On δ- Open Sets in Picture Fuzzy Topological Spaces

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Abstract:-Picture fuzzy sets offer a powerful mathematical tool for capturing uncertainty and imprecision that cannot be adequately handled by classical or intuitionistic fuzzy set theories. Motivated by this framework, the present work explores the concept of δ-open sets in the context of picture fuzzy topology. Building upon this foundation, the δ-interior and δ-closure operators are formulated and their essential characteristics are examined in detail. In addition, four new generalized categories of δ-open sets are put forward, specifically δ-preopen sets, δ-semiopen sets $\delta\alpha$-open sets and $\delta\beta$-open sets, all of which are thoroughly analyzed under the picture fuzzy topological framework. The mutual relationships between these newly defined classes and the already established picture fuzzy open sets are explored, leading to several significant characterization results. Suitable illustrative examples are also constructed to validate and support the theoretical findings presented in this study.

***Keywords**:* picture fuzzy δ-open sets, picture fuzzy δ-interior, picture fuzzy δ-closure, picture fuzzy δ-preopen sets, picture fuzzy δ-semiopen sets, picture fuzzy δα-open sets and picture fuzzy δβ-open sets.

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1. Introduction

The mathematical theory of fuzzy sets, first proposed by Zadeh [15] in 1965, laid the groundwork for representing uncertainty and vagueness encountered in practical problems. Building on this foundational contribution, numerous extended models have been developed to handle various forms of imprecise information more efficiently. Among these extensions, Atanassov [2] generalized fuzzy sets by introducing intuitionistic fuzzy sets, which simultaneously account for both membership and non-membership degrees, thus providing a richer and more adaptable representation of uncertainty. The fusion of topological ideas with fuzzy frameworks was pioneered by Chang [3], who constructed fuzzy topological spaces as a generalization of classical topology, thereby creating new possibilities for examining continuity, convergence and separation properties in uncertain settings. Following this, Coker [4] put forward intuitionistic fuzzy topological spaces, which further deepened the study of topology in uncertain environments by providing more sophisticated analytical instruments. In recent years, picture fuzzy sets, originally introduced by Cuong et.al [5, 6], have drawn significant research interest owing to their capacity to encode uncertainty through three mutually independent components, namely positive, neutral and negative membership degrees. The inclusion of the neutrality component renders picture fuzzy sets particularly well-suited for capturing the complexity inherent in decision-making processes and information-based systems. Driven by these merits, picture fuzzy topological spaces have attracted growing scholarly attention, with notable contributions from Razaq et al. [11], who examined a variety of topological properties within this enriched setting. Recent advances in picture fuzzy topology include the work of Alshammari et al. [1], who investigated picture fuzzy nano topological spaces and designed a MADM algorithm in that context. Kishorekumar et al. [9] put forward interval-valued picture fuzzy topological spaces and studied their utility in multi-criteria decision-making scenarios. Additionally, Unver [12] developed q-rung orthopair picture fuzzy topology and analyzed parameter-dependent continuity arising in control system applications. Taken together, these works underscore the expanding relevance of picture fuzzy topology as a vibrant domain of mathematical inquiry.

On the other hand, generalized open sets have long occupied a central position in topological research, as they yield deeper structural insights into topological spaces and are indispensable in the study of continuity and associated properties. Of particular interest is the notion of δ-open sets, which has been investigated across a wide range of topological frameworks. The concept was originally put forth by Velicko [14] in the context of classical topology. It was subsequently carried over to intuitionistic fuzzy topology by Eom et.al [7], who developed the corresponding δ-interior and δ-closure operators and derived several noteworthy results. Latif [10] further advanced this line of research by analyzing δ-open sets alongside δ-continuous functions. Moreover, Vadivel et al. [13] successfully transplanted the idea of δ-open sets into the neutrosophic topological setting. Nevertheless, in spite of the remarkable progress witnessed in picture fuzzy topology, the notion of δ-open sets has remained largely unexplored within this framework, and it is precisely this gap that the present work aims to address.

In this article, the concept of δ-open sets is defined and systematically examined within picture fuzzy topology. Grounded in this concept, associated δ-interior and δ-closure operations are constructed and their key characteristics are thoroughly investigated. Beyond this, several broader families of δ-open sets, comprising δ-preopen, δ-semiopen, $\delta\alpha$-open and $\delta\beta$-open sets, are defined and analyzed in depth. The interconnections between these families and the pre-existing classes of picture fuzzy open sets are systematically established, and relevant characterization results are derived, all of which are reinforced by carefully chosen illustrative examples.

2. Preliminaries

Definition 2.1 [15] Let $\widetilde{\mathfrak{W}}$ be a universe of discourse. A fuzzy set $\widetilde{A_0}$ in $\mathfrak{W}$ is expressed as $\widetilde{A_0} = \{\langle w, \mu_{\widetilde{A_0}}(w)\rangle \mid w \in \widetilde{\mathfrak{W}}\}$, where $\mu_{\widetilde{A_0}}(w) \in [0, 1]$ assigns the degree to which the element w belongs to $\widetilde{A_0}$.

Definition 2.2 [2] An intuitionistic fuzzy set $\widetilde{A_0}$ in $\widetilde{\mathfrak{W}}$ is specified by a pair of mappings $\mu_{\widetilde{A_0}}$ and $\nu_{\widetilde{A_0}}$ representing the grades of membership and non-membership, respectively. Thus, $\widetilde{A_0}$ may be written as $\{\langle w, \mu_{\widetilde{A_0}}(w), \nu_{\widetilde{A_0}}(w)\rangle \mid w \in \widetilde{\mathfrak{W}}\}$where $\mu_{\widetilde{A_0}}(w), \nu_{\widetilde{A_0}}(w) \in [0, 1]$ and $0 \leq \mu_{\widetilde{A_0}}(w) + \nu_{\widetilde{A_0}}(w) \leq 1$. The quantity $\pi_{\widetilde{A_0}}(w) = 1 - \mu_{\widetilde{A_0}}(w) - \nu_{\widetilde{A_0}}(w)$ is called the hesitation (or indeterminacy) degree of w.

Definition 2.3 [2] Let $\widetilde{Q_0} = (\mu_{\widetilde{Q_0}}, \nu_{\widetilde{Q_0}})$ and $\widetilde{K_0} = (\mu_{\widetilde{K_0}}, \nu_{\widetilde{K_0}})$ be two intuitionistic fuzzy sets defined over $\widetilde{\mathfrak{W}}$. The following set-theoretic relations and operations are defined for all $w \in \widetilde{\mathfrak{W}}$:

a) $\widetilde{Q_0} \leq \widetilde{K_0}$ iff $\mu_{\widetilde{Q_0}}(w) \leq \mu_{\widetilde{K_0}}(w)$ & $\nu_{\widetilde{Q_0}}(w) \geq \nu_{\widetilde{K_0}}(w)$.

b) $\widetilde{Q_0} = \widetilde{K_0} \Leftrightarrow \widetilde{Q_0} \leq \widetilde{K_0}$ and $\widetilde{K_0} \leq \widetilde{Q_0}$.

c) $\widetilde{Q_0}^{\,c} = (\nu_{\widetilde{Q_0}}, \mu_{\widetilde{Q_0}})$.

d) $\widetilde{Q_0} \cap \widetilde{K_0} = (\mu_{\widetilde{Q_0}} \wedge \mu_{\widetilde{K_0}}, \nu_{\widetilde{Q_0}} \vee \nu_{\widetilde{K_0}})$.

e) $\widetilde{Q_0} \cup \widetilde{K_0} = (\mu_{\widetilde{Q_0}} \vee \mu_{\widetilde{K_0}}, \nu_{\widetilde{Q_0}} \wedge \nu_{\widetilde{K_0}})$.

f) The intuitionistic fuzzy empty set is $\breve{0} = \langle w, 0, 1\rangle$ and the intuitionistic fuzzy whole set is $\breve{1} = \langle w, 1, 0\rangle$.

Definition 2.4 [4] Let $\widetilde{\mathfrak{W}}$ be a nonempty set. A subfamily τ of the class of all intuitionistic fuzzy sets defined on $\widetilde{\mathfrak{W}}$is called an intuitionistic fuzzy topology provided that::

a) $\breve{0}, \breve{1}$ are members of τ ,

b) for any $\widetilde{G_1}, \widetilde{G_2} \in \tau$, the set $\widetilde{G_1} \cap \widetilde{G_2}$ also belongs to τ

c) whenever $\{\widetilde{G_\iota}: i \in J\} \subseteq \tau$, one has $\cup_{i\in J}\ \widetilde{G_\iota}: \in \tau$.

The structure $(\widetilde{\mathfrak{W}}, \tau)$ is termed an intuitionistic fuzzy topological space (IFts). The members of τ are referred to as intuitionistic fuzzy open sets.

Definition 2.5 [4] Let $(\widetilde{\mathfrak{W}}, \tau)$ be an IFts and $\widetilde{A_0}$ be an intuitionistic fuzzy set in $\mathfrak{W}$. Then

(a) The intuitionistic fuzzy interior of $\widetilde{A_0}$ (in short, IFint($\widetilde{A_0}$)) is given by

$$\text{IFint}(\widetilde{A_0}) = \cup\{\tilde{G} : \tilde{G} \subseteq \widetilde{A_0} \text{ and } \tilde{G} \text{ is an IFos in } \widetilde{\mathfrak{W}}\}.$$

(b) The intuitionistic fuzzy closure of $\widetilde{A_0}$ (in short, IFcl($\widetilde{A_0}$)) is defined as

$$\text{IFcl}(\widetilde{A_0}) = \cap\{\tilde{K} : \tilde{K} \supseteq \widetilde{A_0} \text{ and and } \tilde{G} \text{ is an IFcs in } \widetilde{\mathfrak{W}}\}.$$

Definition 2.6 [8] Let $(\widetilde{\mathfrak{W}}, \tau)$ be an IFts and $\widetilde{A_0}$ be an intuitionistic fuzzy set in $\widetilde{\mathfrak{W}}$. $\widetilde{A_0}$ is said to be an intuitionistic fuzzy regular open set whenever $\widetilde{A_0}$ = IFint(IFcl($\widetilde{A_0}$)) holds. The complement of such a set is referred to as an intuitionistic fuzzy regular closed set.

Definition 2.7 [5] A picture fuzzy set $\widetilde{A_0}$ in $\widetilde{\mathfrak{W}}$ takes the form $\widetilde{A_0} = \{\langle w, \mu_{\widetilde{A_0}}(w), \eta_{\widetilde{A_0}}(w), \nu_{\widetilde{A_0}}(w)\rangle \mid w \in \mathfrak{W}\}$ where $\mu_{\widetilde{A_0}}(w)$, $\eta_{\widetilde{A_0}}(w)$ and $\nu_{\widetilde{A_0}}(w)$ stand for the positive, neutral and negative membership grades respectively, subject to $0 \leq \mu_{\widetilde{A_0}}(w) + \eta_{\widetilde{A_0}}(w) + \nu_{\widetilde{A_0}}(w) \leq 1$, $\forall$ w $\in \widetilde{\mathfrak{W}}$. The refusal degree is accordingly given by $\rho_{\widetilde{A_0}}(w) = 1 - \mu_{\widetilde{A_0}}(w) - \eta_{\widetilde{A_0}}(w) - \nu_{\widetilde{A_0}}(w)$.

Definition 2.8 [5] Let $\widetilde{K_0} = \langle w, \mu_{\widetilde{K_0}}(w), \eta_{\widetilde{K_0}}(w), \nu_{\widetilde{K_0}}(w)\rangle$ and $\widetilde{Q_0} = \langle w, \mu_{\widetilde{Q_0}}(w), \eta_{\widetilde{Q_0}}(w), \nu_{\widetilde{Q_0}}(w)\rangle$ be two picture fuzzy sets in $\mathfrak{W}$. Then the following operations hold:

(i) $\widetilde{K_0} \subseteq \widetilde{Q_0}$ if $\mu_{\widetilde{K_0}}(w) \leq \mu_{\widetilde{Q_0}}(w)$, $\eta_{\widetilde{K_0}}(w) \leq \eta_{\widetilde{Q_0}}(w)$ and $\nu_{\widetilde{K_0}}(w) \geq \nu_{\widetilde{Q_0}}(w)$ for all w $\in \mathfrak{W}$.

(ii) $\widetilde{K_0} = \widetilde{Q_0}$ if $\widetilde{K_0} \subseteq \widetilde{Q_0}$ and $\widetilde{Q_0} \subseteq \widetilde{K_0}$.

(iii) $\widetilde{K_0}^c = \langle w, \nu_{\widetilde{K_0}}(w), \eta_{\widetilde{K_0}}(w), \mu_{\widetilde{K_0}}(w)\rangle$.

(iv) $\widetilde{K_0} \cup \widetilde{Q_0} = \langle w, \mu_{\widetilde{K_0}}(w) \vee \mu_{\widetilde{Q_0}}(w), \eta_{\widetilde{K_0}}(w) \wedge \eta_{\widetilde{Q_0}}(w)), \nu_{\widetilde{K_0}}(w) \wedge \nu_{\widetilde{Q_0}}(w)\rangle$.

(v) $\widetilde{K_0} \cap \widetilde{Q_0} = \langle w, \mu_{\widetilde{K_0}}w) \wedge \mu_{\widetilde{Q_0}}(w), \eta_{\widetilde{K_0}}(w) \wedge \eta_{\widetilde{Q_0}}(w), \nu_{\widetilde{K_0}}(w) \vee \nu_{\widetilde{Q_0}}(w)\rangle$.

(vi) The picture fuzzy empty set is $0_{\mathbb{P}_{ic}\mathbb{F}} = \langle w, 0, 0, 1\rangle$ and the picture fuzzy whole set is $1_{\mathbb{P}_{ic}\mathbb{F}} = \langle w, 1, 0, 0\rangle$.

Definition 2.9 [11] A picture fuzzy topology on $\widetilde{\mathfrak{W}}$ is a collection τ of picture fuzzy sets in $\widetilde{\mathfrak{W}}$ fulfilling the conditions below:

(i) $0_{\mathbb{P}_{ic}\mathbb{F}}$, $1_{\mathbb{P}_{ic}\mathbb{F}}$, $\in \tau$.

(ii) If $\widetilde{K_0}$, $\widetilde{Q_0}$, $\in \tau$, then $\widetilde{K_0}, \cap \widetilde{Q_0} \in \tau$.

(iii) If $\{\widetilde{K_\alpha} : \alpha \in \Lambda\} \subseteq \tau$, then $\cup_{\alpha\in\Lambda} \widetilde{K_\alpha} \in \tau$.

The ordered pair $(\widetilde{\mathfrak{W}}, \tau)$ constitutes a picture fuzzy topological space, denoted $\mathbb{P}_{ic}\mathbb{F}$ts. The elements of τ are designated as picture fuzzy open sets (*shortly*, $\mathbb{P}_{ic}\mathbb{F}$os) and their complements are designated as picture fuzzy closed sets (*shortly*, $\mathbb{P}_{ic}\mathbb{F}$cs).

3 Picture Fuzzy δ-open Sets in $\mathbb{P}_{ic}\mathbb{F}ts$.

In section 3 and 4, let $(\widetilde{\mathfrak{W}}, \tau)$ be any $\mathbb{P}_{ic}\mathbb{F}ts$. Let $\tilde{A}$, $\tilde{D}$, $\tilde{K}$, $\tilde{H}$, $\tilde{B}$, $\tilde{Q}$, $\tilde{P}$, $\tilde{S}$ and $\tilde{T}$ be picture fuzzy sets in $\mathbb{P}_{ic}\mathbb{F}ts$.

Definition 3.1 Let $(\widetilde{\mathfrak{W}}, \tau)$ be a $\mathbb{P}_{ic}\mathbb{F}ts$ and $\tilde{A}$ be a picture fuzzy set in $\widetilde{\mathfrak{W}}$. Then the picture fuzzy

(i) interior of $\tilde{A}$ (shortly, $\mathbb{P}_{ic}\mathbb{F}$int($\tilde{A}$)) is characterized by

$\mathbb{P}_{ic}\mathbb{F}$int($\tilde{A}$) = $\cup\{\tilde{D} : \tilde{D} \subseteq \tilde{A}$ and $\tilde{D}$ is a $\mathbb{P}_{ic}\mathbb{F}os\ in\ \widetilde{\mathfrak{W}}\}$.

(ii) closure of $\tilde{A}$ (shortly, $\mathbb{P}_{ic}\mathbb{F}$cl($\tilde{A}$)) is characterized by

$\mathbb{P}_{ic}\mathbb{F}$cl($\tilde{A}$) = $\cap\{\tilde{D} : \tilde{D} \supseteq \tilde{A}$ and $\tilde{D}$ is a $\mathbb{P}_{ic}\mathbb{F}cs\ in\ \widetilde{\mathfrak{W}}\}$.

Definition 3.2 Let $(\widetilde{\mathfrak{W}}, \tau)$ be a $\mathbb{P}_{ic}\mathbb{F}ts$. A $\mathbb{P}_{ic}\mathbb{F}S\ \tilde{A}$ is called

(i) regular open set (shortly, $\mathbb{P}_{ic}\mathbb{F}$ Ros) if $\tilde{A}$ = $\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}$cl($\tilde{A}$)).

(ii) preopen set (shortly, $\mathbb{P}_{ic}\mathbb{F}$ Pos) if $\tilde{A} \subseteq \mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}\mathbb{F}$cl($\tilde{A}$)).

(iii) semiopen set (shortly, $\mathbb{P}_{ic}\mathbb{F}$Sos) if $\tilde{A} \subseteq \mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}$int($\tilde{A}$)).

(iv) α-open set (shortly, $\mathbb{P}_{ic}\mathbb{F}\alpha$os) if $\tilde{A} \subseteq \mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}$int($\tilde{A}$))).

(v) β-open set (shortly, $\mathbb{P}_{ic}\mathbb{F}\beta$os) if $\tilde{A} \subseteq \mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}$ int($\mathbb{P}_{ic}\mathbb{F}$cl($\tilde{A}$))).

The complement of a $\mathbb{P}_{ic}\mathbb{F}$Ros (resp. $\mathbb{P}_{ic}\mathbb{F}$Pos, $\mathbb{P}_{ic}\mathbb{F}$Sos, $\mathbb{P}_{ic}\mathbb{F}\alpha$os and $\mathbb{P}_{ic}\mathbb{F}\beta$os) is called the picture fuzzy regular (resp. pre, semi, α and β) closed set (shortly, $\mathbb{P}_{ic}\mathbb{F}$Rcs (resp. $\mathbb{P}_{ic}\mathbb{F}$Pcs, $\mathbb{P}_{ic}\mathbb{F}$Scs, $\mathbb{P}_{ic}\mathbb{F}\alpha$cs and $\mathbb{P}_{ic}\mathbb{F}\beta$cs)) in $\widetilde{\mathfrak{W}}$.

The family of all $\mathbb{P}_{ic}\mathbb{F}$Ros(resp. $\mathbb{P}_{ic}\mathbb{F}$Rcs, $\mathbb{P}_{ic}\mathbb{F}$Pos, $\mathbb{P}_{ic}\mathbb{F}$Pcs, $\mathbb{P}_{ic}\mathbb{F}$Sos, $\mathbb{P}_{ic}\mathbb{F}Scs$, $\mathbb{P}_{ic}\mathbb{F}\alpha$os, $\mathbb{P}_{ic}\mathbb{F}\alpha$cs, $\mathbb{P}_{ic}\mathbb{F}\beta$os and $\mathbb{P}_{ic}\mathbb{F}\beta$cs) of $\widetilde{\mathfrak{W}}$ is represented by $\mathbb{P}_{ic}\mathbb{F}$Ros($\widetilde{\mathfrak{W}}$) (resp. $\mathbb{P}_{ic}\mathbb{F}$Rcs($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}$Pos($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}$Pcs($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}$Sos($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}$Scs($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}\alpha$os($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}\alpha$cs($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}\beta$os($\widetilde{\mathfrak{W}}$) and $\mathbb{P}_{ic}\mathbb{F}\beta$cs($\widetilde{\mathfrak{W}}$)).

Definition 3.3 Let $\tilde{A}$ be a $\mathbb{P}_{ic}\mathbb{F}S$. Then

(i) picture fuzzy δ-interior of $\tilde{A}$ (shortly, $\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{A}$)) is characterized by

$\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{A}$) = $\cup\{\tilde{D} : \tilde{D} \subseteq \tilde{A}$ and $\tilde{D}$ is a $\mathbb{P}_{ic}\mathbb{F}$Ros in $\widetilde{\mathfrak{W}}\}$.

(ii) picture fuzzy δ-closure of $\tilde{A}$ (shortly, $\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$)) is characterized by

$\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$) = $\cap\{\tilde{D} : \tilde{D} \supseteq \tilde{A}$ and $\tilde{D}$ is a $\mathbb{P}_{ic}\mathbb{F}$Rcs in $\widetilde{\mathfrak{W}}\}$.

Definition 3.4 A $\mathbb{P}_{ic}\mathbb{F}S\ \tilde{Q}$ is known as picture fuzzy

a) δ-open set (shortly, $\mathbb{P}_{ic}\mathbb{F}\delta os$) if $\tilde{Q} = \mathbb{P}_{ic}\mathbb{F}\delta int(\tilde{Q})$.

b) δ-pre open set (shortly, $\mathbb{P}_{ic}\mathbb{F}\delta Pos$) if $\tilde{Q} \subseteq \mathbb{P}_{ic}\mathbb{F}int(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{Q}))$.

c) δ-semi open set (shortly, $\mathbb{P}_{ic}\mathbb{F}\delta$Sos) if $\tilde{Q} \subseteq \mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{Q}$)).

d) $\delta\alpha$ open set or a-open set (shortly, $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$os or $\mathbb{P}_{ic}\mathbb{F}$aos) if $\tilde{Q} \subseteq \mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{Q}$))).

e) $\delta\beta$ open set or e*-open set (shortly, $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os or $\mathbb{P}_{ic}\mathbb{F}$e*os) if $\tilde{Q} \subseteq \mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{Q}$))).

The complement of a $\mathbb{P}_{ic}\mathbb{F}\delta$os (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Pos, $\mathbb{P}_{ic}\mathbb{F}\delta$Sos, $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$os and $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os) is called the picture fuzzy δ (resp. δ-pre, δ-semi, $\delta\alpha$ and $\delta\beta$) closed set (shortly, $\mathbb{P}_{ic}\mathbb{F}\delta$cs (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Pcs, $\mathbb{P}_{ic}\mathbb{F}\delta$Scs, $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$cs and $\mathbb{P}_{ic}\mathbb{F}\delta\beta$cs)) in $\widetilde{\mathfrak{W}}$.

The collection of all $\mathbb{P}_{ic}\mathbb{F}\delta$Pos (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Pcs, $\mathbb{P}_{ic}\mathbb{F}\delta$Sos, $\mathbb{P}_{ic}\mathbb{F}\delta$Scs, $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$os, $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$cs, $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os and $\mathbb{P}_{ic}\mathbb{F}\delta\beta$cs) of $\widetilde{\mathfrak{W}}$ is characterized by $\mathbb{P}_{ic}\mathbb{F}\delta$Pos($\widetilde{\mathfrak{W}}$) (resp, $\mathbb{P}_{ic}\mathbb{F}\delta$Pcs($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}\delta$Sos($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}\delta$Scs($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$os($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$cs($\widetilde{\mathfrak{W}}$), $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os($\widetilde{\mathfrak{W}}$) and $\mathbb{P}_{ic}\mathbb{F}\delta\beta$cs($\widetilde{\mathfrak{W}}$)).

Definition 3.5 A $\mathbb{P}_{ic}\mathbb{F}S$ $\tilde{A}$ is known as $\mathbb{P}_{ic}\mathbb{F}\delta$-pre(resp. $\mathbb{P}_{ic}\mathbb{F}\delta$-semi, $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$ and $\mathbb{P}_{ic}\mathbb{F}\delta\beta$) interior of $\tilde{A}$ (shortly, $\mathbb{P}_{ic}\mathbb{F}\delta$Pint($\tilde{A}$)(resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Sint($\tilde{A}$), $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$int($\tilde{A}$) and $\mathbb{P}_{ic}\mathbb{F}\delta\beta$int($\tilde{A}$))) is the union of all $\mathbb{P}_{ic}\mathbb{F}\delta$Pos (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Sos, $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$os and $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os) contained in $\tilde{A}$.

Definition 3.6 A $\mathbb{P}_{ic}\mathbb{F}S$ $\tilde{A}$ is known as $\mathbb{P}_{ic}\mathbb{F}\delta$-pre (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$-semi, $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$ and $\mathbb{P}_{ic}\mathbb{F}\delta\beta$) closure of $\tilde{A}$ (shortly, $\mathbb{P}_{ic}\mathbb{F}\delta$Pcl($\tilde{A}$)(resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Scl($\tilde{A}$), $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$cl($\tilde{A}$) and $\mathbb{P}_{ic}\mathbb{F}\delta\beta$cl($\tilde{A}$))) is the intersection of all $\mathbb{P}_{ic}\mathbb{F}\delta$Pcs(resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Scs, $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$cs & $\mathbb{P}_{ic}\mathbb{F}\delta\beta$cs) containing $\tilde{A}$.

Preposition 3.1 The picture fuzzy δ-interior operator satisfies

a) $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K}) \subseteq \widetilde{K}$.
b) $\widetilde{K} \subseteq \widetilde{H} \Rightarrow \mathbb{P}_{ic}\mathbb{F}\ \delta\text{int}(\widetilde{K}) \subseteq \mathbb{P}_{ic}\mathbb{F}\ \delta\text{int}(\widetilde{H})$.
c) $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K} \cap \widetilde{H}) = \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K}) \cap \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{H})$.
d) $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K})$ is the greatest $\mathbb{P}_{ic}\mathbb{F}\delta$os contained in $\widetilde{K}$.
e) $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K}) = \widetilde{K}$ iff $\widetilde{K}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$os.
f) $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K})) = \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K})$.

g) $\widetilde{\mathfrak{W}} - \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K}) = \mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\widetilde{\mathfrak{W}} - \widetilde{K})$.

Proof.

a) $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K}) = \cup\{\tilde{S} : \tilde{S} \subseteq \widetilde{K}$ and $\tilde{S}$ is a $\mathbb{P}_{ic}\mathbb{F}Ros\}$. Thus, $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K}) \subseteq \widetilde{K}$.

b) $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{H}) = \cup\{\tilde{S} : \tilde{S} \subseteq \widetilde{H}$ and $\tilde{S}$ is a $\mathbb{P}_{ic}\mathbb{F}Ros\} \supseteq \cup\{\tilde{S} : \tilde{S} \subseteq \widetilde{K}$ and $\tilde{S}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$Ros$\} \supseteq \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K})$. Thus, $\mathbb{P}_{ic}\mathbb{F}\ \delta\text{int}(\widetilde{K}) \subseteq \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{H})$.

c) $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K} \cap \widetilde{H}) = \cup\{\tilde{S} : \tilde{S} \subseteq \widetilde{K} \cap \widetilde{H}$ and $\tilde{S}$ is a $\mathbb{P}_{ic}\mathbb{F}Ros\}$ = $(\cup\{\tilde{S} : \tilde{S} \subseteq \widetilde{K}$ and $\tilde{S}$ is a $\mathbb{P}_{ic}\mathbb{F}Ros\}) \cap (\cup\{\tilde{S} : \tilde{S} \subseteq \widetilde{H}$ and $\tilde{S}$ is a $\mathbb{P}_{ic}\mathbb{F}Ros\})$ = $\mathbb{P}_{ic}\mathbb{F}\ \delta\text{int}(\widetilde{K}) \cap \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{H})$. Thus, $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K} \cap \widetilde{H}) = \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K}) \cap \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{H})$.

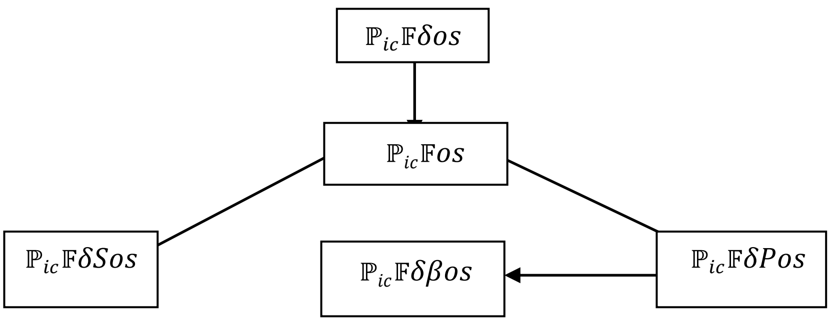
d) If $\widetilde{T}$ is any $\mathbb{P}_{ic}\mathbb{F}\delta$os contained in $\widetilde{K}$, then $\widetilde{T} \subseteq \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K})$. Hence, $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K})$ is the greatest $\mathbb{P}_{ic}\mathbb{F}\delta$os contained in $\widetilde{K}$.

e) Let $\widetilde{K}$ be a $\mathbb{P}_{ic}\mathbb{F}\delta$os of $\widetilde{\mathfrak{W}}$. Since $\widetilde{K}$ itself is the largest $\mathbb{P}_{ic}\mathbb{F}\delta$os contained in $\widetilde{K}$, it follows that $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K}) = \widetilde{K}$.

f) By (d), the greatest $\mathbb{P}_{ic}\mathbb{F}\delta$os contained in $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K})$ is itself. Hence, $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K})) = \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K})$.

g) $\mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K})$ is the greatest $\mathbb{P}_{ic}\mathbb{F}\delta$os contained in $\widetilde{K}$. The complement is the smallest $\mathbb{P}_{ic}\mathbb{F}\delta$cs containing $\widetilde{\mathfrak{W}} - \widetilde{K}$. Therefore, $\widetilde{\mathfrak{W}} - \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{K}) = \mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\widetilde{\mathfrak{W}} - \widetilde{K})$.

Remark 3.1 The following diagram illustrates the relationship among various classes of picture fuzzy δ-open sets in $\mathbb{P}_{ic}\mathbb{F}ts$.



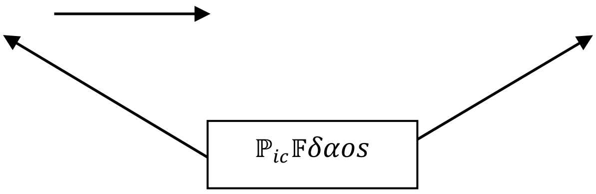


Diagram : I - $\mathbb{P}_{ic}\mathbb{F}\delta$o sets in $\mathbb{P}_{ic}\mathbb{F}ts$

Remark 3.1 The examples presented below establish that none of the reverse implications appearing in the above diagram hold not true in general.

Example 3.1 Let $\widetilde{\mathfrak{W}}$ = { $x_{1,}$ $x_{2,}$ x_3 }. Let $\widetilde{A_1}$, $\widetilde{B_1}$ and $\widetilde{C_1}$ are $\mathbb{P}_{ic}\mathbb{F}$S defined as follows:

$$\widetilde{A_1} = \langle\left(\frac{\mu_{x_{1,}}}{0.1}, \frac{\sigma_{x_1}}{0.1}, \frac{\nu_{x_1}}{0.6}\right), \left(\frac{\mu_{x_2}}{0}, \frac{\sigma_{x_2}}{0.1}, \frac{\nu_{x_2}}{0.7}\right), \left(\frac{\mu_{x_3}}{0.1}, \frac{\sigma_{x_3}}{0.2}, \frac{\nu_{x_3}}{0.5}\right)\rangle$$

$$\widetilde{B_1} = \langle\left(\frac{\mu_{x_{1,}}}{0.2}, \frac{\sigma_{x_1}}{0.1}, \frac{\nu_{x_1}}{0.5}\right), \left(\frac{\mu_{x_2}}{0.3}, \frac{\sigma_{x_2}}{0.1}, \frac{\nu_{x_2}}{0.5}\right), \left(\frac{\mu_{x_3}}{0.3}, \frac{\sigma_{x_3}}{0.2}, \frac{\nu_{x_3}}{0.4}\right)\rangle$$

$$\widetilde{C_1} = \langle\left(\frac{\mu_{x_{1,}}}{0.3}, \frac{\sigma_{x_1}}{0.1}, \frac{\nu_{x_1}}{0.4}\right), \left(\frac{\mu_{x_2}}{0.4}, \frac{\sigma_{x_2}}{0.1}, \frac{\nu_{x_2}}{0.5}\right), \left(\frac{\mu_{x_3}}{0.4}, \frac{\sigma_{x_3}}{0.2}, \frac{\nu_{x_3}}{0.4}\right)\rangle$$

Here, τ = { $0_{\mathbb{P}_{ic}\mathbb{F}}$, $1_{\mathbb{P}_{ic}\mathbb{F}}$, $\widetilde{A_1}$, $\widetilde{B_1}$, $\widetilde{C_1}$} be a $\mathbb{P}_{ic}\mathbb{F}ts$. Then the following results hold:

(i) The $\mathbb{P}_{ic}\mathbb{F}S's$ $\widetilde{A_1}$ and $\widetilde{B_1}$ are $\mathbb{P}_{ic}\mathbb{F}$os but not $\mathbb{P}_{ic}\mathbb{F}\delta$os

(ii) The $\mathbb{P}_{ic}\mathbb{F}S$ $\widetilde{A_1}$ is $\mathbb{P}_{ic}\mathbb{F}\delta\beta os$ but not $\mathbb{P}_{ic}\mathbb{F}\delta$Sos.

(iii) The $\mathbb{P}_{ic}\mathbb{F}S's\widetilde{A_1}$ and $\widetilde{B_1}$ are $\mathbb{P}_{ic}\mathbb{F}\delta$Pos but not $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$os.

Example 3.2 In Example 3.1, let $\widetilde{D_1}$ and $\widetilde{E_1}$ be $\mathbb{P}_{ic}\mathbb{F}$ S defined as follows:

$$\widetilde{D_1} = \langle\left(\frac{\mu_{x_{1,}}}{0.6}, \frac{\sigma_{x_1}}{0.2}, \frac{\nu_{x_1}}{0.1}\right), \left(\frac{\mu_{x_2}}{0.7}, \frac{\sigma_{x_2}}{0.2}, \frac{\nu_{x_2}}{0}\right), \left(\frac{\mu_{x_3}}{0.5}, \frac{\sigma_{x_3}}{0.3}, \frac{\nu_{x_3}}{0.1}\right)\rangle$$

$$\widetilde{E_1} = \langle\left(\frac{\mu_{x_{1,}}}{0.4}, \frac{\sigma_{x_1}}{0.1}, \frac{\nu_{x_1}}{0.3}\right), \left(\frac{\mu_{x_2}}{0.5}, \frac{\sigma_{x_2}}{0.1}, \frac{\nu_{x_2}}{0.4}\right), \left(\frac{\mu_{x_3}}{0.4}, \frac{\sigma_{x_3}}{0.2}, \frac{\nu_{x_3}}{0.4}\right)\rangle$$

Then the following results hold:

(i) The $\mathbb{P}_{ic}\mathbb{F}S$ $\widetilde{D_1}$ is $\mathbb{P}_{ic}\mathbb{F}\delta$Pos but not $\mathbb{P}_{ic}\mathbb{F}$os.

(ii) The $\mathbb{P}_{ic}\mathbb{F}S$ $\widetilde{E_1}$ is $\mathbb{P}_{ic}\mathbb{F}\delta$Sos but not $\mathbb{P}_{ic}\mathbb{F}$os.

(iii) The $\mathbb{P}_{ic}\mathbb{F}S$ $\widetilde{E_1}$ is $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os but not $\mathbb{P}_{ic}\mathbb{F}\delta$Pos.

(iv) The $\mathbb{P}_{ic}\mathbb{F}S$ $\widetilde{E_1}$ is $\mathbb{P}_{ic}\mathbb{F}\delta$Sos but not $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$os.

Preposition 3.2 The picture fuzzy δ-closure operator satisfies

(i) $\tilde{A} \subseteq \mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\tilde{A})$.

(ii) $\tilde{A} \subseteq \widetilde{D} \Rightarrow \mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\tilde{A}) \subseteq \mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\widetilde{D})$.

(iii) $\mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\tilde{A} \cup \widetilde{D}) = \mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\tilde{A}) \cup \mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\widetilde{D})$.

(iv) $\mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\tilde{A})$ is the smallest $\mathbb{P}_{ic}\mathbb{F}\delta$cs containing $\tilde{A}$.

(v) $\mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\tilde{A})$ = A iff $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$cs.

(vi) $\mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\tilde{A})) = \mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\tilde{A})$.

(vii) $\widetilde{\mathfrak{W}} - \mathbb{P}_{ic}\mathbb{F}\delta\text{cl}(\tilde{A}) = \mathbb{P}_{ic}\mathbb{F}\delta\text{int}(\widetilde{\mathfrak{W}} - \tilde{A})$.

(viii) x ∈ $\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$) iff ($\tilde{A} \cap \tilde{Q}$) ≠ ∅ for every $\mathbb{P}_{ic}\mathbb{F}\delta$os $\tilde{Q}$ containing x.

Proof: (i) -- (vii) follow directly from Proposition 3.1.

(viii) Suppose x ∈ $\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$). Let $\tilde{Q}$ be a $\mathbb{P}_{ic}\mathbb{F}\delta$os containing x. If $\tilde{A} \cap \tilde{Q}$ = ∅, then $\widetilde{\mathfrak{W}} - \tilde{Q}$ is a $\mathbb{P}_{ic}\mathbb{F}\mathbb{F}\delta$cs containing $\tilde{A}$ and so x ∉ $\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$), a contradiction. Therefore, $\tilde{A} \cap \tilde{Q} \neq \emptyset$. If x ∉ $\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$) then ∃ a $\mathbb{P}_{ic}\mathbb{F}\delta$cs Q^c containing $\tilde{A}$ ∋ x ∉ $\tilde{Q}^c$. Then Q = $\widetilde{\mathfrak{W}} - \tilde{Q}^c$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$os containing x ∋ $\tilde{A} \cap \tilde{Q}$ = ∅, a contradiction. Therefore, x ∈ $\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$).

Remark 3.3 The results of Propositions 3.1 and 3.2 are valid for $\mathbb{P}_{ic}\mathbb{F}\delta$Pos, $\mathbb{P}_{ic}\mathbb{F}\delta$Sos, $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$os and $\mathbb{P}_{ic}\mathbb{F}\delta\beta os$ in terms of their associated interior and closure operators.

Preposition 3.3 The following statements hold in $\mathbb{P}_{ic}\mathbb{F}$ts.

a) Each $\mathbb{P}_{ic}\mathbb{F}\delta$os (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$cs) is a $\mathbb{P}_{ic}\mathbb{F}$ os (resp. $\mathbb{P}_{ic}\mathbb{F}$cs).

b) Each $\mathbb{P}_{ic}\mathbb{F}$os (resp. $\mathbb{P}_{ic}\mathbb{F}$cs) is a $\mathbb{P}_{ic}\mathbb{F}\delta$Sos (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Scs).

c) Every $\mathbb{P}_{ic}\mathbb{F}\delta$os (resp. $\mathbb{P}_{ic}\mathbb{F}$cs) is a $\mathbb{P}_{ic}\mathbb{F}\delta$Pos (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Pcs).

d) Each $\mathbb{P}_{ic}\mathbb{F}\delta$Sos (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Scs) is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os (resp. $\mathbb{P}_{ic}\mathbb{F}\delta\beta$cs).

e) Each $\mathbb{P}_{ic}\mathbb{F}$ δPos (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Pcs) is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os (resp. $\mathbb{P}_{ic}\mathbb{F}\delta\beta$cs).

f) Each $\mathbb{P}_{ic}\mathbb{F}$ δαos (resp. $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$cs) is a $\mathbb{P}_{ic}\mathbb{F}$ δSos (resp. $\mathbb{P}_{ic}\mathbb{F}$ δScs).

g) Each $\mathbb{P}_{ic}\mathbb{F}\delta$αos (resp. $\mathbb{P}_{ic}\mathbb{F}\delta\alpha$cs) is a $\mathbb{P}_{ic}\mathbb{F}\delta$Pos (resp. $\mathbb{P}_{ic}\mathbb{F}\delta$Pcs).

The converse does not necessarily follow.

Proof:

a) $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$os, then $\tilde{A}$ = $\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{A}$) ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\tilde{A}$). Therefore, $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}$os.

b) $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}$os then $\tilde{A}$ ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\tilde{A}$). So $\tilde{A}$ ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\tilde{A}$) ⊆ $\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}\mathbb{F}\delta$int($\tilde{A}$)). Therefore, $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$Sos.

c) $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}$os then $\tilde{A}$ ⊆ $\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{A}$). So $\tilde{A}$ ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\tilde{A}$) ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$)). Therefore, $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$Pos.

d) $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$Sos then $\tilde{A}$ ⊆ $\mathbb{P}_{ic}\mathbb{F}\delta$cl($\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{A}$)) ⊆ $\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$))). Therefore, $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os.

e) $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$Pos then $\tilde{A}$ ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$)) ⊆ $\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$))). Therefore, $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os.

f) $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$αos then $\tilde{A}$ ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{A}$))). So $\tilde{A}$ ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{A}$))) ⊆ $\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{A}$)). Therefore, $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$Sos.

g) $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta$αos then $\tilde{A}$ ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{A}$))). So $\tilde{A}$ ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}\delta$int($\tilde{A}$))) ⊆ $\mathbb{P}_{ic}\mathbb{F}$int($\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}$)). Therefore, $\tilde{A}$ is a $\mathbb{P}_{ic}\mathbb{F}$ δPos.

Similarly, the statements also hold true for closed sets.

Proposition 3.4 Arbitrary unions of $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os($\widetilde{\mathfrak{W}}$) are $\mathbb{P}_{ic}\mathbb{F}\delta\beta os$($\widetilde{\mathfrak{W}}$) and arbitrary intersections of $\mathbb{P}_{ic}\mathbb{F}\delta\beta$cs($\widetilde{\mathfrak{W}}$) are $\mathbb{P}_{ic}\mathbb{F}\delta\beta$cs($\widetilde{\mathfrak{W}}$).

Proof. Suppose {$\tilde{A}_{a}$: a ∈ ($\widetilde{\mathfrak{W}}$, τ)} be a collection of $\mathbb{P}_{ic}\mathbb{F}\delta\beta$os's.

For every a ∈ ($\widetilde{\mathfrak{W}}$, τ), $\tilde{A}_a$ ⊆ $\mathbb{P}_{ic}\mathbb{F}$cl($\mathbb{P}_{ic}\mathbb{F}$ int($\mathbb{P}_{ic}\mathbb{F}\delta$cl($\tilde{A}_a$))).

$$\bigcup_{a\in(\widetilde{\mathfrak{W}},\tau)} \tilde{A}_a \subseteq \bigcup_{a\in(\widetilde{\mathfrak{W}},\tau)} \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}\left(\mathbb{P}_{ic}\mathbb{F}\mathrm{int}\left(\mathbb{P}\mathbb{F}\delta\mathrm{cl}(\tilde{A}_a)\right)\right)$$

$$\subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\cup \tilde{A}_a)))$$

Another case follows a similar approach. The Proposition 3.4 is also true for $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Sos}(\widetilde{\mathfrak{W}})$, $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Scs}(\widetilde{\mathfrak{W}})$ (resp. $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Pos}(\widetilde{\mathfrak{W}})$ & $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Pcs}(\widetilde{\mathfrak{W}})$).

4 Properties of $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{os}$

Proposition 4.1 If $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{os}$ and $\tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta\mathrm{os}$, then $\tilde{B} \cap \tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta\mathrm{os}$.

Proof:

$$\tilde{B} \cap \tilde{P} \subseteq \tilde{B} \cap \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{P}))) \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\tilde{B}) \cap \mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{P})) \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{B} \cap \tilde{P})))$$

Therefore, $\tilde{B} \cap \tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta\mathrm{os}$.

Remark 4.1 Proposition 4.1 remains valid if $\tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Sos}$, $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Pos}$ and $\mathbb{P}_{ic}\mathbb{F}\delta\alpha\mathrm{os}$.

Proposition 4.2 If $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Pos}$ and $\tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\alpha\mathrm{os}$, then $\tilde{B} \cap \tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Pos}$.

Proof:

$$\tilde{B} \cap \tilde{P} \subseteq \mathbb{P}_{ic}\mathbb{F}int\left(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{B})\right) \cap \mathbb{P}_{ic}\mathbb{F}int\left(\mathbb{P}_{ic}\mathbb{F}cl\left(\mathbb{P}_{ic}\mathbb{F}\delta int(\tilde{P})\right)\right)$$

$$\subseteq \mathbb{P}_{ic}\mathbb{F}int\left(\mathbb{P}_{ic}\mathbb{F}int\left(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{B})\right)\right) \cap \mathbb{P}_{ic}\mathbb{F}cl\left(\mathbb{P}_{ic}\mathbb{F}\delta int(\tilde{P})\right)$$

$$\subseteq \mathbb{P}_{ic}\mathbb{F}int\left(\mathbb{P}_{ic}\mathbb{F}cl\left(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{B})\right)\right) \cap \mathbb{P}_{ic}\mathbb{F}cl\left(\mathbb{P}_{ic}\mathbb{F}\delta int(\tilde{P})\right)$$

$$\subseteq \mathbb{P}_{ic}\mathbb{F}int\left(\mathbb{P}_{ic}\mathbb{F}cl\left(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{B})\right)\right) \cap \mathbb{P}_{ic}\mathbb{F}\delta int(\tilde{P})$$

$$\subseteq \mathbb{P}_{ic}\mathbb{F}int\left(\mathbb{P}_{ic}\mathbb{F}cl\left(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{B} \cap \tilde{P})\right)\right)$$

$$= \mathbb{P}_{ic}\mathbb{F}int(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{B} \cap \tilde{P}))$$

Therefore, $\tilde{B} \cap \tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Pos}$.

Corollary 4.1 If $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Pcs}$ and $\tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\alpha\mathrm{os}$, then $\tilde{B} \cup \tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Pcs}$.

Proposition 4.3 Let $\tilde{P}$ be a $\mathbb{P}_{ic}\mathbb{F}\mathrm{S}$ of $\widetilde{\mathfrak{W}}$ and $\tilde{B}$ be a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Pos}$ on $\widetilde{\mathfrak{W}}$ such that $\tilde{B} \subseteq \tilde{P} \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{int}(\tilde{B}))$. Then $\tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta\mathrm{os}$.

Proof: Since $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Pos}$, $\tilde{B} \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{B}))$. Now,

$$\tilde{P} \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{int}(\tilde{B})) \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{B})))) = \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{P})))$$

$$\tilde{P} \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{P}))).$$

Therefore, $\tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta\mathrm{os}$.

Proposition 4.4 If $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta\mathrm{os}$ which is also a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Scs}$, then it is a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Sos}$.

Proof: Let $\tilde{B}$ be a $\mathbb{P}_{ic}\mathbb{F}\delta\beta\mathrm{os}$ and $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Scs}$. Then, $\tilde{B} \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\ \mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\ \delta\mathrm{cl}(\tilde{B})))$ and $\mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{B})) \subseteq \tilde{B}$. Therefore, $\mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{B})) \subseteq \tilde{B}$ and so, $\mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{B}))) \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{int}(\tilde{B}))$. Hence $\tilde{B} \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\mathrm{int}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{cl}(\tilde{B}))) \subseteq \mathbb{P}_{ic}\mathbb{F}\mathrm{cl}(\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{int}(\tilde{B}))$. $\therefore$ $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\mathrm{Sos}$.

Proposition 4.5 If $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta cs$ & $\mathbb{P}_{ic}\mathbb{F}\delta Sos$, then it is a $\mathbb{P}_{ic}\mathbb{F}\delta Scs$.

Proof: Since $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta cs$ and $\mathbb{P}_{ic}\mathbb{F}\delta Sos$, then $\widetilde{\mathfrak{W}} - \tilde{B}$ is $\mathbb{P}_{ic}\mathbb{F}\delta\beta os$ and $\mathbb{P}_{ic}\mathbb{F}\delta Scs$ and so by Proposition 4.4, $\widetilde{\mathfrak{W}} - \tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta Sos$. Therefore, $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta Scs$.

Proposition 4.6 If $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta os$ which is also a $\mathbb{P}_{ic}\mathbb{F}\delta\alpha cs$. Then $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta cs$.

Proof: Let $\tilde{B}$ be a $\mathbb{P}_{ic}\mathbb{F}\delta\beta os$ and $\mathbb{P}_{ic}\mathbb{F}\delta\alpha cs$. Then, $\tilde{B} \subseteq \mathbb{P}_{ic}\mathbb{F}cl(\mathbb{P}_{ic}\mathbb{F}int(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{B})))$ and $\mathbb{P}_{ic}\mathbb{F}cl(\mathbb{P}_{ic}\mathbb{F}int(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{B}))) \subseteq \tilde{B}$. Therefore, $\mathbb{P}_{ic}\mathbb{F}cl(\mathbb{P}_{ic}\mathbb{F}int(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{B}))) \subseteq \tilde{B} \subseteq \mathbb{P}_{ic}\mathbb{F}cl(\mathbb{P}_{ic}\mathbb{F}int(\mathbb{P}_{ic}\mathbb{F}\delta cl(\tilde{B})))$.

So, $\tilde{B} = \mathbb{P}_{ic}\mathbb{F}cl(\mathbb{P}_{ic}\mathbb{F}int(\mathbb{P}_{ic}\mathbb{F}\ \delta cl(\tilde{B})))$.Therefore, $\tilde{B}$ is a $\mathbb{P}_{ic}\mathbb{F}\ \delta cs$.

Corollary 4.2 If $\tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta\beta cs$ which is also a $\mathbb{P}_{ic}\mathbb{F}\delta\alpha os$. Then $\tilde{P}$ is a $\mathbb{P}_{ic}\mathbb{F}\delta os$.

5 Conclusion

In this article, the notion of picture fuzzy δ-open sets has been formally established within the framework of $\mathbb{P}_{ic}\mathbb{F}ts$. Building upon this foundation, the concept of picture fuzzy δ-interior and picture fuzzy δ-closure were carefully constructed and their core properties were thoroughly analyzed. Utilizing these constructions, several broader families of picture fuzzy δ-open sets were put forward and examined in detail, encompassing picture fuzzy δ-preopen sets, picture fuzzy δ-semiopen sets, picture fuzzy $\delta\alpha$-open sets and picture fuzzy $\delta\beta$-open sets.

Furthermore, the interrelations between these newly introduced classes and the existing families of picture fuzzy open sets were systematically explored, yielding a number of significant characterization results. The findings of this work contribute meaningfully to the advancement of picture fuzzy topological theory. Looking ahead, it is anticipated that the ideas developed here can be naturally extended to explore δ-continuous mappings, δ-irresolute mappings and other allied structures within the setting of $\mathbb{P}_{ic}\mathbb{F}ts$.

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