

Error Estimation of Functions Belonging to the Lipschitz Class by Borel-Euler Summability Means of the Conjugate Series of Fourier Series

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Abstract: In this paper, we have estimated the error of functions belonging to the Lipschitz class by Borel-Euler ($\widetilde{BE}_\lambda^1$) product means.

Keywords and Phrases: Error estimation of function, Lipschitz class, conjugate Fourier series, Borel-Euler product means, Borel mean, Euler mean.

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1. Introduction

Let $f(x)$ be a function which is 2π periodic and Lebesgue integrable, then the Fourier series of $f(x)$ is given by ([4]).

$$s_n(f; x) = \frac{1}{2}a_0 + \sum_{v=0}^n (a_v \cos vx + b_v \sin vx) = \sum_{v=0}^n A_v(f; x) \quad (1)$$

The conjugate series of Fourier series (3) is given by

$$\sum_{v=1}^{\infty} (a_v \sin vx - b_v \cos vx) = - \sum_{v=0}^{\infty} T_v(f; x) \quad (2)$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with partial sum s_n which is said to be (E,1) summable if

$$E_n^1 = \frac{1}{2^n} \sum_{v=0}^n \binom{n}{v} s_v \rightarrow s \quad \text{as } n \rightarrow \infty$$

The Borel B-summability is defined as

$$B_\lambda = e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} s_m$$

The Borel-Euler product means of the infinite series is defined as

$$\begin{aligned} BE_\lambda^1 &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} E_n^1 \\ &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left(\frac{1}{2^m} \sum_{v=0}^m \binom{m}{v} s_v \right) \rightarrow s \quad \text{as } n \rightarrow \infty \end{aligned} \quad (3)$$

The Euler's mean of the conjugate series (2) of the Fourier series is given by

$$\tilde{E}_n^1 = \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos^n t/2 \cos\left(\frac{n+1}{2}t\right)}{\sin t/2} dt$$

Now taking the Borel transformation of $\tilde{E}_n^1$, we get

$$\begin{aligned} \widetilde{BE}_n^1 &= \frac{1}{\pi} e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left\{ \int_0^\pi \psi(t) \frac{\cos^m t/2 \cos\left(\frac{p+1}{2}t\right)}{\sin t/2} dt \right\} \\ &= \frac{1}{2\pi} e^{-\lambda} \int_0^\pi \frac{\psi(t)}{\sin t/2} \left\{ \sum_{m=0}^{\infty} \frac{\lambda^m \cos^m t/2 \cos\left(\frac{p+1}{2}t\right)}{m!} dt \right\} \end{aligned}$$

Now we consider

$$\bar{C}' = \cos\left(\frac{t}{2}\right) + \frac{\lambda \cos\left(\frac{t}{2}\right) \cos\left(\frac{2t}{2}\right)}{1!} + \frac{\lambda^2 \cos\left(\frac{t}{2}\right) \cos\left(\frac{3t}{2}\right)}{2!} + \dots$$

Also

$$\bar{S}' = \sin\left(\frac{t}{2}\right) + \frac{\lambda \cos\left(\frac{t}{2}\right) \sin\left(\frac{2t}{2}\right)}{1!} + \frac{\lambda^2 \cos\left(\frac{t}{2}\right) \sin\left(\frac{3t}{2}\right)}{2!} + \dots$$

So,

$$\begin{aligned} C' + iS' &= e^{it/2} + \frac{\lambda}{1!} \cos \frac{t}{2} e^{\frac{2it}{2}} + \frac{\lambda^2}{2!} \cos^2 \frac{t}{2} e^{\frac{3it}{2}} + \dots \\ &= e^{it/2} \left[1 + \lambda \cos \frac{t}{2} e^{\frac{it}{2}} + \frac{\lambda^2}{2!} \cos^2 \frac{t}{2} e^{\frac{2it}{2}} + \dots \right] \\ &= e^{it/2} \exp(\lambda \cos t/2 e^{it/2}) \\ &= e^{it/2} \exp(\lambda \cos t/2 (\cos t/2 + i \sin t/2)) \\ &= e^{it/2} \exp\{\lambda \cos^2 t/2 + i\lambda \sin t/2 \cos t/2\} \\ &= e^{\lambda \cos^2 \frac{t}{2}} \cdot e^{\frac{i(t+\lambda \sin t)}{2}} \end{aligned}$$

Or

$$\begin{aligned} C' &= e^{\lambda \cos^2 \frac{t}{2}} \cos\left(\frac{t+\lambda \sin t}{2}\right) \\ S' &= e^{\lambda \cos^2 \frac{t}{2}} \sin\left(\frac{t+\lambda \sin t}{2}\right) \end{aligned}$$

From above, the Borel-Euler product means of the infinite series becomes

$$\begin{aligned} \widetilde{BE}_\lambda^1 &= \frac{1}{2\pi} e^{-\lambda} \int_0^\pi \frac{\psi(t)}{\sin t/2} e^{\lambda \cos^2 t/2} \cos\left(\frac{t+\lambda \sin t}{2}\right) dt \\ &= \frac{1}{2\pi} \int_0^\pi \frac{\psi(t)}{\sin t/2} \exp\left\{-\lambda \left(\frac{1-\cos t}{2}\right)\right\} \cos\left(\frac{t+\lambda \sin t}{2}\right) dt \\ \widetilde{BE}_\lambda^1 &= \frac{1}{2\pi} \int_0^\pi \psi(t) \tilde{K}(\lambda, t) dt \end{aligned}$$

The infinite series $\sum_{n=0}^{\infty} u_n$ is said to be $\widetilde{BE}_\lambda^1$ summable to $\tilde{s}$ if,

$$\widetilde{B}E_{\lambda}^1 \rightarrow \tilde{s} \quad \text{as } \lambda \rightarrow \infty.$$

A function $f \in Lip \alpha$. If, $f(x+t) - f(x) = o(|t|^\alpha)$ for $0 < \alpha \leq 1$.

The error estimation of a function f by a trigonometric polynomial t_n of order n under the norm $\|\cdot\|_\infty$ is defined by Zygmund [2] with

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}$$

And the best approximation $E_n(f)$ of a function $f \in L_p$ is defined by

$$E_n(f) = \min_{t_n} \|t_n - f\|_p.$$

We use the following notation throughout the paper:

$$\psi(t) = f(x+t) - f(x-t)$$

$$\tilde{f}(x) = -\frac{1}{2\pi} \int_0^\pi \psi(t) \cot\left(\frac{t}{2}\right) dt$$

$$\tilde{K}(\lambda, t) = \frac{1}{\sin t/2} \exp\left\{-\lambda \left(\frac{1 - \cos t}{2}\right) \cos\left(\frac{t + \lambda \sin t}{2}\right)\right\}$$

2. Lemma:

We need the following lemma to prove our theorem: For $\frac{\pi}{m+1} < t < \pi$

$$\begin{aligned} |\tilde{K}(\lambda, t)| &= Re. p. f \left| \frac{1}{2\pi} \frac{e^{-\lambda \cos^2 t/2}}{\sin t/2} e^{i\left(\frac{t+\lambda \sin t}{2}\right)} \right| \\ &\cong O\left(\frac{1}{t}\right) \end{aligned}$$

3. Theorem:

If f is a 2π periodic function belonging to $lip\alpha$ class, then its error estimation of functions by the Borel-Euler product summability method of the conjugate series of Fourier series (3) is given by

$$\|\tilde{t}_{E_1}^{B_\lambda} - \tilde{f}\| \cong O\left(\frac{1}{(m+1)^\alpha}\right), \quad 0 < \alpha < 1.$$

Proof of the Theorem

The n^{th} partial sum of the conjugate series of Fourier series (2) is given by

$$\begin{aligned} \tilde{s}_n &= -\frac{1}{2\pi} \int_0^\pi \cot\left(\frac{t}{2}\right) \psi(t) dt + \frac{1}{2\pi} \int_0^\pi \frac{\cos\left(n + \frac{1}{2}\right)}{\sin \frac{t}{2}} \psi(t) dt \\ \tilde{s}_n &- \left[-\frac{1}{2\pi} \int_0^{\pi/(m+1)} \cot\left(\frac{t}{2}\right) \psi(t) dt - \frac{1}{2\pi} \int_{\pi/(m+1)}^\pi \cot\left(\frac{t}{2}\right) \psi(t) dt \right] \\ &= \frac{1}{2\pi} \left(\int_0^{\pi/(m+1)} + \int_{\pi/(m+1)}^\pi \right) \frac{\cos\left(n + \frac{1}{2}\right) t}{\sin\left(\frac{t}{2}\right)} \psi(t) dt \end{aligned}$$

$$\tilde{s}_n(x) - \tilde{s}_m(x) = \frac{1}{2\pi} \int_0^{\pi/(m+1)} \left(\frac{\cos\left(n + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} - \cot\left(\frac{t}{2}\right) \right) \psi(t) dt + \frac{1}{2\pi} \int_{\pi/(m+1)}^{\pi} \frac{\cos\left(n + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \psi(t) dt$$

Taking Borel-Euler transforms, we get

$$\begin{aligned} \tilde{t}_{E_1}^{B(m+1)} - \tilde{f} &= \frac{1}{2\pi} e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left\{ \int_0^{\pi/(m+1)} \frac{\psi(t) \cos\left(m + \frac{1}{2}\right)t \cos\left(\frac{m}{2}\right)t}{\sin t/2} dt \right\} \\ &\quad + \frac{1}{2\pi} e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left\{ \int_{\pi/(m+1)}^{\pi} \frac{\psi(t) \cos\left(m + \frac{1}{2}\right)t \cos\left(\frac{m}{2}\right)t}{\sin t/2} dt \right\} \\ &= \frac{1}{2\pi} e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left\{ \int_0^{\pi/(m+1)} \psi(t) \left(\frac{\cos\left(m + \frac{1}{2}\right)t}{\sin \frac{t}{2}} - \cot \frac{t}{2} \right) dt \right\} \\ &\quad + \frac{1}{2\pi} e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left\{ \int_{\pi/(m+1)}^{\pi} \psi(t) \left(\frac{\cos\left(m + \frac{1}{2}\right)t}{\sin \frac{t}{2}} \right) dt \right\} \\ &= \frac{1}{2\pi} e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left\{ \int_0^{\pi/(m+1)} \psi(t) \left(\frac{\cos\left(m + \frac{1}{2}\right)t - \cos \frac{t}{2}}{\sin \frac{t}{2}} \right) dt \right\} \\ &\quad + \int_{\pi/(m+1)}^{\pi} \psi(t) \tilde{K}(\lambda, t) dt \\ \left| \tilde{t}_{E_1}^{B\lambda} - \tilde{f} \right| &\leq O(m+1) \left\{ \int_0^{\pi/(m+1)} \psi(t) dt \right\} + \int_{\pi/(m+1)}^{\pi} \psi(t) \tilde{K}(\lambda, t) dt = \tilde{I}_1 + \tilde{I}_2 \end{aligned}$$

Therefore,

$$\begin{aligned} |\tilde{I}_1| &= \left| \int_0^{\pi/(m+1)} \psi(t) O(m+1) dt \right| \\ &= O(m+1) \int_0^{\pi/(m+1)} t^\alpha dt \\ &= O(m+1) \left[\frac{t^\alpha}{\alpha} \right]_0^{\pi/(m+1)} \\ &\cong O\left(\frac{1}{(m+1)^\alpha}\right) \\ |\tilde{I}_2| &= \left| \int_{\pi/(m+1)}^{\pi} \psi(t) \tilde{K}(\lambda, t) dt \right| \\ &= \left| \int_{\pi/(m+1)}^{\pi} t^\alpha \cdot \frac{1}{t} dt \right| \end{aligned}$$

$$= \left| \int_{\pi/(m+1)}^{\pi} t^{\alpha-1} dt \right|$$

$$\cong O\left(\frac{1}{(m+1)^\alpha}\right), \quad 0 < \alpha < 1$$

Therefore, the error estimation of functions belonging to the Lipschitz class by the Borel-Euler summability method of the conjugate series of Fourier series (2) is given by

$$\|\tilde{t}_{E_1}^{B_\lambda} - \tilde{f}\| \cong O\left(\frac{1}{(m+1)^\alpha}\right), \quad 0 < \alpha < 1$$

4. Conclusion-

In this paper, we have introduced the product summability of conjugate series of Fourier using Borel-Euler product means. The present theorem extends, generalizes, and improves many existing results on summability of conjugate series of Fourier series, also by using functions that belong to the Lipschitz class. These results may be a motivation to other researchers to carry out the outcomes in the field of summability theory.

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