Performance Analysis of Discrete – Time Geo^x/ G/1 Retrial Queue – Two Vacation Types with Impatient Customers

C.Christy Anitha[1], N.Nithyapriya [2]

[1,2] *Department of Mathematics, D.K.M College for Women (Autonomous), Vellore – 1, (Affiliated to Thiruvalluvar University ,Serkkadu, Vellore – 632 115), Tamil Nadu , India.*

Abstract:-A Stochastic system where bulk (batch) of customers arrives according to a Geometric distribution retrial time follows a general distribution. We examine a retrial queueing system featuring two vacation types. This paper investigates a discrete – time $Geo^x/G/$ 1 retrial queue under urgent and normal vacations with impatient customers. We attain some efficiency characteristics of the probability generating function and orbit size of the system.

Keywords: *Batch arrival, Retrial queue, impatient customers, single server, Vacations.*

Mathematics subject Classification :*60K25, 68M20, 90B22.*

1. Introduction

Queueing is an unavoidable aspect of daily life. Retrial and classical queues are indispensable tools for modeling and analyzing stochastic systems.

Retrial queues have played a crucial role in various systems, including computers, telephone networks, and telecommunications. These systems have been extensively studied for their practical applications in modelling. In a classical queue, customers arrive, wait in line, and eventually receive service. However, in a retrial queue, if a customer cannot be served immediately, they don't simply leave. Instead, they re-join a pool of waiting customers (often called an 'orbit') and try their service again later. This behaviour is characteristic of retrial queues. In these system, customers often arrive in large numbers and may retry for service a random time if their initial attempt is unsuccessful. To optimize system performance and reduce costs, server may take vacations to rejuvenate or perform maintenance tasks. Literature and main results on this paper are referred to [2,3,6,8 &13].

This paper focuses on bulk retrial queues with two vacation types, where customers arrive in bulk and the server takes either a urgent or normal vacation. We examine the steady-state behavior of this system, focusing on critical performance measures such a marginal generating functions and orbit size.

2.Model Description

In this part of our study, we examine a discrete-time retrial queue of Geo ^{X}/G/1 with two types of server vacations. Batch of Customers arrive in geometric process with the parameter λ, where (0< λ<1) is the probability of an arrival event will take place in a slot. If server is ready to accept (i.e) free, the arriving customer vacates the system immediately after receive the service. Batch of L customers arrived one customer receive the service from the free server immediately in orbit other customers waits. The server is not free, busy or on vacation the bulk arrival of customers wait in orbit and try for the service again after some time, the retry time is expected to be a general distribution of probabilities $\{r_i\}_{i=0}^{\infty}$ with the generating function R(x) = $\sum_{i=0}^{\infty} r_i\, x^i$, 0≤ x ≤1.

The identically distributed random variable with the probability mass function of batch size L is b_i= (P=L=i), i ≥ 1, with probability generating function B(X) = $\sum_{h=1}^{\infty} b_h\, x^h$, 0≤ x ≤1. General probability distribution variable of the service time is $\{t_i\}_{i=0}^{\infty}$ with the generating function T(x) = $\sum_{i=0}^{\infty} t_i\, x^i$, 0≤ x ≤1, the moment of first is T_1 and the moment of second factorial is $T_2 = \sum_{i=0}^{\infty} i(i-1)t_i$.

The server with two varieties of vacations are assumed. First type is the non-exhaustive vacation (i.e) an urgent vacation of the server with the probability $\bar{\mu}$ = 1-μ , when the server is serving the customer , where μ is the probability that the server fails to take an urgent vacation. In the instance of urgent vacation of server customer may enter in orbit and wait for the interrupted service retry to receive the service. When the orbit is empty during a normal exhaustive vacation, the server immediately takes a vacation. Upon completing both vacation types, the server enters an idle state, awaiting incoming customers from the orbit or outside.

The urgent vacation time of the general probability distribution is $\{\mathbb{v}_{1,i}\}_{i=0}^{\infty}$ with generating function $V_1(x)$ = $\sum_{i=0}^{\infty} \mathbb{v}_{1,i}\, x^i$, 0≤ x ≤1, the first moment $V_{1,1}$ and the second factorial moment $V_{1,2}$= $\sum_{i=0}^{\infty} i(i-1)\mathbb{v}_{1,i}$. Normal vacation of the general distribution is considered to be $V_2(x)$= $\sum_{i=0}^{\infty} \mathbb{v}_{2,i}\, x^i$, 0≤ x ≤1, first moment $V_{2,1}$, and moment of second factorial is $V_{2,2}$=$\sum_{i=0}^{\infty} i(i-1)\mathbb{v}_{2,i}$. It is assumed that the system's stochastic processes involved independently of one another.

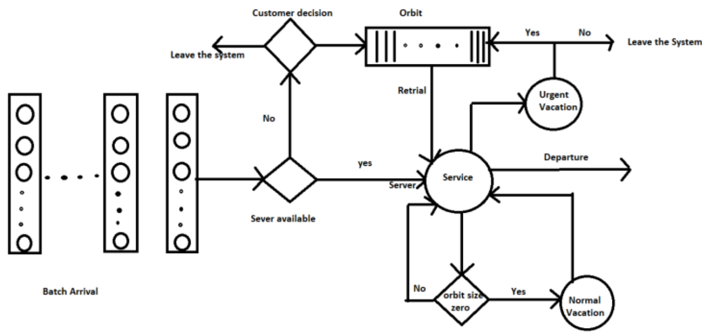

Figure 1. Schematic diagram - Model

3. The Steady – State Analysis

The steady state analysis for the retrial queue of Geo ^{X}/G/1 with impatient consumers will be presented in this part.

At the time m+, let $\mathcal{C}_m$ represents the state of server, $\mathcal{C}_m$ = 0,1,2,or 3 depending on whether the server is available (free), occupied (busy), on an urgent vacation or on a regular (normal) vacation. Let $\mathcal{N}_m$ be the total number of customers within the orbit.

If $\mathcal{C}_m$ = 0, $\mathcal{E}_m$ symbolizes the amount of time remaining for the retrial.

If $\mathcal{C}_m$ = 1, $\mathcal{E}_m$ symbolizes the amount of time remaining service for the customers currently in service.

If $\mathcal{C}_m$ = 2, $\mathcal{E}_m$ symbolizes the amount of time remaining for the urgent vacation.

If $\mathcal{C}_m$ = 3, $\mathcal{E}_m$ symbolizes the amount of time remaining for the normal vacation.

Thus, at the time m $^+$, the process described for the system is $\mathcal{Y}_m = (\mathcal{C}_m, \mathcal{E}_m, \mathcal{N}_m)$. $\{\mathcal{Y}_m$, m = 0,1,2……$\}$ may be shown as a Markov chain using the state space as follows:

$$\pi = \{ (0, 0) \} \cup \{ (0, i, k) : i \geq 1, k \geq 1 \}$$

$$\cup \{ (1, i, k) : i \geq 1 , k \geq 0 \}$$

$$\cup \{ (2, i, k) : i \geq 1 , k \geq 1 \}$$

$$\cup \ \{ (3, i, k) : i \geq 1, \ k \geq 0 \}. \quad (1)$$

The stationary probability of the Markov chain { $\mathcal{Y}_m$, = 0, 1, 2,3,...} is initially defined as follows:

$$\Omega_{0,0} = \lim_{m\to\infty} P\{ \mathcal{C}_m = 0, \mathcal{N}_m = 0\},$$

$$\Omega_{0,i,k} = \lim_{m\to\infty} P\{ \mathcal{C}_m = 0, \ \mathcal{E}_m = i, \ \mathcal{N}_m = k\} ; \quad i \geq 1, k \geq 1,$$

$$\Omega_{1,i,k} = \lim_{m\to\infty} P\{ \mathcal{C}_m = 1, \ \mathcal{E}_m = i, \ \mathcal{N}_m = k\} ; \quad i \geq 1, k \geq 0,$$

$$\Omega_{2,i,k} = \lim_{m\to\infty} P\{ \mathcal{C}_m = 2, \ \mathcal{E}_m = i, \ \mathcal{N}_m = k\} ; \quad i \geq 1, k \geq 1,$$

$$\Omega_{3,i,k} = \lim_{m\to\infty} P\{ \mathcal{C}_m = 3, \ \mathcal{E}_m = i, \ \mathcal{N}_m = k\} ; \quad i \geq 1, k \geq 0. \quad (2)$$

The following are the result of obtaining the Kolmogorov equations:

$$\Omega_{0,0} = \bar{\lambda}\, \Omega_{0,0} + \lambda\, \Omega_{3,1,0}, \quad (3)$$

$$\Omega_{0,i,k} = \bar{\lambda}\, \Omega_{0,i+1,k} + \bar{\lambda}\ t_i\, \Omega_{1,1,k} + \ \bar{\lambda} t_i\ \ \Omega_{2,1,k} + \bar{\lambda} t_i\ \ \Omega_{3,1,k}, \quad i \geq 1, k \geq 1, \quad (4)$$

$$\begin{aligned}\Omega_{1,i,k} = {} & \delta_{0,k}\lambda\mu\, b_{k+1} t_i\, \Omega_{0,0} + (1-\delta_{0,k})\, \lambda\mu\, t_i \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{0,i,k-l+1} \\ & +\bar{\lambda}\ \mu t_i\, \Omega_{0,1,k+1} + \lambda\mu t_i \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{1,1,k} + \bar{\lambda}\ \mu r_0 t_i \Omega_{1,1,k+1} \\ & +(1-\delta_{0,k})\, \lambda\mu\ \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{1,i+1,k-l} + \bar{\lambda}\ \mu\ \Omega_{1,i+1,k} \\ & +\bar{\lambda}\ \mu r_0 t_i\ \Omega_{2,1,k+1} + (1-\delta_{0,k})\, \lambda\mu\ t_i \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{2,1,k-l+1} \\ & + \lambda\mu\, t_i \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{3,1,k-l+1} + \bar{\lambda}\ \mu r_0 t_i\ \Omega_{3,1,k+1}, \quad i \geq 1, k \geq 0,\end{aligned} \quad (5)$$

$$\begin{aligned}\Omega_{2,i,k} = {} & \delta_{1,k}\lambda\bar{\mu} b_k v_{1,i}\, \Omega_{0,0} + (1-\delta_{1,k})\, \lambda\bar{\mu}\, t_i\, v_{1,i} \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{0,i,k-l} \\ & +\bar{\lambda}\, \bar{\mu}\, v_{1,i}\, \Omega_{0,1,k} + \lambda\bar{\mu}\, v_{1,i} \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{1,1,k-l} \\ & +\bar{\lambda}\ \bar{\mu} r_0 v_{1,i}\, \Omega_{1,1,k} + (1-\delta_{1,k})\, \lambda \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{2,i+1,k-l} \\ & +\bar{\lambda}\ \Omega_{2,i+1,k} + \bar{\lambda}\, \bar{\mu}\, r_0 v_{1,i}\, \Omega_{2,1,k} + (1-\delta_{1,k})\, \lambda\bar{\mu}\ v_{1,i} \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{2,1,k-l} \\ & + \lambda\bar{\mu}\, v_{1,i} \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{3,1,k-l} + \bar{\lambda}\ \bar{\mu} r_0 v_{1,i} \Omega_{3,1,k} \\ & + (1-\delta_{1,k})\ \lambda\bar{\mu} v_{1,i} \textstyle\sum_{l=1}^{k} b_l \sum_{j=2}^{\infty} \Omega_{2,j,k-l-1} + \bar{\lambda}\, \bar{\mu}\, v_{1,i}\, \Omega_{i,j,k-1},\ i \geq 1, k \geq 1,\end{aligned} \quad (6)$$

$$\Omega_{3,i,k} = \bar{\lambda}\ \ \Omega_{3,i+1,k} + (1-\delta_{0,k})\, \lambda \textstyle\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Omega_{3,i+1,k-l} + \delta_{0,k}\, \bar{\lambda}\ \ v_{2,i}\ \Omega_{1,1,0}. \quad \geq 1, k \ \geq 0. \quad (7)$$

The state of normalization is

$$\Omega_{0,0} + \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (\Omega_{0,i,k}\ \Omega_{2,i,k}) + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (\Omega_{1,i,k}\ \Omega_{3,i,k}) = 1, \quad (8)$$

where $\delta_{0,k} = 1$, if k=0; otherwise, $\delta_{0,k} = 0$, if $k \neq 0$.

We introduce the generating functions in order to solve (3) to (8).

$$\eta_0\, (x,z) = \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \Omega_{0,i,k}\ x^{\,i}\ z\ ,$$

$$\eta_1\, (x, z) = \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \Omega_{1,i,k} x^{\,i}\ z\ ,$$

$$\eta_2\, (x, z) = \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \Omega_{2,i,k} x^{\,i}\ z^{\,k},$$

$$\eta_3\, (x, z) = \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \Omega_{3,i,k} x^{\,i}\ z\ , \quad (9)$$

and the auxiliary generating functions

$$\eta_{0,i}(z) = \sum_{k=1}^{\infty} \Omega_{0,i,k}\, z\,, \quad i \geq 1,$$

$$\eta_{1,i}(z) = \sum_{k=0}^{\infty} \Omega_{1,i,k}\, z^k, \quad i \geq 1,$$

$$\eta_{2,i}(z) = \sum_{k=1}^{\infty} \Omega_{2,i,k}\, z\,, \quad i \geq 1,$$

$$\eta_{3,i}(z) = \sum_{k=1}^{\infty} \Omega_{3,i,k}\, z\,, \quad i \geq 1 \qquad (10)$$

We can now use the generating function technique to solve (3) to (8).We will use the following lemma so solve the theorem.

Lemma 3.1:

The following conditions are hold:

(i) $T(x) \leq x$ for $0 \leq x \leq 1$,

(ii) $V_1(x) \leq x$ for $0 \leq x \leq 1$. (11)

Lemma 3.2

If $(\lambda\mu + \bar{\mu} + \lambda V_{1,1}\bar{\mu}) < \tau_1$ then the inequality

$$[B(z)+\bar{\lambda}A(\bar{\lambda})(1-B(z))]\Omega(z) - z\,\gamma(z)(1-\mu(\gamma(z))\,, \qquad (12)$$

hold for $0 \leq z < 1$, where $\gamma(z) = \bar{\lambda} + \lambda\, B(z)$,

$$\tau_1 = \frac{\bar{\mu}\left(\lambda + \bar{\lambda}A(\bar{\lambda})\right)}{1-T(\mu)}\,, \qquad (13)$$

$$\Omega(z) = (1-\mu(z))T(\mu\gamma(z)) + z\,[1-T(\mu\,\gamma(z))]\,\bar{\mu}\,V_1(\gamma(z)).$$

Proof

Let us define the function $g(z) = \bar{\lambda}A(\bar{\lambda})(1-B(z))]\,\Omega(z) + z\mu\,\gamma^2(z)$ and $h(z) = z\,\gamma(z)$. We can check that $g'(1) = (1-T(\mu))(\lambda\mu + \bar{\mu} + \lambda V_{1,1}\bar{\mu}) - 1 - \lambda\mu - \bar{\mu}\bar{\lambda}A(\bar{\lambda})$ and $h'(1) = 1+\lambda$. Thus $g'(1) < h'(1)$ it is equivalent to $\lambda\mu + \bar{\mu} + \lambda V_{1,1}\bar{\mu} < \tau_1$. Since $f''(z) = (1-\bar{\lambda}A(\bar{\lambda}))\,\Omega(z) + (z + \bar{\lambda}A(\bar{\lambda})(1-B(z))\,\Omega'(z) + \mu\,\gamma^2(z) + 2\lambda\mu z\,\gamma(z) >$ 0 and $h''(z) = 2\lambda > 0$, we have that g(z) and h(z) are convex. So we have $g(z) > h(z)$, for $0 \leq z \leq 1$.

Lemma 3.3

If $\lambda\mu + \bar{\mu} + \lambda V_{1,1}\bar{\mu} < \tau_1$ the following limits exist:

$$\lim_{z\to 1}\frac{C(Z)(1-\mu\,\gamma(z)}{\Lambda(z)} = \frac{\bar{\mu}(\lambda\,(1-V_{2,1}) - A(\bar{\lambda})(\lambda + V_2(\bar{\lambda}))}{(1-T(\mu))\,(\lambda\mu + \bar{\mu} + \lambda V_{1,1}\bar{\mu}) - \bar{\mu}\,(\lambda + \bar{\lambda}A(\bar{\lambda}))}$$

$$\lim_{z\to 1}\frac{(1-\mu\,\gamma(z))\Gamma(z) - \lambda + V_2(\bar{\lambda}))\,\Omega(z)}{\Lambda(z)} = \frac{\bar{\mu}(\lambda\,(1+V_2(\bar{\lambda}) - \bar{\lambda}\,V_{2,1}) + \left(\lambda + V_2(\bar{\lambda})\right)(1-T(\mu))(\lambda\mu + \bar{\mu} + \lambda V_{1,1}\bar{\mu})}{(1-T(\mu))\,(\lambda\mu + \bar{\mu} + \lambda V_{1,1}\bar{\mu}) - \bar{\mu}\,(\lambda + \bar{\lambda}A(\bar{\lambda}))}\,, \qquad (14)$$

where

$$\Gamma(z) = \gamma(z)(1+V_2(\bar{\lambda})) - \lambda V_2(\gamma(z))\,,$$

$$(z) = B(z)(\gamma(z) - V_2(\gamma(z))) + A(\bar{\lambda})(1-B(z)\,\Gamma(z),$$

$$\Lambda(z) = [B(z) + \bar{\lambda}A(\bar{\lambda})(1-B(z))]\,\Omega(z) - z\,\gamma(z)(1-\mu(\gamma(z)),$$

$$\Omega(z) = (1-\mu(z))T(\mu\gamma(z)) + z[1-T(\mu\,\gamma(z))]\,\bar{\mu}\,V_1(\gamma(z)). \qquad (15)$$

Stability Condition

A necessary condition for system stability with $\Omega_{0,0} > 0$ $\lambda\mu + \bar{\mu} + \lambda V_{1,1}\bar{\mu} < \tau_1$. When "urgent vacations are not allowed", this simplifies to $\lambda T_1 - \lambda < \bar{\lambda}A(\bar{\lambda})$. The LHS gives the mean external arrivals per service, while the

RHS gives the mean number of repeated customers entering service at a service start. This is consistent with the known result for discrete-time exhaustive vacation retrial queues.

Theorem 3.1

The Probability Generating Functions of the steady-state probability distribution of the Markov chain $\mathcal{Y}_m = (\mathcal{C}_m, \mathcal{E}_m, \mathcal{N}_m)$ at an arbitrary slot is

$$\eta_0\,(x,z) = \frac{A(x) - A(\bar{\lambda})}{x-\bar{\lambda}} \; \frac{z(1-\mu\,\gamma(z))\Gamma(z) - B(z)(\lambda + V_2(\bar{\lambda}))\,\Omega(z)}{\Lambda(z)\,V_2(\bar{\lambda})}\,\lambda\, x\Omega_{0,0},$$

$$\eta_1\,(x,z) = \frac{T(x) - T(\mu\,\gamma(z))}{x - \mu\,\gamma(z)} \; \frac{C(z)(1-\mu\,\gamma(z))}{\Lambda(z)\,V_2(\bar{\lambda})}\,\lambda\,\mu x\Omega_{0,0},$$

$$\eta_2\,(x,z) = \frac{V_1(x) - V_1(\gamma(z))}{x - \gamma(z)} \; \frac{C(z)[1 - T(\mu\gamma(z))]}{\Lambda(z)\,V_2(\bar{\lambda})}\, x\,\lambda\,\bar{\mu} z\Omega_{0,0}\,,$$

$$\eta_3\,(x,z) = \frac{\lambda\, x\,[V_2(x) - V_2(\gamma(z))]}{[x - \gamma(z)]}\,\Omega_{0,0}\,, \tag{16}$$

where

$$\Omega_{0,0} = \frac{(1 - T(\mu))\,(\lambda\mu + \bar{\mu} + \lambda V_{1,1}\bar{\mu}) - \bar{\mu}\,(\lambda + \bar{\lambda} A(\bar{\lambda}))}{\bar{\mu}\, T(\mu)[\,\lambda\,(1 - V_{2,1}) - A(\bar{\lambda})\,(\lambda + V_2(\bar{\lambda}))]}\, V_2(\bar{\lambda})\,. \tag{17}$$

Proof

The following equations are obtained by multiplying (4) to (7) by z^k and summing up over k:

$$\eta_{0,i}\,(z) = \bar{\lambda}\,\eta_{0,i+1}\,(z) + \bar{\lambda}\, t_i(\eta_{1,1}(z) + \eta_{2,1}(z) + \eta_{3,1}(z)) - \bar{\lambda}\, r_i(\Omega_{1,1,0} + \Omega_{3,1,0})\,,\ i \geq 1, \tag{18}$$

$$\begin{aligned}\eta_{1,i}\,(z) &= \lambda\mu\, t_i\Omega_{0,0}\frac{B(z)}{z} + \lambda\mu t_i\eta_0(1,z)\,\frac{B(z)}{z}\\ &+ (\bar{\lambda} + \lambda B(z))\,\mu\eta_{1,i+1}(z) + \frac{\bar{\lambda}}{z}\,\mu t_i\,\eta_{0,1}(z)\\ &+ \left(\frac{\lambda\, B(z) + \bar{\lambda}\, r_0}{z}\right)\mu t_i(\eta_{1,1}(z) + \eta_{2,1}(z) + \eta_{3,1}(z))\\ &- \frac{\bar{\lambda}}{z}\, r_0\mu t_i(\Omega_{1,1,0} + \Omega_{3,1,0})\,, \qquad i \geq 1,\end{aligned} \tag{19}$$

$$\begin{aligned}\eta_{2,i}\,(z) &= \lambda\bar{\mu}\, V_{1,i}B(z)\Omega_{0,0} + \lambda\bar{\mu}\, V_{1,i}\,\eta_0(1,z)\,B(z)\\ &+ \bar{\lambda}\,\bar{\mu}\, V_{1,i}\eta_{0,1}(z) + (\bar{\lambda} + \lambda\, B(z))\,\eta_{2,i+1}(z)\\ &+ (\bar{\lambda}\, r_0 + \lambda\, B(z))\,\mu\, V_{1,i}\,(\eta_{1,1}(z) + \eta_{2,1}(z) + \eta_{3,1}(z)\\ &+ (\bar{\lambda} + \lambda\, B(z))\,\bar{\mu}\, V_{1,i}\,\eta_{1,1}(z) - \bar{\lambda}\, t_{2,i} r_0\bar{\mu}\,(\Omega_{1,1,0} + \Omega_{3,1,0}), \quad i \geq 1,\end{aligned} \tag{20}$$

$$\eta_{3,i}\,(z) = (\bar{\lambda} + \lambda\, B(z))\,\eta_{3,i+1}(z) + \bar{\lambda} V_{2,i}\,\Omega_{1,1,0}, \quad i \geq 1. \tag{21}$$

Multiplying equation (21) by x^i and summing over i, letting $\gamma(z) = \bar{\lambda} + \lambda\, B(z)$, we get,

$$\frac{x - \gamma(z)}{x}\,\eta_3\,(x,z) = -\gamma\,(z)\,\eta_{3,1}(z) + \bar{\lambda}\, V_2\,(x)\,\Omega_{1,1,0}. \tag{22}$$

Setting $x = (z)$ in (22), we have got,

$$\eta_{3,1}(z) = \bar{\lambda}\,\frac{V_2(\gamma(z))}{\gamma(z)}\,\Omega_{1,1,0}. \tag{23}$$

Substituting (23) into (22), we obtain ,

$$\eta_3(x,z) = x\,\bar{\lambda}\left(\frac{[V_2(x) - V_2(\gamma(z))]}{x - \gamma(z)}\right)\Omega_{1,1,0}\,. \tag{24}$$

Differentiating (24) with respect to x and set x=z=0

$$\Omega_{3,1,0}= V_2(\bar{\lambda})\,\Omega_{1,1,0}. \tag{25}$$

Substituting the value of (25) in (3), we get,

$$\Omega_{1,1,0}=\frac{\lambda}{\bar{\lambda}v_2(\bar{\lambda})}\,\Omega_{0,0}. \tag{26}$$

Using x to Multiply (18) to (20) , summing over i, and we get it by using (23), (25), and (26).

$$\begin{aligned}\frac{x-\bar{\lambda}}{x}\eta_0(x,) &= \bar{\lambda}(A(x)-r_0)(\eta_{1,1}(z)+\eta_{2,1}(z))\\ &\quad -\bar{\lambda}\eta_{0,1}(z)-\lambda(A(x)-r_0)\Big(\frac{\gamma(z)(1+V_2(\bar{\lambda})-\bar{\lambda}V_2(\gamma(z))}{\gamma(z)V_2(\bar{\lambda})}\,\Omega_{0,0}, \end{aligned}\tag{27}$$

$$\begin{aligned}\frac{x-\gamma(z)\mu}{x}\eta_1(x,) &= \frac{[\lambda B(z)+\bar{\lambda}r_0]}{z}[\mu T(x)-\mu\gamma(z)]\,\eta_{1,1}(z)+\lambda\mu\,T(x)\frac{B(z)}{z}\,\eta_0(1,z)\\ &\quad +\frac{\bar{\lambda}}{z}\mu\,T(x)\eta_{0,1}(z)+\frac{[\lambda B(z)+\bar{\lambda}r_0]}{z}\mu\,T(x)\,\eta_{2,1}(z)\\ &\quad +\frac{K(z)-r_0\gamma(z)(1+V_2(\bar{\lambda}))}{z\gamma(z)V_2(\bar{\lambda})}\lambda\mu\,T(x)\,\Omega_{0,0}, \end{aligned}\tag{28}$$

$$\begin{aligned}\frac{x-\gamma(z)}{x}\eta_2(x,) &= [\lambda B(z)+\bar{\lambda}r_0]\bar{\mu}\,V_1(x)-\gamma(z)]\,\eta_{2,1}(z)+\bar{\lambda}\,\bar{\mu}\,V_1(x)\,\eta_{0,1}(z)\\ &\quad +\lambda\,\bar{\mu}\,V_1(x)\,B(z)\,\eta_0(1,z)+[[\lambda B(z)+\bar{\lambda}r_0]-z\gamma(z)]\,\bar{\mu}\,V_1(x)\,\eta_{1,1}(z)\\ &\quad +z\gamma(z)\bar{\mu}\;V_1(x)\;\eta_1(1,)+\frac{K(z)-r_0\gamma(z)(1+V_2(\bar{\lambda})}{\gamma(z)V_2(\bar{\lambda})}\lambda\bar{\mu}\,V_1(x)\;\Omega_{0,0}, \end{aligned}$$

(29)

here $K(z)=B(z)\gamma(z)\,V_2(\bar{\lambda})+(\lambda B(z)+\bar{\lambda}r_0)\,V_2(\gamma(z))$.

To find the value of $\eta_0(1,)$ in (28), we put x = 1 in (27) and we get,

$$\begin{aligned}\lambda\,\eta_0(x,) &= (1-r_0)\,\bar{\lambda}(\eta_{1,1}(z)+\eta_{2,1}(z))-\bar{\lambda}\eta_{0,1}(z)\\ &\quad -\lambda(1-r_0)\Big(\frac{\gamma(z)((1+V_2(\bar{\lambda}))-\bar{\lambda}V_2(\gamma(z)))}{\gamma(z)V_2(\bar{\lambda})}\,\Omega_{0,0}. \end{aligned}\tag{30}$$

Substituting (30) into (28), we obtain,

$$\begin{aligned}\frac{x-\gamma(z)\mu}{x}\quad\eta_1(x,) &= \Big[\frac{[B(z)+\bar{\lambda}r_0(1-B(z)]}{z}\mu T(x)-\mu\gamma(z)]\,\eta_{1,1}(z)\\ &\quad +\frac{(1-B(z))}{z}\bar{\lambda}\mu T(x)\,\eta_{0,1}(z)+\frac{[B(z)+\bar{\lambda}r_0(1-B(z)]}{z}\mu T(x)\,\eta_{2,1}(z)\\ &\quad -\frac{B(z)[\gamma(z)-V_2(\gamma(z))]+r_0(1-B(z)\Gamma(z)}{zV_2(\bar{\lambda})\gamma(z)}\lambda\mu\,T(x)\,\Omega_{0,0}. \end{aligned}\tag{31}$$

By setting the x=1 in (31) , we obtain $\eta_1(1,z)$; then substituting $\eta_0(1,z)$ and $\eta_1(1,z)$ in (29), we get

$$\begin{aligned}\frac{x-\gamma(z)}{x}\eta_2(x,) &= \frac{[B(z)+\bar{\lambda}r_0(1-B(z)]}{z}\bar{\mu}\,V_1(x)-\gamma(z)]\,\eta_{2,1}(z)\\ &\quad +\frac{\bar{\lambda}(1-B(z)}{1-\mu\gamma(z)}\bar{\mu}\,V_1(x)\,\eta_{0,1}(z)+\frac{[B(z)+\bar{\lambda}r_0(1-B(z)-z\gamma(z)]}{1-\mu\gamma(z)}\bar{\mu}\,V_1(x)\,\eta_{1,1}(z)\\ &\quad -\frac{B(z)[\gamma(z)-V_2(\gamma(z))]+r_0(1-B(z))\Gamma(z)}{V_2(\bar{\lambda})\gamma(z)(1-\mu\gamma(z))}\lambda\bar{\mu}\,V_1(x)\,\Omega_{0,0}, \end{aligned}\tag{32}$$

where $\Gamma(z)=(\gamma(z)(1+V_2(\bar{\lambda}))-\bar{\lambda}V_2(\gamma(z))$.

$\eta_{0,1}(z)$, $\eta_{1,1}(z)$ and $\eta_{2,1}(z)$ can be obtain by setting $x=\bar{\lambda}$ in (27), $x=\mu\,\gamma(z)$ in (31) and $x=\gamma(z)$ in (32) respectively. The generating functions can be obtained by solving these equations:

$$\eta_{0,1}(z) = \lambda\,(\;(\bar{\lambda}) - \mathrm{r}_0)\frac{(z(1-\mu\,\gamma\,(z))\,\Gamma\,(z) - B(z)(\lambda + V_2(\bar{\lambda}))}{\Lambda(z)\;V_2(\bar{\lambda})}\frac{\Omega_{0,0}}{\bar{\lambda}},$$

$$\eta_{1,1}(z) = \frac{C(z)(\,1-\mu\,\gamma\,(z))\,\lambda\,T(\mu\,\gamma\,(z))}{\Lambda(z)\,V_2(\bar{\lambda})\,\gamma\,(z)}\,\Omega_{0,0}\,,$$

$$\eta_{2,1}(z) = \frac{C(z)z[1-T\,(\mu\gamma(z))]\,\lambda\bar{\mu}\,V_1(\gamma\,(z))}{\Lambda(z)\,V_2(\bar{\lambda})\gamma\,(z)}\,\Omega_{0,0} \tag{33}$$

We obtain $\eta_3(x,\,z)$ by substituting (26) into (24). Similarly, we obtain η_0 (x,) η_1 (x,z) η_2 (x,z) and η_3 (x,z) by substituting (33) into (27), (31) and (32). The unknown constant $\Omega_{0,0}$ can be obtained by applying the normalizing condition. Hence completes the proof.

3.2 Marginal Generating Functions

Using in the theorem lim x ⟶ 1, the marginal generating functions of the orbit size under the stability condition when the server is in various states are as follows:

1. The number of customer in the orbit when the server is free(idle) or on vacation is represented by the following marginal generating function

$$\Omega_{0,0} + \eta_0\,(1,z) + \eta_2\,(x,z)\; + \eta_3\,(x,z)$$

$$= \frac{[(\lambda\mu\,(1-B(z)) + \bar{\mu}(1-z)\,)T\,(\mu\gamma(z)) - z\lambda\mu\,\;(1-B(z))}{(1-B(z))\Lambda(z)V_2(\bar{\lambda})}\,C(z)\;\Omega_{0,0}. \tag{34}$$

2. The number of customers in the orbit when the server is busy is represented by the following marginal generating function

$$\eta_1\,(1,) \;= \frac{[1-T\,(\mu\,\gamma\,(z))]\;C(z)}{\Lambda(z)\,V_2\,(\bar{\lambda})}\;\;\lambda\mu\;\Omega_{0,0}. \tag{35}$$

3. The number of customers in the orbit when the server is on urgent vacation is represented by the following marginal generating function

$$\eta_2\,(1,)\;= \frac{1-V_1\,(\gamma\,(z))}{1-B(z)}\;\frac{C(z)\;\;[1-T(\mu\,\gamma\,(z))]}{\Lambda(z)\,V_2\,(\bar{\lambda})}\;\;\bar{\mu}\,z\,\Omega_{0,0}. \tag{36}$$

4. The number of customers in the orbit when the server is on vacation is represented by the following marginal generating function

$$\eta_3\,(1,) = \frac{1-V_2(\gamma\,(z))}{(1-B(z))\,V_2\,(\bar{\lambda})}\,\Omega_{0,0}. \tag{37}$$

5. The number of customers in the orbit of the generating function is given by

$$\chi(z)\;\; = \Omega_{0,0} + \eta_0\,(1,) + \eta_1\,(1,z)\;\; + \eta_2\,(1,z)\;\; + \eta_3\,(1,z)$$

$$= \frac{[\bar{\mu}\;T(\gamma\,(z))(1-z) + \lambda\mu\,(1-z)(1-B(z))]\;C(z)}{(1-B(z))\Lambda(z)\,V_2\,(\bar{\lambda})}\,\Omega_{0,0}. \tag{38}$$

6. The number of customers in the system of the probability generating function is given by

$$\varphi\,(z)\;\; = \Omega_{0,0} + \eta_0\,(1,) + z\eta_1\,(1,z)\;\; + \eta_2\,(1,z)\;\; + \eta_3\,(1,z)$$

$$= \frac{(1-z)[\bar{\mu}\;\; + \lambda\mu\,(1-B(z))]\;T(\mu\gamma\,(z))\;C(z)}{(1-B(z)\Lambda(z)\,V_2\,(\bar{\lambda})}\,\Omega_{0,0}. \tag{39}$$

7. If the system is empty, then the probability of the system is given by

$$\Omega_{0,0} = \;\; \frac{(1-T(\mu))\;(\,\bar{\mu} + \lambda\mu + \lambda V_{1,1}\bar{\mu}) - \bar{\mu}\;(\lambda + \bar{\lambda}A\;(\bar{\lambda}))}{\bar{\mu}\;T(\mu)[\,\lambda\,(1-V_{2,1}) - A(\bar{\lambda})\;(\lambda + V_2\,(\bar{\lambda})]}\;\;V_2\,(\bar{\lambda})\,. \tag{40}$$

4.1 Performance Measures

1. If the server is idle, then the probability is

$$\Omega_{0,0}+ \eta_0(1,1)+\eta_2(1,1)+\eta_3(1,1) = 1-\frac{\lambda\mu\,(1-T(\mu))}{\bar{\mu}\,T(\mu)}\,. \tag{41}$$

2. If the server is busy, then the probability is given by

$$\eta_1(1,1)=\frac{\lambda\mu\,(1-T(\mu))}{\bar{\mu}\,T(\mu)}\,. \tag{42}$$

3. If the server is on urgent vacation, then the probability is given by

$$\eta_2(1,1) = \frac{\lambda V_{1,1}(1-T(\mu))}{T(\mu)}\,. \tag{43}$$

4. If the server is on normal vacation then the probability is given by

$$\eta_3(1,1) = \lambda V_{1,1}\,\frac{(1-T(\mu)(\bar{\mu}+\lambda\mu+\lambda V_{1,1}\bar{\mu})-\bar{\mu}\,(\lambda+\bar{\lambda}A(\bar{\lambda}))}{\bar{\mu}\,T(\mu)[\lambda(1-V_{2,1})-A(\bar{\lambda})(\lambda+V_2(\bar{\lambda})]}\,. \tag{44}$$

5. The mean number of customers in the orbit with B(1)= B_1 = 1 is

$$E(N)=\chi'(z)_{z=1}=\frac{2\lambda\mu\,(\bar{\mu}T'(\mu)-1)\big(\lambda(1-V_{2,1})\big)-A(\bar{\lambda})(\lambda+V_2(\bar{\lambda})]+2\bar{\mu}\,T(\mu)O-\lambda^2V_{2,2}}{2\bar{\mu}\,T(\mu)[\lambda(1-V_{2,1})-A(\bar{\lambda})(\lambda+V_2(\bar{\lambda})]}$$

$$-\frac{\bar{\mu}(1-T(\mu)\lambda^2V_{1,2}+2\big((1-\bar{\lambda}A(\bar{\lambda})\big)P+2Q}{2\big((1-T(\mu))(\bar{\mu}+\lambda\mu+\lambda V_{1,1}\bar{\mu}\big)-\bar{\mu}(\lambda+\bar{\lambda}A(\bar{\lambda})}\,. \tag{45}$$

6. When B(1)= B_1 = 1=z then the mean number of customers in the system is given by

$$E(L)=\varphi'(z)_{z=1}=\frac{2\lambda\mu\,(\bar{\mu}T'(\mu)-1)\big(\lambda(1-V_{2,1})\big)-A(\bar{\lambda})(\lambda+V_2(\bar{\lambda})]+2\bar{\mu}\,T(\mu)O-\lambda^2V_{2,2}}{2\bar{\mu}\,T(\mu)(\lambda(1-V_{2,1})-A(\bar{\lambda})(\lambda+V_2(\bar{\lambda})}$$

$$+\frac{\lambda\mu\,(1-T(\mu))}{\bar{\mu}\,T(\mu)}+\frac{\bar{\mu}(1-T(\mu)\lambda^2V_{1,2}+2\big((1-\bar{\lambda}A(\bar{\lambda})\big)P+2Q}{2\big((1-T(\mu))(\bar{\mu}+\lambda\mu+\lambda V_{1,1}\bar{\mu}\big)-\bar{\mu}(\lambda+\bar{\lambda}\,(A(\bar{\lambda}))}\,. \tag{46}$$

where

$$O=\lambda\,(1-V_{2,1})-A(\bar{\lambda})\,(\lambda+\lambda\,V_2\,(\bar{\lambda}))-\lambda\bar{\lambda}V_{2,1})\,,$$

$$P=\bar{\mu}\,(1+\lambda\,V_{1,1})(1-T(\mu))-\lambda\,\mu T(\mu),$$

$$Q=\bar{\mu}\,(1-T(\mu))\,\lambda\,V_{1,1}-2\lambda\mu S'(\mu)\,(\lambda\mu+\bar{\mu}+\lambda\,V_{1,1}\bar{\mu})-\lambda\,(\bar{\mu}-\mu-\lambda\mu)\,. \tag{47}$$

The results of (41),(45) and (46) which are similar to Feng Zhang and Zhifeng Zhu [7].

4.2 Numerical Examples

Two performance measures are investigated: If the system is empty then the probability of the system and the mean of orbit size (N). Let us take the service time, retrial time, the normal vacation and the urgent vacation time are all geometric distribution with their parameters r, q, V_2 and V_1 respectively. Their corresponding generating functions are defined by

$$A(x)=\frac{1-r}{1-rx},$$

$$T(x)=\frac{(1-q)\,x}{1-qx}\,,$$

$$V_1(x)=\frac{(1-V_1)\,x}{1-V_1x},$$

$$V_2(x)=\frac{(1-V_2)\,x}{1-V_2x}\qquad \text{for } 0\le x\le 1. \tag{48}$$

In each of the numerical cases, we select the arrival rate λ = 0.3 , the service rate q = 0.3 the retrial rate r = 0.2 and B_1 = 1. The probability that the system is empty is displayed against the parameter μ in Figure 2 and 2, where μ represents the probability that the server does not take an urgent vacation. In the figure 1 we select $V_{1,1}$ = 7/4, 7/3 and 2 and $V_{2,1} = 2$, and in the figure 3 we select $V_{2,1} = 3,6$ and 10.We can identify from the figure 2 and 3 that $\Omega_{0,0}$ significantly increases with parameter μ.

In figure 3 , we can identify that increasing values of $V_{1,1}$ with $\Omega_{0,0}$ decreases for the mean of urgent vacation time with different values $V_{1,1}$. The probability that the system is empty $\Omega_{0,0}$decreases as the mean of an urgent vacation time increases so does the expected waiting time for a customer increases. We can identify that the increasing values of $V_{2,1}$ with $\Omega_{0,0}$ decreases shown in the figure 3. The customers mean number E(N) in the retrial queue (orbit) displayed against the μ in the figure 4 and 5. In Figure 4 we particularly take $_{1,1}$ = 7/4, 7/3 and 2 and in figure 5 we take $V_{2,1} = 3$, 6 and 10. We can identify that the increasing of parameter μ with the (N) exhibits a stable decreasing shown in the figure 4 and 5.

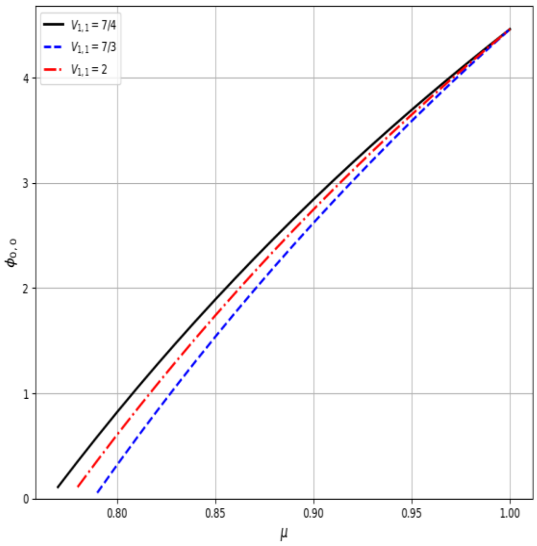

Figure 2

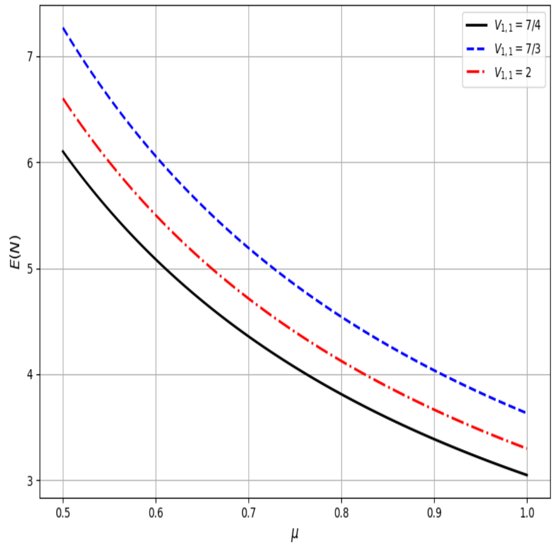

Figure 4

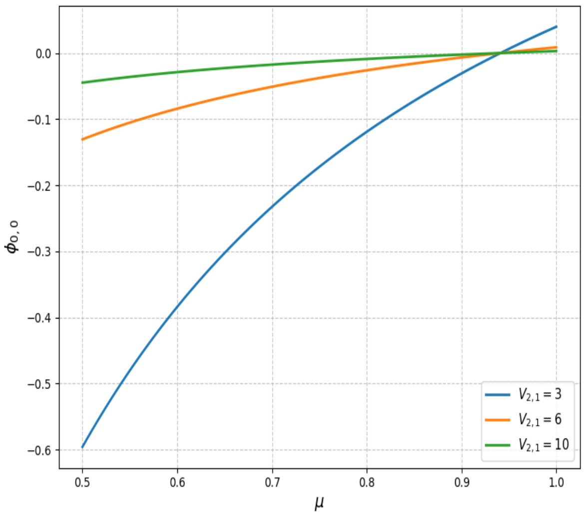

Figure 3

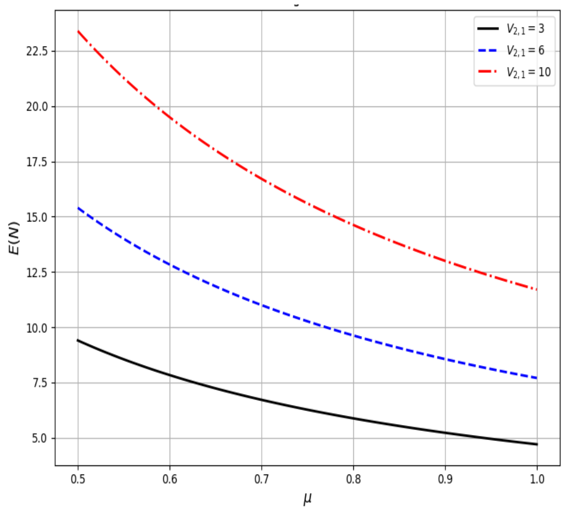

Figure 5

Conclusion

This paper mainly helps researchers make decisions regarding server allocation. Feng Zhang and Zhifeng Zhu [7] obtained results for a single customer. Rajasudha R, Arumuganathan R and Dharmaraja S [12] obtained the result for the batch of customers in the retrial queue with normal vacation. Our findings give insights into decision-making for batch arrivals of customers their service and waiting in orbit for urgent and normal vacations. This investigation took into consideration of a discrete Geo ^{X}/G/1 queueing system with two vacation types: urgent and normal. In the bulk retrial queueing system we investigated the underlying Markov chain. We derived the probability generating functions and the orbit size of the marginal generating functions.

Acknowledgments:

I would like to express my profound appreciation to "D.K.M College for Women (Autonomous), Vellor-1, affiliated to Thiruvalluvar University, Serkkadu, Vellore- 632115, Tamilnadu, India" for giving us resources and support we needed to complete this work. I am especially thankful of the Department of Mathematics great advice and encouragement throughout the course of their work. This paper would not have been possible without their constant support.

References

[1] Aissani A, "An MX/G/1 energetic retrial queue with vacations and it's control," Electronic Notes in Theoretical Computer Science,Vol.253, No.3, 33–44, 2009.
[2] Artalejo J.R, "Analysis of anM/G/1 queue with constant repeated attempts and server vacations," Computers & Operations Research, Vol. 24, No. 6, 493–504, 1997.
[3] Artalejo J.R and Gomez-Corral A, "Retrial Queueing Systems: A Computational Approach", Springer, Berlin, Germany, 2008.
[4] Artalejo J.R, "Accessible bibliography on retrial queues: progress in 2000–2009," Mathematical and Computer Modelling, Vol. 51, No. 9-10, 1071–1081, 2010.
[5] Chang F.M and Ke J.C, "On a batch retrial model with J vacations," Journal of Computational and Applied Mathematics, Vol. 232, No. 2, 402–414, 2009.
[6] Falin G.I, "A survey of retrial queues," Queueing Systems. Theory and Applications, Vol.7, No.2, 127–167,1990.
[7] Feng Zhang and Zhifeng Zhu "A Discrete-Time Geo/G/1 Retrial Queue with Two Different Types of Vacations" Hindawi Publishing Corporation Mathematical Problems in Engineering Volume 2015.
[8] Gomez-Corral A, "A bibliographical guide to the analysis of retrial queues through matrix analytic techniques," Annals of Operations Research, Vol.141, 163–191, 2006.
[9] Ke J.C and Chang F.M, "Modified vacation policy for M/G/1retrial queue with balking and feedback," Computers and Industrial Engineering, Vol.57, No.1, 433–443,2009.
[10] Kumar B.K and Arivudainambi D, "The M/G/1 retrial queue with Bernoulli schedules and general retrial times," Computers & Mathematics with Applications, Vol. 43, No. 1-2, 15–30, 2002.
[11] Kumar B.K, Rukmani R and Thangaraj V,"An M/M/Cretrial queueing system with Bernoulli vacations," Journal of Systems Science and Systems Engineering, Vol. 18, No. 2, 222–242, 2009.
[12] Rajasudha R, Arumuganathan R and Dharmaraja S. "Performance Analysis Of Discrete-Time $GeoX/G/1$ Retrial Queue With Various Vacation Policies And Impatient Customers" RAIRO-Oper. Res. 56 (2022) 1089–1117.
[13] Yang T and Templeton J.G.C, "A survey of retrial queues", Queueing Systems,Vol.2, No.3, 201–233,1987.
[14] Yang T and Li H, "On the steady-state queue size distribution of the discrete time Geo/G/1 queue with repeated customers," Queueing Systems, Vol 21, No1-2, 199-215,1995.