideals of \( X \times Y \)
Suppose that \( \mu_{AXB, \alpha}, \gamma_{AXB, \beta} \neq \emptyset \) for any \( \alpha, \beta \in [0,1) \)
It is clear that \((0,0) \in \mu_{AXB, \alpha} \) and \( \gamma_{AXB, \beta} \)
Since \( \mu_{AXB}(0,0) \geq \mu_{AXB}(x, y) \geq \alpha \)
Also \( \gamma_{AXB}(0,0) \leq \gamma_{AXB}(x, y) \leq \beta \)
(ii) Let \((x_1, y_1) \ast (x_2, y_2) \in \mu_{AXB, \alpha} \) and \((x_2, y_2) \in \mu_{AXB, \alpha} \)
\[ \mu_{AXB}((x_1, y_1) \ast (x_2, y_2)) \geq \alpha \]
\[ \mu_{AXB}((x_2, y_2)) \geq \alpha \]
Therefore \((x_1, y_1) \in \mu_{AXB, \alpha} \)
Hence \( \mu_{AXB, \alpha} \) are BG-ideals of \( X \times Y \).
Also \((x_1, y_1) \ast (x_2, y_2) \in \gamma_{AXB, \beta} \) and \((x_2, y_2) \in \gamma_{AXB, \beta} \)
\[ \gamma_{AXB}((x_1, y_1) \ast (x_2, y_2)) \leq \beta \]
\[ \gamma_{AXB}((x_2, y_2)) \leq \beta \]
Therefore \((x_1, y_1) \in \gamma_{AXB, \beta} \)
Hence \( \gamma_{AXB, \beta} \) are BG-ideals of \( X \times Y \).
Therefore \( \mu_{AXB, \alpha} \) and \( \gamma_{AXB, \beta} \) are BG-ideals of \( X \times Y \).
(iii) Clearly,
\[
\mu_{AXB}(x_1, y_1) \geq \alpha \quad \text{and} \quad \mu_{AXB}(x_2, y_2) \geq \alpha
\]
That is \(\mu_{AXB}(x_1, y_1) \ast (x_2, y_2) \in \mu_{AXB, \alpha}\)
Hence \(\mu_{AXB, \alpha}\) are BG-ideals of \(X \times Y\).

Similarly,
\[
\gamma_{AXB}(x_1, y_1) \ast (x_2, y_2) \leq \max \{\gamma_{AXB}(x_1, y_1), \gamma_{AXB}(x_2, y_2)\}
\]
Therefore \(\gamma_{AXB, \beta}\) are BG-ideals of \(X \times Y\).
Hence \(\mu_{AXB, \alpha}\) and \(\gamma_{AXB, \beta}\) are intuitionistic fuzzy BG-ideals of \(X \times Y\).

**Theorem 3.3**

For any intuitionistic fuzzy set \(A \times B = (\mu_{AXB}, \gamma_{AXB})\) is an intuitionistic fuzzy BG-ideal of \(X \times Y\) iff the non-empty upper \(\alpha\) – level cut \(\mu_{AXB}: \alpha\) and the non-empty lower \(\beta\) – level cut of \(\gamma_{AXB}: \beta\) are ideals of \(X \times Y\) for any \(\alpha, \beta \in [0, 1]\)

**Solution:**

Let \(A \times B = (\mu_{AXB}, \gamma_{AXB})\) be an intuitionistic fuzzy BG-ideal of \(X \times Y\).

(i) \(\mu_{AXB}(0,0) \geq \mu_{AXB}((x_1, y_1))\) and
\[
\gamma_{AXB}(0,0) \leq \gamma_{AXB}((x_1, y_1)) \forall (x_1, y_1) \in X \times Y
\]
(ii) \(\mu_{AXB}(x_1, y_1) \geq \min \{\mu_{AXB}((x_1, y_1) \ast (x_2, y_2)), \mu_{AXB}(x_2, y_2)\}\)
and
\[
\gamma_{AXB}(x_1, y_1) \leq \max \{\gamma_{AXB}((x_1, y_1) \ast (x_2, y_2)), \gamma_{AXB}(x_2, y_2)\}
\]
for any \(\alpha, \beta \in [0, 1]\), if \(\mu_{AXB}(x_1, y_1) \geq \alpha\)
That is,
\[
\min \{\mu_{AXB}((x_1, y_1) \ast (x_2, y_2)), \mu_{AXB}(x_2, y_2)\} \geq \alpha
\]
This implies \((x_1, y_1) \in \mu_{AXB, \alpha}\)
Clearly \((x_1, y_1) \ast (x_2, y_2) \in \mu_{AXB, \alpha}\)
\[
(x_2, y_2) \in \mu_{AXB, \alpha}
\]
Now \(\mu_{AXB}(x_1, y_1) \geq \min \{\mu_{AXB}((x_1, y_1) \ast (x_2, y_2)), \mu_{AXB}(x_2, y_2)\}\)
\[
\geq \min \{\alpha, \alpha\}
\]
\[
\geq \alpha
\]
This implies \((x_1, y_1) \in \mu_{AXB, \beta}\)
Thus \(\mu_{AXB}(x_1, y_1) \leq \beta\)
Then
\[
\max \{\gamma_{AXB}((x_1, y_1) \ast (x_2, y_2)), \gamma_{AXB}(x_2, y_2)\} \leq \beta
\]
This implies \((x_1, y_1) \in \gamma_{AXB, \beta}\)
Clearly \((x_1, y_1) \ast (x_2, y_2) \in \gamma_{AXB, \beta}\)
\[
(x_2, y_2) \in \gamma_{AXB, \beta}
\]
Now \(\gamma_{AXB}(x_1, y_1) \leq \max \{\gamma_{AXB}((x_1, y_1) \ast (x_2, y_2)), \gamma_{AXB}(x_2, y_2)\}\)
\[
\leq \max \{\beta, \beta\}
\]
\[
\leq \beta
\]
This implies \((x_1, y_1) \in \gamma_{AXB, \beta}\)
Thus \(\beta \in [0, 1]\), is an intuitionistic fuzzy BG-ideal of \(X \times Y\).
Hence \(\alpha, \beta \in [0, 1]\) is an intuitionistic fuzzy BG-ideal of \(X \times Y\).

Conversely,
Let \((x_1, y_1), (x_2, y_2) \in X \times Y\) such that
\[
\mu_{AXB}(x_1, y_1) = \alpha
\]
\[
\gamma_{AXB}(x_1, y_1) = \beta
\]
This implies \((x_1, y_1) \in \mu_{AXB, \alpha}\) and \((x_1, y_1) \in \gamma_{AXB, \beta}\)
Therefore

\[ \mu_{A \times B}(x_1, y_1) \geq \alpha \]
\[ \gamma_{A \times B}(x_1, y_1) \leq \beta \]

This gives

\[ \mu_{A \times B}((x_1, y_1)) \geq \min \{\mu_{A \times B}((x_1, y_1) * (x_2, y_2)), \mu_{A \times B}(x_2, y_2)\} \]

and

\[ \gamma_{A \times B}((x_1, y_1)) \leq \max \{\gamma_{A \times B}((x_1, y_1) * (x_2, y_2)), \gamma_{A \times B}(x_2, y_2)\} \]

Hence \( A \times B \) is a intuitionistic fuzzy BG-ideal of \( X \times Y \).

Reference