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A Hotel Model's Generalized Ulam-Hyers Stability of System of Additive Functional Equations

Arunkumar Mohan, Sathya Elumalai

Department of Mathematics, Kalaignar Karunanidhi Government ArtsCollege, Tiruvannamalai-606603, Tamilnadu, India.

Abstract:- The generalized Ulam-hyers stability of the system of additive functional equations in Banach spaces are explored in this work. Additionally, an application of the functional equations are examioned

Keywords: *Additive functiona;l equation, generalized Ulam-Hyers stability, Banach space, Hotel.*

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I . INTRODUCTION

The fundamental concept of Stability of Hyers-Ulam for functional equations (FES) may be traced back to a well-known group homomorphism problem that S.M. Ulam [21] and D.H. Hyers resolved [13]. Many articles addressing the stability issue with FES have already been published in recent decades, and numerous significant issues in this area have been researched [2,12,17,18,20]. As a result, numerous articles [1,14,15,16,19] have identified the most effective techniques, including the direct approach, the shadowing approach, the invariant mean approach, and others. Specifically, the primary study aid for examining various kinds of functional equations is always the direct technique.

One of the famous FE is the Cauchy additive FE

$$H(S+T)=H(S)+H(T). \tag{1.1}$$

Several additive FESD and its stability issues in various normed spaces have been comprehensively studied by several authors, and many motivating results have been gained on this topic (see [3-11]).

Usually while travelling in car or went to tour, we take food on the hotels with my friends. One day, we got a idea that we want to find which hotel serves fine and tasty food and also we preferto inform my friends about the hotels. The quality of the hotel observed by the parking, thequality of food, service of the weightier and some may give tips. We found Fivetypes of the following Hotels:

H_1 : We went to a first hotel. If parking is available then we decided go andtake food. Servers first gave some water, take the food order. After some initially time the food came andthey served charmingly. Food was very tasty. So we decided to help for themand then gave some tips for them.

H_2 : We went to second hotel. Their also parking is available, so we decided goand take food. The food was tasty and they served pleasantly. But the impressionwas not as in the first hotel. So we did not help them and we avoided to givetips.

H_3 : Again, we went to a third hotel to take food. In that hotel also parking isavailable, But this one was self-serviced. So there were no servers and there wasno need of giving tips. Here also the food was fine.

H_4 : Later, we went to fourth hotel. Here parking is not available. But this onewas self-serviced. So there were no servers and there was no need of giving tips.Here also the food was fine.

H_5 : Finally, we went to fifth hotel. Their also parking is not available. Butas usual we ordered food. The liked food is not available, the available only we want to eat. Butthe food was tasty and we did not like it. Even though they served in goodmanner we did not gave any tips for them.

Based on the above data, let us have the following assumptions:

P_K denotes parking availability;

F_K denotes the foods;

Q_K denotes quality;

S_K denotes service;

T_K denotes tips; respectively.

If it is satisfactory wetake it as + **(PLUS)** and notsatisfactory wetake it as **- (MINUS)**.

The aforementioned data can be converted into a system of additive FES

$$H_1\left(P_1+\frac{F_1+F_2}{2}+Q_1+S_1+T_1\right)=H_1(P_1)+\frac{1}{2}\{H_1(F_1)+H_1(F_2)\}+H_1(Q_1)+H_1(S_1)+H_1(T_1) \tag{1.2}$$

$$H_2\left(P_2+\frac{F_1+F_2}{2}+Q_2+S_2-T_2\right)=H_2(P_2)+\frac{1}{2}\{H_2(F_1)+H_2(F_2)\}+H_2(Q_2)+H_2(S_2)-H_2(T_2) \tag{1.3}$$

$$H_3\left(P_3+\frac{F_1+F_2}{2}+Q_3-S_3-T_3\right)=H_3(P_3)+\frac{1}{2}\{H_3(F_1)+H_3(F_2)\}+H_3(Q_3)-H_3(S_3)-H_3(T_3) \tag{1.4}$$

$$H_4\left(-P_4+\frac{F_1+F_2}{2}+Q_4-S_4-T_4\right)=-H_4(P_4)+\frac{1}{2}\{H_4(F_1)+H_4(F_2)\}+H_4(Q_4)-H_4(S_4)-H_4(T_4) \tag{1.5}$$

$$H_5\left(-P_5-\frac{F_1+F_2}{2}+Q_5+S_5-T_5\right)=-H_5(P_5)-\frac{1}{2}\{H_5(F_1)+H_5(F_2)\}+H_5(Q_5)+H_5(S_5)-H_5(T_5). \tag{1.6}$$

The Ulam-Hyers stability of the system of additive FES (1.2), (1.3), (1.4), (1.5), and (1.6) in Banach Space is investigated in this study.

The general solution to the FES (1.2), (1.3), (1.4), (1.5), and (1.6) is given in section II.

The stability theorems for the FES (1.2), (1.3), (1.4), (1.5), and (1.6) in Banach space are provided in section.III.The applications of the FES (1.2), (1.3), (1.4), (1.5), and (1.6) are covered in section IV.

II. SOLUTION OF THE FES

We provide the general solution of the FES (1.2), (1.3), (1.4), (1.5), (1.6)in this section by assuming I, J are real vector spaces.

Theorem II.1 If functions $H; H_K : I \to J$ $(K = 1,2,3,4,5)$ satisfies (1.1) to (1.6), then they are equivalent for all $P_K, F_1, F_2, Q_K, S_K, T_K, S, T \in I$.

Proof. First, assume a function $H : I \to J$ satisfies (1.1). Replace (S,T) as $(0,0)$; $(-T,T)$; (T,T); $(T,2T)$ and for a positive integer C, we have

$$H(0)=H(0);\ H(-T)=-H(T);\ H(2T)=2H(T);\ H(3T)=3H(T); H(CT)=CH(T); H\left(\frac{T}{C}\right)=\frac{1}{C}H(T); \tag{2.1}$$

for all $T \in I$. Again replacing

$$(S,T)=\left(P_1+Q_1+S_1+T_1,\frac{F_1}{2}+\frac{F_2}{2}\right)$$

in (1.1), using (1.1), (2.1) and take $H=H_1$, we get (1.2).

Second, assume a function $H_1 : I \to J$ satisfies (1.2). Change $\left(P_1,Q_1,S_1,T_1,\frac{F_1}{2},\frac{F_2}{2}\right)$ as $(0,0,0,0,0,0)$; $(T,T,-T,0,0,0)$; $(T,T,0,0,0,0)$; $(T,T,T,0,0,0)$ and for a $c>0$, we obtain

$$H_1(0)=H_1(0);\ H_1(-T)=-H_1(T);H_1(2T)=2H_1(T);\ H_1(3T)=3H_1(T);H_1(cT)=cH_1(T);H_1\left(\frac{T}{c}\right)=\frac{1}{c}H_1(T); \tag{2.2}$$

for all $T \in I$. Again, change

$$\left(P_1,Q_1,S_1,T_1,\frac{F_1}{2},\frac{F_2}{2}\right)=\left(P_2,Q_2,S_2,-T_2,\frac{F_1}{2},\frac{F_2}{2}\right)$$

in (1.2), using (2.2) and take $H_1=H_2$, we arrive (1.3). Third, assume a function $H_2 : I \to J$ satisfies (1.3). Set $\left(P_2,Q_2,S_2,T_2,\frac{F_1}{2},\frac{F_2}{2}\right)$ as $(0,0,0,0,0,0)$; $(T,0,0,-T,0,0)$; $(T,T,0,0,0,0)$; $(T,T,T,0,0,0)$ and for a $c>0$, we have

$$H_3(0)=H_3(0);H_3(-T)=-H_3(T);H_3(2T)=2H_3(T);H_3(3T)=3H_3(T);H_3(cT)=cH_3(T);H_3\left(\frac{T}{c}\right)=\frac{1}{c}H_3(T); \tag{2.3}$$

for all $T \in I$. Again, set

$$\left(P_2,Q_2,S_2,T_2,\frac{F_1}{2},\frac{F_2}{2}\right)=\left(P_3,Q_3,-S_3,-T_3,\frac{F_1}{2},\frac{F_2}{2}\right)$$

in (1.3), using (2.3) and take $H_2=H_3$, we get (1.4). Fourth, assume a function $H_3 : I \to J$ satisfies (1.4).

Put $\left(P_3,Q_3,S_3,T_3,\frac{F_1}{2},\frac{F_2}{2}\right)$ as $(0,0,0,0,0,0,0)$; $(T,0,0,-T,0,0)$; $(T,T,0,0,0,0)$; $(T,T,0,0,T,T)$ and for a $c>0$, we obtain

$$H_3(0)=H_3(0);H_3(-T)=-H_3(T);H_3(2T)=2H_3(T);\ H_3(3T)=3H_3(T);\ H_3(cT)=cH_4(T);\ H_3\left(\frac{T}{c}\right)=\frac{1}{c}H_3(T); \tag{2.4}$$

for all $T \in I$. Again, put

$$\left(P_3,Q_3,S_3,T_3,\frac{F_1}{2},\frac{F_2}{2}\right)=\left(-P_4,Q_4,-S_4,-T_4,\frac{F_1}{2},\frac{F_2}{2}\right)$$

in (1.4), using (2.4) and take $H_3=H_4$, we arrive (1.5). Fifth, assume a function $H_4 : I \to J$ satisfies (1.5).

Substitute $\left(P_4,Q_4,S_4,T_4,\frac{F_1}{2},\frac{F_2}{2}\right)$ as $(0,0,0,0,0,0,0)$; $(T,T,0,0,T,T)$; $(0,T,0,0,T,T)$; $(0,2T,0,0,T,T)$ and for a $c>0$, we have

$$H_4(0)=H_4(0);H_4(-T)=-H_4(T);H_4(2T)=2H_4(T);\ H_4(3T)=3H_4(T);H_4(cT)=cH_4(T);H_4\left(\frac{T}{c}\right)=\frac{1}{c}H_4(T); \tag{2.5}$$

for all $T \in I$. Again, substitute

$$\left(P_4, Q_4, S_4, T_4, \frac{F_1}{2}, \frac{F_2}{2}\right) = \left(P_5, Q_5, -S_5, T_5, \frac{-F_1}{2}, \frac{-F_2}{2}\right)$$

in (1.5), using (2.5) and take $H_4 = H_5$, we get (1.6). Finally, assume a function $H_5 : I \to J$ satisfies (1.6).

Switch $\left(P_5, Q_5, S_5, T_5, \frac{F_1}{2}, \frac{F_2}{2}\right)$ as $(0,0,0,0,0,0)$; $(0,T,T,T,0,0)$; $(0,T,T,0,0,0)$; $(0,T,T,-T,0,0)$ and for a $c > 0$, we obtain

$$H_5(0) = H_5(0); H_5(-T) = -H_5(T); H_5(2T) = 2H_5(T); H_5(3T) = 3H_5(T); H_5(cT) = cH_5(T); H_5\left(\frac{T}{c}\right) = \frac{1}{c}H_5(T); \tag{2.6}$$

for all $T \in I$. Again, switch

$$\left(P_5, Q_5, S_5, T_5, \frac{F_1}{2}, \frac{F_2}{2}\right) = (0, S, T, 0, 0, 0)$$

in (1.6), using (2.6) and take $H_5 = H$, we arrive (1.1).

III BANACH SPACE : STABILITY

The stability theorems of FES (1.2), (1.3), (1.4), (1.5),and (1.6) in Banach space are provided in this division. We make the assumptions that Z is a Banach space and Y is normedspace. To validate the theorems, let's take

$$H_1(P_1; F_1; F_2; Q_1; S_1; T_1) = H_1\left(P_1 + \frac{F_1 + F_2}{2} + Q_1 + S_1 + T_1\right) - \left\{H_1(P_1) + \frac{1}{2}\{H_1(F_1) + H_1(F_2)\} + H_1(Q_1) + H_1(S_1) + H_1(T_1)\right\}$$

$$H_2(P_2; F_1; F_2; Q_2; S_2; T_2) = H_2\left(P_2 + \frac{F_1 + F_2}{2} + Q_2 + S_2 - T_2\right) - \left\{H_2(P_2) + \frac{1}{2}\{H_2(F_1) + H_2(F_2)\} + H_2(Q_2) + H_2(S_2) - H_2(T_2)\right\}$$

$$H_3(P_3; F_1; F_2; Q_3; S_3; T_3) = H_3\left(P_3 + \frac{F_1 + F_2}{2} + Q_3 - S_3 - T_3\right) - \left\{H_3(P_3) + \frac{1}{2}\{H_3(F_1) + H_3(F_2)\} + H_3(Q_3) - H_3(S_3) - H_3(T_3)\right\}$$

$$H_4(P_4; F_1; F_2; Q_4; S_4; T_4) = H_4\left(-P_4 + \frac{F_1 + F_2}{2} + Q_4 - S_4 - T_4\right) - \left\{-H_4(P_4) + \frac{1}{2}\{H_4(F_1) + H_4(F_2)\} + H_4(Q_4) - H_4(S_4) - H_4(T_4)\right\}$$

$$H_5(P_5; F_1; F_2; Q_5; S_5; T_5) = H_5\left(-P_5 - \frac{F_1 + F_2}{2} + Q_5 + S_5 - T_5\right) - \left\{-H_5(P_5) - \frac{1}{2}\{H_5(F_1) + H_5(F_2)\} + H_5(Q_5) + H_5(S_5) - H_5(T_5)\right\}$$

Theorem III.1 If functions $H_K : Y \to Z$ $(K = 1,2,3,4,5)$ satisfies the following set of inequalities

$$\left\| H_K(P_K; F_1; F_2; Q_K; S_K; T_K) \right\| \le R_K(P_K, F_1, F_2, Q_K, S_K, T_K) \tag{3.1}$$

Where $R_K : Y^6 \to [0,\infty)$ $(K = 1,2,3,4,5)$ be functions with conditions

$$\lim_{L\to\infty} \frac{R_K\left(A_K^{LN}P_K, A_K^{LN}F_1, A_K^{LN}F_2, A_K^{LN}Q_K, A_K^{LN}S_K, A_K^{LN}T_K\right)}{A_K^{LN}} = 0 \tag{3.2}$$

with $N = \pm 1$ and for all $P_K, F_1, F_2, Q_K, S_K, T_K \in Y$ such that $A_K(F_K) : Y \to Z$ are unique additive functions for $(K = 1,2,3,4,5)$ which satisfies the FES (1.2), (1.3), (1.4), (1.5), (1.6) respectively and the inequalities

$$\left\| H_K\left(F_K\right) - B_K\left(F_K\right)\right\| \le \frac{1}{A_K}\sum_{M=\frac{1-N}{2}}^{\infty}\frac{R_K\left(A_K^{MN}F_K\right)}{A_K^{MN}} \tag{3.3}$$

where A_K; $B_K\left(F_K\right)$ and $R_K\left(A_K^{MN}F_K\right)$ are respectively obtained by

$$A_K = \begin{cases} 5 & K=1; \\ 4 & K=2; \\ 3 & K=3; \\ 2 & K=4,5; \end{cases} \tag{3.4}$$

$$B_K\left(F_K\right) = \lim_{L\to\infty}\frac{H_K\left(A_K^{LN}F_K\right)}{A_K^{LN}};\left(K=1,2,3,4,5\right) \tag{3.5}$$

and

$$R_1\left(A_1^{MN}F_1\right)=R_1\left(A_1^{MN}F_1,A_1^{MN}F_1,A_1^{MN}F_1,A_1^{MN}F_1,A_1^{MN}F_1,A_1^{MN}F_1\right) \tag{3.6}$$

$$R_2\left(A_2^{MN}F_2\right)=R_2\left(A_2^{MN}F_2,A_2^{MN}F_2,A_2^{MN}F_2,A_2^{MN}F_2,A_2^{MN}F_2,0\right) \tag{3.7}$$

$$R_3\left(A_3^{MN}F_3\right)=R_3\left(A_3^{MN}F_3,A_3^{MN}F_3,A_3^{MN}F_3,A_3^{MN}F_3,0,0\right) \tag{3.8}$$

$$R_4\left(A_4^{MN}F_4\right)=R_4\left(0,A_4^{MN}F_4,A_4^{MN}F_4,A_4^{MN}F_4,0,0\right) \tag{3.9}$$

$$R_5\left(A_5^{MN}F_5\right)=R_5\left(0,0,0,A_5^{MN}F_5,A_5^{MN}F_5,0\right) \tag{3.10}$$

for all $F_K \in Y\left(K=1,2,3,4,5\right)$.

Proof. Placing

$\left(P_1,F_1,F_2,Q_1,S_1,T_1\right)=\left(F_1,F_1,F_1,F_1,F_1,F_1\right)$ in (3.1) for $K=1$

$\left(P_2,F_1,F_2,Q_2,S_2,T_2\right)=\left(F_2,F_2,F_2,F_2,F_2,0\right)$ in (3.1) for $K=2$

$\left(P_3,F_1,F_2,Q_3,S_3,T_3\right)=\left(F_3,F_3,F_3,F_3,0,0\right)$ in (3.1) for $K=3$

$\left(P_4,F_1,F_2,Q_4,S_4,T_4\right)=\left(0,F_4,F_4,F_4,0,0\right)$ in (3.1) for $K=4$

$\left(P_5,F_1,F_2,Q_5,S_5,T_5\right)=\left(0,0,0,F_5,F_5,0\right)$ in (3.1) for $K=5$

we arrive the following inequalities

$$\left\| H_1\left(5F_1\right)-5H_1\left(F_1\right)\right\| \le R_1\left(F_1,F_1,F_1,F_1,F_1,F_1\right) \tag{3.11}$$

$$\left\| H_2\left(4F_2\right)-4H_2\left(F_2\right)\right\| \le R_2\left(F_2,F_2,F_2,F_2,F_2,0\right)$$

(3.12)

$$\left\| H_3\left(3F_3\right)-3H_3\left(F_3\right)\right\| \le R_3\left(F_3,F_3,F_3,F_3,0,0\right) \tag{3.13}$$

$$\left\| H_4\left(2F_4\right)-2H_4\left(F_4\right)\right\| \le R_4\left(0,F_4,F_4,F_4,0,0\right) \tag{3.14}$$

$$\left\| H_5\left(2F_5\right)-2H_5\left(F_5\right)\right\| \le R_5\left(0,0,0,F_5,F_5,0\right) \tag{3.15}$$

for all $F_K \in Y(K = 1,2,3,4,5)$. From the above inequalities, we have

$$\left\| \frac{H_1(5F_1)}{5} - H_1(F_1) \right\| \le \frac{1}{5} R_1(F_1,F_1,F_1,F_1,F_1,F_1) \tag{3.16}$$

$$\left\| \frac{H_2(4F_2)}{4} - H_2(F_2) \right\| \le \frac{1}{4} R_2(F_2,F_2,F_2,F_2,F_2,0) \tag{3.17}$$

$$\left\| \frac{H_3(3F_3)}{3} - H_3(F_3) \right\| \le \frac{1}{3} R_3(F_3,F_3,F_3,F_3,0,0) \tag{3.18}$$

$$\left\| \frac{H_4(2F_4)}{2} - H_4(F_4) \right\| \le \frac{1}{2} R_4(0,F_4,F_4,F_4,0,0) \tag{3.19}$$

$$\left\| \frac{H_5(2F_5)}{2} - H_5(F_5) \right\| \le \frac{1}{2} R_5(0,0,0,F_5,F_5,0) \tag{3.20}$$

for all $F_K \in Y(K = 1,2,3,4,5)$. For any $L > 0$, the above inequalities can be generalized as

$$\left\| \frac{H_1(5^L F_1)}{5^L} - H_1(F_1) \right\| \le \frac{1}{5} \sum_{M=0}^{L-1} \frac{R_1(5^M F_1,5^M F_1,5^M F_1,5^M F_1,5^M F_1,5^M F_1)}{5^M} \tag{3.21}$$

$$\left\| \frac{H_2(4^L F_2)}{4^L} - H_2(F_2) \right\| \le \frac{1}{4} \sum_{M=0}^{L-1} \frac{R_2(4^M F_2,4^M F_2,4^M F_2,4^M F_2,4^M F_2,0)}{4^M} \tag{3.22}$$

$$\left\| \frac{H_3(3^L F_3)}{3^L} - H_3(F_3) \right\| \le \frac{1}{3} \sum_{M=0}^{L-1} \frac{R_3(3^M F_3,3^M F_3,3^M F_3,3^M F_3,0,0)}{3^M} \tag{3.23}$$

$$\left\| \frac{H_4(2^L F_4)}{2^L} - H_4(F_4) \right\| \le \frac{1}{2} \sum_{M=0}^{L-1} \frac{R_4(0,2^M F_4,2^M F_4,2^M F_4,0,0)}{2^M} \tag{3.24}$$

$$\left\| \frac{H_5(2^L F_5)}{2^L} - H_5(F_5) \right\| \le \frac{1}{2} \sum_{M=0}^{L-1} \frac{R_5(0,0,0,2^M F_5,2^M F_5,0)}{2^M} \tag{3.25}$$

for all $F_K \in Y(K = 1,2,3,4,5)$.

Placing $F_K = A_K^{L1} F_K \; (K = 1,2,3,4,5)$ in above inequalities, we get

$$\begin{aligned} \left\| \frac{H_1(5^{L+L1} F_1)}{5^{L+L1}} - \frac{H_1(5^{L1} F_1)}{5^{L1}} \right\| &= \frac{1}{5^{L1}} \left\| \frac{H_1(5^L 5^{L1} F_1)}{5^L} - H_1(5^{L1} F_1) \right\| \\ &\le \frac{1}{5} \sum_{M=0}^{L-1} \frac{R_1(5^{M+L1} F_1,5^{M+L1} F_1,5^{M+L1} F_1,5^{M+L1} F_1,5^{M+L1} F_1,5^{M+L1} F_1)}{5^{M+L1}}; \end{aligned} \tag{3.26}$$

$$\begin{aligned} \left\| \frac{H_2(4^{L+L1} F_2)}{4^{L+L1}} - \frac{H_2(4^{L1} F_2)}{4^{L1}} \right\| &= \frac{1}{4^{L1}} \left\| \frac{H_2(4^L 4^{L1} F_2)}{4^L} - H_2(4^{L1} F_2) \right\| \\ &\le \frac{1}{4} \sum_{M=0}^{L-1} \frac{R_2(4^{M+L1} F_2,4^{M+L1} F_2,4^{M+L1} F_2,4^{M+L1} F_2,4^{M+L1} F_2,0)}{4^{M+L1}}; \end{aligned} \tag{3.27}$$

$$\begin{aligned} \left\| \frac{H_3(3^{L+L1} F_3)}{3^{L+L1}} - \frac{H_3(3^{L1} F_3)}{3^{L1}} \right\| &= \frac{1}{3^{L1}} \left\| \frac{H_3(3^L 3^{L1} F_3)}{3^L} - H_3(3^{L1} F_3) \right\| \\ &\le \frac{1}{3} \sum_{M=0}^{L-1} \frac{R_3(3^{M+L1} F_3,3^{M+L1} F_3,3^{M+L1} F_3,3^{M+L1} F_3,0,0)}{3^{M+L1}}; \end{aligned} \tag{3.28}$$

$$\left\|\frac{H_4\left(2^{L+L1}F_4\right)}{2^{L+L1}}-\frac{H_4\left(2^{L1}F_4\right)}{2^{L1}}\right\|=\frac{1}{2^{L1}}\left\|\frac{H_4\left(2^{L}2^{L1}F_4\right)}{2^{L}}-H_4\left(2^{L1}F_4\right)\right\| \le \frac{1}{2}\sum_{M=0}^{L-1}\frac{R_4\left(0,2^{M+L1}F_4,2^{M+L1}F_4,2^{M+L1}F_4,0,0\right)}{2^{M+L1}}; \tag{3.29}$$

$$\left\|\frac{H_5\left(2^{L+L1}F_5\right)}{2^{L+L1}}-\frac{H_5\left(2^{L1}F_5\right)}{2^{L1}}\right\|=\frac{1}{2^{L1}}\left\|\frac{H_5\left(2^{L}2^{L1}F_5\right)}{2^{L}}-H_5\left(2^{L1}F_5\right)\right\| \le \frac{1}{2}\sum_{M=0}^{L-1}\frac{R_5\left(0,0,0,2^{M+L1}F_5,2^{M+L1}F_5,0\right)}{2^{M+L1}}; \tag{3.30}$$

for all $F_K \in Y\left(K=1,2,3,4,5\right)$.

Allowing $L_1 \to \infty\left(K=1,2,3,4,5\right)$ in the (3.26) to (3.30), we obtain the sequences

$$\left\{\frac{H_K\left(A_K^L F_K\right)}{A_K^L}\right\}\left(K=1,2,3,4,5\right)$$

are Cauchy sequences and converges to $B_K\left(F_K\right)$. So, we define

$$B_K\left(F_K\right)=\lim_{L\to\infty}\frac{H_K\left(A_K^L F_K\right)}{A_K^L};\left(K=1,2,3,4,5\right) \tag{3.31}$$

for all $F_K \in Y\left(K=1,2,3,4,5\right)$. Over again Allowing $L \to \infty\left(K=1,2,3,4,5\right)$ in inequalities (3.21) to (3.25) and using (3.31), we arrive (3.3) holds.

To prove $B_K\left(F_K\right)\left(K=1,2,3,4,5\right)$ satisfies the FES (1.2) to (1.6), changing $\left(P_K,F_1,F_2,Q_K,S_K,T_K\right)=\left(A_K^L P_K,A_K^L F_1,A_K^L F_2,A_K^L Q_K,A_K^L S_K,A_K^L T_K\right)$ in (3.1) and divided by A_K^L, we have

$$\left\|\frac{1}{A_K^L}H_K\left(P_K;F_1;F_2;Q_K;S_K;T_K\right)\right\|\le\frac{1}{A_K^L}R_K\left(A_K^L P_K;A_K^L F_1;A_K^L F_2;A_K^L Q_K;A_K^L S_K;A_K^L T_K\right) \tag{3.32}$$

for all $P_K,F_1,F_2^L,Q_K,S_K,T_K \in Y$. Allowing $L\to\infty$ for $\left(K=1,2,3,4,5\right)$ and using (3.31) for $\left(K=1,2,3,4,5\right)$, we have $B_K\left(F_K\right)\left(K=1,2,3,4,5\right)$ satisfies the FES (1.2) to (1.6).

Finally, to prove B_K is unique, let us undertake B_K^1 for all $\left(K=1,2,3,4,5\right)$ satisfies (1.2) to (1.6), (3.3) and (3.5) for all $F_K \in Y\left(K=1,2,3,4,5\right)$.Now,

$$\left\|B_K\left(F_K\right)-B_K^1\left(F_K\right)\right\| \le \frac{1}{A_K^{L1}}\left\{\left\|B_K\left(A_K^{L1}F_K\right)-H_K\left(A_K^{L1}F_K\right)\right\|+\left\|H_K\left(A_K^{L1}F_K\right)-B_K^1\left(A_K^{L1}F_K\right)\right\|\right\} \le \frac{2}{A_K}\sum_{M=0}^{\infty}\frac{R_K\left(A_K^{M+L1}F_K\right)}{A_K^{M+L1}}\to 0 \text{ as } L1\to\infty$$

for all $F_K \in Y\left(K=1,2,3,4,5\right)$. Therefore B_K'*s* are unique. Therefore, the theorem is valid for $N=1$.

Changing $F_K=\frac{F_K}{A_K}\left(K=1,2,3,4,5\right)$ in (3.11) to (3.15), we obtain

$$\left\|H_1\left(F_1\right)-5H_1\left(\frac{F_1}{5}\right)\right\|\le R_1\left(\frac{F_1}{5},\frac{F_1}{5},\frac{F_1}{5},\frac{F_1}{5},\frac{F_1}{5},\frac{F_1}{5}\right) \tag{3.33}$$

$$\left\|H_2\left(F_2\right)-4H_2\left(\frac{F_2}{4}\right)\right\|\leq R_2\left(\frac{F_2}{4},\frac{F_2}{4},\frac{F_2}{4},\frac{F_2}{4},\frac{F_2}{4},0\right) \tag{3.34}$$

$$\left\|H_3\left(F_3\right)-3H_3\left(\frac{F_3}{3}\right)\right\|\leq R_3\left(\frac{F_3}{3},\frac{F_3}{3},\frac{F_3}{3},\frac{F_3}{3},0,0\right) \tag{3.35}$$

$$\left\|H_4\left(F_4\right)-2H_4\left(\frac{F_4}{2}\right)\right\|\leq R_4\left(0,\frac{F_4}{2},\frac{F_4}{2},\frac{F_4}{2},0,0\right) \tag{3.36}$$

$$\left\|H_5\left(F_5\right)-2H_5\left(\frac{F_5}{2}\right)\right\|\leq R_5\left(0,0,0,\frac{F_5}{2},\frac{F_5}{2},0\right) \tag{3.37}$$

for all $F_K \in Y\left(K=1,2,3,4,5\right)$. The remaining proof is comparable to the case of $N=1$. Thus, the proof is finished.

Corollary III.2 If functions $H_K : Y \rightarrow Z$ $\left(K=1,2,3,4,5\right)$ satisfies the following set of inequalities

$$\left\|H_K\left(P_K;F_1;F_2;Q_K;S_K;T_K\right)\right\|\leq X \tag{3.38}$$

for all $P_K, F_1, F_2^L, Q_K, S_K, T_K \in Y$ where $X>0$ such that $A_K\left(F_K\right): Y \rightarrow Z$ are unique additive functions for $\left(K=1,2,3,4,5\right)$ which satisfies the FES (1.2), (1.3), (1.4), (1.5), (1.6) respectively and the inequalities

$$\left\|H_K\left(F_K\right)-B_K\left(F_K\right)\right\|\leq \frac{X}{\left|A_K-1\right|} \tag{3.39}$$

for all $F_K \in Y\left(K=1,2,3,4,5\right)$.

Corollary III.3 If functions $H_K : Y \rightarrow Z$ $\left(K=1,2,3,4,5\right)$ satisfies the following set of inequalities

$$\left\|H_K\left(P_K;F_1;F_2;Q_K;S_K;T_K\right)\right\|\leq X\left\{\left|P_K\right|^W+\left|F_1\right|^W+\left|F_2\right|^W+\left|Q_K\right|^W+\left|S_K\right|^W+\left|T_K\right|^W\right\} \tag{3.40}$$

for all $P_K, F_1, F_2^L, Q_K, S_K, T_K \in Y$ where $X>0; W\neq 1$ such that $A_K\left(F_K\right): Y \rightarrow Z$ are unique additive functions for $\left(K=1,2,3,4,5\right)$ which satisfies the FES (1.2), (1.3), (1.4), (1.5), (1.6) respectively and the inequalities

$$\left\|H_K\left(F_K\right)-B_K\left(F_K\right)\right\|\leq \frac{DX\left|F_K\right|^W}{\left|A_K-A_K^W\right|} \tag{3.41}$$

for all $F_K \in Y\left(K=1,2,3,4,5\right)$ where

$$D=\begin{cases}6 & K=1;\\ 5 & K=2;\\ 4 & K=3;\\ 3 & K=4;\\ 2 & K=5.\end{cases} \tag{3.42}$$

Corollary III.4 If a function $H_1 : Y \rightarrow Z$ satisfies the inequality

$$\left\|H_1\left(P_1;F_1;F_2;Q_1;S_1;T_1\right)\right\|\leq X\left\{\left|P_1\right|^W\times\left|F_1\right|^W\times\left|F_2\right|^W\times\left|Q_1\right|^W\times\left|S_1\right|^W\times\left|T_1\right|^W\right\} \tag{3.43}$$

for all $P_1, F_1, F_2^L, Q_1, S_1, T_1 \in Y$ where $X > 0;\ 6W \neq 1$ such that $A_1(F_1): Y \to Z$ are unique additive function which satisfies the FE (1.2) and the inequality

$$\left\| H_1(F_1) - B_1(F_1) \right\| \leq \frac{X\left|F_1\right|^{6W}}{A_1\left|A_1 - A_1^{6W}\right|} \tag{3.44}$$

for all $F_1 \in Y$.

Corollary III.5 If a function $H_1 : Y \to Z$ satisfies the inequality

$$\left\| H_1(P_1; F_1; F_2; Q_1; S_1; T_1) \right\| \leq X\left\{\left\{\left|P_K\right|^{6W} + \left|F_1\right|^{6W} + \left|F_2\right|^{6W} + \left|Q_K\right|^{6W} + \left|S_K\right|^{6W} + \left|T_K\right|^{6W}\right\} + \left\{\left|P_1\right|^{W} \times \left|F_1\right|^{W} \times \left|F_2\right|^{W} \times \left|Q_1\right|^{W} \times \left|S_1\right|^{W} \times \left|T_1\right|^{W}\right\}\right\} \tag{3.45}$$

for all $P_1, F_1, F_2^L, Q_1, S_1, T_1 \in Y$ where $X > 0;\ 6W \neq 1$ such that $A_1(F_1): Y \to Z$ are unique additive function which satisfies the FE (1.2) and the inequality

$$\left\| H_1(F_1) - B_1(F_1) \right\| \leq \frac{6X\left|F_1\right|^{6W}}{A_1\left|A_1 - A_1^{6W}\right|} \tag{3.46}$$

for all $F_K \in Y(K = 1,2,3,4,5)$.

IV APPLICATIONS

From the FES (1.2), (1.3), (1.4), (1.5), (1.6). we collect the following data, with the help of Binary digits, If good we give 1 (one) and Bad we give 0 (zero) (for parking availability, food, quality, service, tips, and $K = 1,2,3,4,5$).

Hotel	Parking	Food	Food	Quality	Service	Tips	Total	Preference
H_1	1	1	1	1	1	1	5	1*
H_2	1	1	1	1	1	0	4	1*
H_3	1	1	1	1	0	0	3	2*
H_4	0	1	1	1	0	0	2	3
H_5	0	0	0	1	1	0	2	4

According to above Table and after tasting the food on 5 hotels, we gave some suggestions about the hotels to my friends based on Food, Quality of Food, Serves Attitude, Issuing Tips and Parking.

We liked first two hotels H_1 and H_2, because parking is available and the food was tasty, also they served very well. So normally people will go to hotels H_1 and H_2 in more number.

Also, third Hotel H_3 was self-serviced, because parking is available and the food was tasty. Some people like the hotel H_3 and some of them will not go there.

In Hotel H_4, parking is not available even though the food is good none of them will go there.

Finally, in Hotel H_5 , parking is not available and the favorite food is not available some of them will leave the hotel.

Now, it follows from Corollary III.2 of the inequalities (3.39), we see that

$$\|H_1(F_1)-B_1(F_1)\| \le \frac{X}{|A_1-1|}=\frac{X}{|5-1|}=\frac{X}{|4|} \tag{4.1}$$

$$\|H_2(F_2)-B_2(F_2)\| \le \frac{X}{|A_2-1|}=\frac{X}{|4-1|}=\frac{X}{|3|} \tag{4.2}$$

$$\|H_3(F_3)-B_3(F_3)\| \le \frac{X}{|A_3-1|}=\frac{X}{|3-1|}=\frac{X}{|2|} \tag{4.3}$$

$$\|H_4(F_4)-B_4(F_4)\| \le \frac{X}{|A_4-1|}=\frac{X}{|2-1|}=\frac{X}{|1|} \tag{4.4}$$

$$\|H_5(F_5)-B_5(F_5)\| \le \frac{X}{|A_5-1|}=\frac{X}{|2-1|}=\frac{X}{|1|} \tag{4.5}$$

So, in the inequalities (4.1), (4.2), we get the better possible upper bound stability analysis in first two hotels only. Also in (4.3) half of thebound only came.

Inrest of the situations that as in (4.4), (4.5), we can't get the upper bound.

V CONCLUSION

"Normally, if parking is not available we can't prefer to go inside the hotel. Some peoples like self-serviced hotels. Maximum peoples like to go parking availability and tasty food hotels."

REFERENCES

[1] J. Aczel and J. Dhombres, Functional Equations in Several Variables, Cambridge Univ, Press, 1989.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2 (1950), 64-66.

[3] M. Arunkumar, Solution and Stability of Arun-Additive functionalequations, International Journal Mathematical Sciences and Engineering Applications, Vol 4, No. 3, (2010), 33-46.

[4] M. Arunkumar and C. Leela Sabari, Solution and stability of a functionalequation originating from a chemical equation, International Journal Mathematical Sciences and Engineering Applications, 5(2) (2011), 1-8.

[5] M. Arunkumar, S. Hema latha, C. Devi Shaymala Mary, Functional equation originating from arithmetic Mean of consecutive terms of an arithmetic Progression are stable in Banach space: Direct and fixed point method, JP Journal of Mathematical Sciences, Vol 3(1), (2012), 27-43.

[6] M. Arunkumar, P. Agilan, Additive functional equation and inequality are Stable in Banach space and its applications, Malaya Journal of Matematik (MJM), Vol 1, Issue 1, (2013), 10-17.

[7] M. Arunkumar, E.Sathya, S. Ramamoorthi, General Solution And Generalized Ulam – Hyers Stability Of A Additive Functional Equation Originating From N Observations Of An Arithmetic Mean In Banach Spaces Using Various Substitutions In Two Different Approaches, Malaya Journal of Matematik, 5(1) (2017), 4–18.

[8] M.Arunkumar, E.Sathya, S.Karthikeyan, G. Ganapathy, T. Namachivayam, Stability of System of Additive Functional Equations in Various Banach Spaces: Classical Hyers Methods, Malaya Journal of Matematik, Volume 6, Issue 1, 2018, 91-112.

[9] M. Arunkumar, E.Sathya, C.Pooja, Stability of a functional equation originates from a corona Model with additive solution in Banach space using direct and fixed point methods, Recent Advancement of Mathematics in Science and Technology, JPS Scientific Publication India, First Edition, Book Chapter, 103-113, 2021, (ISBN): 978-81-950475-0-5.

[10] M. Arunkumar, S. Tamilarasan, R. Kondandan, E. Sathya, Stability analysis of system of additive functional equations from a hotel model Via Fixed Point Method, Recent Advancement of Mathematics in Science and Technology, JPS Scientific Publication India, First Edition, Book Chapter, 25-34, 2021, (ISBN): 978-81-950475-0-5.

[11] Arunkumar Mohan, John Micheal Rassias, Sathya Elumalai, Chandiran Vadamalai,Alexpandiyan Velmurugan, Velmurugan Tamilselvan, Fixed Point Stabilities of System ofFunctional Equations from a Hotel Model in Various Banach Spaces, Data-Driven AI: AMultidisciplinary Approach Techniques, Applications, and Insights from MultipleDomains, 1st Edition, CRC Press, ISBN 9781041272366, PP 225-234, 2026..

[12] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184 (1994), 431-436.

[13] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci.,U.S.A.,27 (1941) 222-224.

[14] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of functional equations in several variables,Birkhauser, Basel, 1998.

[15] S.M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, PalmHarbor, 2001.

[16] Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer Monographs in Mathematics, 2009.

[17] J.M. Rassias, On approximately of approximately linear mappings by linear mappings, J. Funct. Anal. USA, 46, (1982) 126-130.

[18] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc.Amer.Math. Soc., 72 (1978), 297-300.

[19] Th.M. Rassias, Functional Equations, Inequalities and Applications, Kluwer Acedamic Publishers, Dordrecht, Bostan London, 2003.

[20] K. Ravi, M. Arunkumar and J.M. Rassias, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, International Journal of Mathematical Sciences, Autumn 2008 Vol.3, No. 08, 36-47.

[21] S.M. Ulam, Problems in Modern Mathematics, Science Editions,Wiley, NewYork, 1964.