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Error Estimation of Functions Belonging to the Lipschitz Class by Borel-Euler Summability Means of the Conjugate Series of Fourier Series

[1]Jitendra Kumar Kushwaha∗, [2]Raha Vishwakarma†, [3]Kritika Singh**
[123]Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur

Abstract: In this paper, we have estimated the error of functions belonging to the Lipschitz class by Borel-Euler $(\widetilde{\mathrm{BE}}_{\lambda}^{1})$ product means.

Keywords and Phrases: Error estimation of function, Lipschitz class, conjugate Fourier series, Borel-Euler product means, Borel mean, Euler mean.
Subject Classification: 42B05; 42B08

1. Introduction

Let $f(x)$ be a function which is 2π periodic and Lebesgue integrable, then the Fourier series of $f(x)$ is given by ([4]).

$$s_n(f;x) = \frac{1}{2}a_0 + \sum_{\nu=0}^{n}(a_\nu \cos \nu x \; + b_\nu \sin \nu x) = \sum_{\nu=0}^{n} A_\nu(f;x) \qquad (1)$$

The conjugate series of Fourier series (3) is given by

$$\sum_{\nu=1}^{\infty}(a_\nu \sin \nu x - b_n \cos \nu x) = -\sum_{\nu=0}^{\infty} T_\nu(f;x) \qquad (2)$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with partial sum s_n which is said to be (E,1) summable if

$$E_n^1 = \frac{1}{2^n}\sum_{\nu=0}^{n}\binom{n}{\nu} s_n \longrightarrow s \qquad as \qquad n \longrightarrow \infty$$

The Borel B-summability is defined as

$$B_\lambda = e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}s_n$$

The Borel-Euler product means of the infinite series is defined as

$$BE_\lambda^1 = e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}E_n^1$$

$$= e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\left(\frac{1}{2^m}\sum_{\nu=0}^{m}\binom{n}{\nu}s_m\right) \to s \quad as \quad n \to \infty \qquad (3)$$

The Euler's mean of the conjugate series (2) of the Fourier series is given by

$$\tilde{E}_n^1 = \frac{1}{2\pi}\int_0^{\pi} \psi(t)\frac{cos^n t/2\ \ cos\left(\frac{n+1}{2}\right)t}{\sin t/2}\, dt$$

Now taking the Borel transformation of $\tilde{E}_n^1$, we get

$$\widetilde{BE}_n^1 = \frac{1}{\pi}e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\left\{\int_0^{\pi}\psi(t)\frac{cos^p\ t/2\ \ cos\left(\frac{p+1}{2}\right)t}{\sin t/2}dt\right\}$$

$$= \frac{1}{2\pi}e^{-\lambda}\int_0^{\pi}\frac{\psi(t)}{\sin t/2}\left\{\sum_{m=0}^{\infty}\frac{\lambda^m\ cos^p\ t/2\ \ cos\left(\frac{p+1}{2}\right)t}{m!}\ dt\right\}$$

Now we consider

$$\bar{C}' = cos\left(\frac{t}{2}\right) + \frac{\lambda\cos\left(\frac{t}{2}\right)cos\left(\frac{2t}{2}\right)}{1!} + \frac{\lambda^2\cos\left(\frac{t}{2}\right)\cos\left(\frac{3t}{2}\right)}{2!} + \cdots\ldots$$

Also

$$\bar{S}' = \sin\left(\frac{t}{2}\right) + \frac{\lambda\cos\left(\frac{t}{2}\right)sin\left(\frac{2t}{2}\right)}{1!} + \frac{\lambda^2\cos\left(\frac{t}{2}\right)\sin\left(\frac{3t}{2}\right)}{2!} + \cdots\ldots$$

So,

$$C' + iS' = e^{it/2} + \frac{\lambda}{1!}\cos\frac{t}{2}\ e^{\frac{2it}{2}} + \frac{\lambda^2}{2!}\cos^2\frac{t}{2}\ e^{\frac{3it}{2}} + \cdots\ldots\ldots.$$

$$= e^{it/2}\left[1 + \lambda\cos\frac{t}{2}\ e^{\frac{it}{2}} + \frac{\lambda^2}{2!}\cos^2\frac{t}{2}\ e^{\frac{2it}{2}} + \cdots\ldots\ldots\right]$$

$$= e^{it/2}\exp\left(\lambda\cos t/2\ \ e^{it/2}\right)$$

$$= e^{it/2}\exp\left(\lambda\cos t/2(\cos t/2 +\ i\ sin\ t/2)\right)$$

$$= e^{it/2}\ \exp\left\{\lambda\cos^2 t/2 + i\lambda\sin t/2\ cos\ \ t/2\right\}$$

$$= e^{\lambda\cos^2\frac{t}{2}}.e^{\frac{i(t+\lambda\sin(t))}{2}}$$

Or

$$C' = e^{\lambda\cos^2\frac{t}{2}}\cos\left(\frac{t+\lambda\sin t}{2}\right)$$

$$S' = e^{\lambda\cos^2\frac{t}{2}}\sin\left(\frac{t+\lambda\sin t}{2}\right)$$

From above, the Borel-Euler product means of the infinite series becomes

$$\widetilde{BE}_\lambda^1 = \frac{1}{2\pi}e^{-\lambda}\int_0^{\pi}\frac{\psi(t)}{\sin t/2}e^{\lambda\cos^2 t/2}cos\left(\frac{t+\lambda\sin t}{2}\right)dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\psi(t)}{\sin t/2}exp\left\{-\lambda\left(\frac{1-\cos t}{2}\right)\right\}\cos\left(\frac{t+\lambda\sin t}{2}\right)dt$$

$$\widetilde{BE}_\lambda^1 = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\tilde{K}(\lambda,t)dt$$

The infinite series $\sum_{n=0}^{\infty} u_n$ is said to be $\widetilde{BE}_\lambda^1$ summable to $\tilde{s}$ if,

$$\widetilde{BE}_\lambda^1 \to \tilde{s} \quad \text{as} \quad \lambda \to \infty.$$

A function $f \in Lip\ \alpha$. If, $f(x+t) - f(x) = o(|t|^\alpha)$ for $0 < \alpha \leq 1$.

The error estimation of a function f by a trigonometric polynomial t_n of order n under the norm $\|.\|_\infty$ is defined by Zygmund [2] with

$$\|t_n - f\|_\infty = sup\{|t_n(x) - f(x)| : x \in R\}$$

And the best approximation $E_n(f)$ of a function $f \in L_p$ is defined by

$$E_n(f) = \min_{t_n} \|t_n - f\|_p.$$

We use the following notation throughout the paper:

$$\psi(t) = f(x+t) - f(x-t)$$

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\left(\frac{1}{2}\right)dt$$

$$\tilde{K}(\lambda, t) = \frac{1}{\sin t/2} exp\left\{-\lambda\left(\frac{1-\cos t}{2}\right) cos\left(\frac{t+\lambda \sin t}{2}\right)\right\}$$

2. Lemma:

We need the following lemma to prove our theorem: For $\frac{\pi}{m+1} < t < \pi$

$$\left|\tilde{K}(\lambda, t)\right| = Re.p.f\left|\frac{1}{2\pi}\frac{e^{-\lambda\cos^2 t/2}}{\sin t/2} e^{i\left(\frac{t+\lambda\sin t}{2}\right)}\right|$$

$$\cong O\left(\frac{1}{t}\right)$$

3. Theorem:

If f is a 2π periodic function belonging to $lip\alpha$ class, then its error estimation of functions by the Borel-Euler product summability method of the conjugate series of Fourier series (3) is given by

$$\|\tilde{t}_{E_1}^{B_\lambda} - \tilde{f}\| \cong O\left(\frac{1}{(m+1)^\alpha}\right), \quad 0 < \alpha < 1.$$

Proof of the Theorem

The n^{th} partial sum of the conjugate series of Fourier series (2) is given by

$$\tilde{s}_n = -\frac{1}{2\pi}\int_0^\pi \cot\left(\frac{t}{2}\right)\psi(t)dt + \frac{1}{2\pi}\int_0^\pi \frac{\cos\left(n+\frac{1}{2}\right)}{\sin\frac{t}{2}}\psi(t)dt$$

$$\tilde{s}_n - \left[-\frac{1}{2\pi}\int_0^{\pi/(m+1)} \cot\left(\frac{t}{2}\right)\psi(t)dt - \frac{-1}{2\pi}\int_{\pi/(m+1)}^{\pi} \cot\left(\frac{t}{2}\right)\psi(t)dt\right]$$

$$= \frac{1}{2\pi}\left(\int_0^{\pi/(m+1)} + \int_{\pi/(m+1)}^{\pi}\right)\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\psi(t)dt$$

$$\tilde{s}_n(x)-\tilde{s}_m(x)=\frac{1}{2\pi}\int_0^{\pi/(m+1)}\left(\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}-\cot\left(\frac{t}{2}\right)\right)\psi(t)dt+\frac{1}{2\pi}\int_{\pi/(m+1)}^{\pi}\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\psi(t)dt$$

Taking Borel-Euler transforms, we get

$$\tilde{t}_{E_1}^{B(m+1)}-\tilde{f}=\frac{1}{2\pi}e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\left\{\int_0^{\pi/(m+1)}\frac{\psi(t)\cos\left(m+\frac{1}{2}\right)t\cos\left(\frac{m}{2}\right)t}{\sin t/2}dt\right\}$$

$$+\frac{1}{2\pi}e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\left\{\int_{\pi/(m+1)}^{\pi}\frac{\psi(t)\cos\left(m+\frac{1}{2}\right)t\cos\left(\frac{m}{2}\right)t}{\sin t/2}dt\right\}$$

$$=\frac{1}{2\pi}e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\left\{\int_0^{\pi/(m+1)}\psi(t)\left(\frac{\cos\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}-\cot\frac{t}{2}\right)dt\right\}$$

$$+\frac{1}{2\pi}e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\left\{\int_{\pi/(m+1)}^{\pi}\psi(t)\left(\frac{\cos\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right)dt\right\}$$

$$=\frac{1}{2\pi}e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\left\{\int_0^{\pi/(m+1)}\psi(t)\left(\frac{\cos\left(m+\frac{1}{2}\right)t-\cos\frac{t}{2}}{\sin\frac{t}{2}}\right)dt\right\}$$

$$+\int_{\pi/(m+1)}^{\pi}\psi(t)\widetilde{K}\,(\lambda,t)dt$$

$$|\tilde{t}_{E_1}^{B_\lambda}-\tilde{f}|\leq O(m+1)\left\{\int_0^{\pi/(m+1)}\psi(t)dt\right\}+\int_{\pi/(m+1)}^{\pi}\psi(t)\widetilde{K}(\lambda,t)dt=\widetilde{I_1}+\widetilde{I_2}$$

Therefore,

$$|\widetilde{I_1}|=\left|\int_0^{\pi/(m+1)}\psi(t)O(m+1)dt\right|$$

$$=O(m+1)\int_0^{\pi/(m+1)}t^\alpha\,dt$$

$$=O(m+1)\left[\frac{t^\alpha}{\alpha}\right]_0^{\pi/(m+1)}$$

$$\cong O\left(\frac{1}{(m+1)^\alpha}\right)$$

$$|\widetilde{I_2}|=\left|\int_{\pi/(m+1)}^{\pi}\psi(t)\widetilde{K}(\lambda,t)dt\right|$$

$$=\left|\int_{\pi/(m+1)}^{\pi}t^\alpha.\frac{1}{t}dt\right|$$

$$= \left| \int_{\pi/(m+1)}^{\pi} t^{\alpha-1}\, dt \right|$$

$$\cong O\left(\frac{1}{(m+1)^{\alpha}}\right) \quad , \qquad\qquad 0 < \alpha < 1$$

Therefore, the error estimation of functions belonging to the Lipschitz class by the Borel-Euler summability method of the conjugate series of Fourier series (2) is given by

$$\left\| \tilde{t}_{E_1}^{B_\lambda} - \tilde{f} \right\| \cong O\left(\frac{1}{(m+1)^{\alpha}}\right), \quad 0 < \alpha < 1$$

4. Conclusion-

In this paper, we have introduced the product summability of conjugate series of Fourier using Borel-Euler product means. The present theorem extends, generalizes, and improves many existing results on summability of conjugate series of Fourier series, also by using functions that belong to the Lipschitz class. These results may be a motivation to other researchers to carry out the outcomes in the field of summability theory.

References

[1] A. Zygmund, Trigonometrical series. Warsaw; 1935; I-II.

[2] A. Zygmund, Trigonometric Series, Vol.1, 2nd Rev.ed., Cambridge Univ. Press, Cambridge, New York, 1959.

[3] B.E. Rhaodes , "On the degree of approximation of functions belonging to Lipschitz class by Hausdroff means of its Fourier series", Tamkang J. Math .34 (2003), no. 3,245-247.

[4] E.C. Titchmarch, "The Theory of functions". Oxford Uni Press; 1952.

[5] G.H. Hardy, Divergent series, 1st , Oxford University press, 1949

[6] H. Lebesgue, Recherches Sur la convergence des series de Fourier. Math Annalen. 1905; 61: 251-280.

[7] H.K. Nigam, "A study on Approximation of conjugate of functions belonging to Lipschitz class and generalized Lipschitz class by Product Summability means of Conjugate series of Fourier series", Thai J. Math, 10(2012). no. 2, 275-187.

[8] H.K. Nigam and K. Sarma, "Approximation of conjugate functions belonging to Lip (α, gamma) class and to weighted W(Lr;ξ(t)) class by Product means of Conjugate Fourier series", Eur. J. Pure Appl. Math 3(2011), 276-286.

[9] J.K. Kushwaha, L. Rathour, L. N. Mishra, V.N.Mishra and R.Vishwakarma, "On the degree of Conjugate Functions using Generalized Nörlund - Euler Summability Method" GANITA, Vol.71(1),2021,145-153.

[10] J.K. Kushwaha, L. Rathour, L. N. Mishra, V.N. Mishra and K. Kumar, "Approximation of Lipschitz class by Deferred-Generalized Nörlund product summability means" J. Appl. Math. & Informatics Vol. 41(2023), No. 5, pp. 1057-1069.

[11] J.K. Kushwaha, "Approximation of functions by (C,2) (E,1) product summability method of Fourier series".Ratio Mathematica, Volume 38, 2020, pp.341-348.

[12] K. Saxena and M. Prabhakar,"A new criterion for Borel- Euler summability method of Fourier series", Asian Research journal of Mathematics,8(2): 1-2,2018; Article no. ARJOM. 33839, ISSN: 2456- 477X

[13] K. Saxena, M. Prabhakar, "A study of double Euler summability method of Fourier series and it's conjugate series". Int. Jour.Sci. Innovative Math Res. 2016; 4(1): 46-52.

[14] P Chandra, Trigonometric approximation of functions in Lp Norm, J.Math . Anal. Appl. 275 (2002), 13-26.

[15] S. Izumi, “Study of Cesaro Summability of Fourier series”. Tohoku Math Jour. 1949; 51:144-149

[16] Tom Laying, “A Tauberian Theorem for the general Euler- Borel Summa bility method”. Canad.Jour. Math .1992;44(5).

[17] V.N. Mishra, H.H. Khan, K. Khatri and L.N. Mishra, “On approximation of conjugate of signals (Functions) belonging to the generalized Weighted W (Lr, ξ(t)),(r ≥ 1)- class by Product Summability means of Conjugate series of Fourier series,” international journal of Mathematical Analysis, Vol.6, no. 35, pp.1703-1715,2012.