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Fuzzy Stability Analysis of a Quartic-Additive Mixed TypeFunctional Equation

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Abstract:- In this research proposal, we exhibit the generalized Hyers-Ulam-Rassias stability of a quartic-additve mixed type functional equation in fuzzy normed space using the popular Hyers direct and fixed point methods.

Keywords: *Additive-Quartic functional equation , Ulam-Hyers-Rassias stability of functional equations, Fuzzy banach space, Fixed point method, Hyers, Radus method.*

1. Introduction

The stability of the functional equation is an fascinating topic that has been executed in last several decennium. S.M. Ulam [28] hoisted the rudimentary query for the stability of approximate group homomorphism problem in 1940. In the next year D.H. Hyers [14] provided a partial solution to the Ulam's problem for mappings between banach spaces. This report was generalized by T. Aoki [2] to treat this problem for additive mappings and Th.M.Rassias [25] provided a generalized version of Hyers for approximately linear mappings. J.M. Rassias modified the Hyers stability result by introducing the product of different powers of norms and a mixed product-sum of powers of norms, respectively. A generalization of all the above stability results was obtained by P .Gavruta [13] in 1994 by replacing the unbounded cauchy difference by a general control function $\phi(x,y)$. The stability idea produced by Rassias[25] and Gavruta [13] is currently called as H-U-R stability of functional equations. In 2008 a special category of Gavruta's theorem for the unbounded cauchy difference was discovered by M.Arunkumar et.al.,[26] by considering the summation of both the sum and the product of two p-norms. The generalized Hyers-Ulam-Rassias stability for a 3-dimensional quartic functional equation.

$$I(2d_1+d_2+d_3)+I(-2d_1+d_2+d_3)+I(2d_1-d_2+d_3)+I(2d_1+d_2-d_3)+16I(d_2)+16I(d_3)$$
$$=8[I(d_1+d_2)+I(d_1-d_2)+I(d_1+d_3)+I(d_1-d_3)]+2[I(d_2+d_3)+I(d_2-d_3)]+32I(d_1)$$

(1.1)

in fuzzy normed space was inspected by M.Arunkumar et.al.,[3].The orthogonal mixed type additive–quartic functional equation

$$7[I(2d_1+d_2)+I(2d_1-d_2)]=28[I(d_1+d_2)+I(d_1-d_2)]-3[I(2d_1)-2I(d_1)]+14[I(2d_2)-4I(d_2)]$$

(1.2)

was investigated by M. Arunkumar, S. Hemalatha [5] .The mixed type additive-quartic functional equation

$$I(2d_1+d_2)+I(2d_1-d_2)$$
$$=4[I(d_1+d_2)+I(d_1-d_2)]+12[I(d_1)+I(-d_1)]-3[I(d_2)+I(-d_2)]-2[I(d_1)-I(-d_1)]$$

(1.3)

in random normed space was discussed by M. Arunkumar., et. al.,
In this Dissertation, we scrutinize the generalized Hyers-Ulam stability of a quartic - additive mixed type functional equation

$$
\begin{aligned}
&I(4d_4+3d_3+2d_2+d_1)+I(4d_4-3d_3+2d_2+d_1)+I(4d_4+3d_3-2d_2+d_1)+I(4d_4+3d_3+2d_2-d_1)\\
&+I(4d_4-3d_3-2d_2+d_1)+I(4d_4-3d_3+2d_2-d_1)+I(4d_4+3d_3-2d_2-d_1)+I(4d_4-3d_3-2d_2-d_1)\\
&=64[I(d_4+d_1)+I(d_4-d_1)]-30[I(d_4+d_1)-I(-d_4-d_1)+I(d_4-d_1)-I(d_1-d_4)]\\
&+256[I(d_4+d_2)+I(d_4-d_2)]+8[I(d_1+d_2)+I(-d_1-d_2)+I(d_1-d_2)+I(d_2-d_1)]\\
&-126[I(d_4+d_2)+I(-d_4-d_2)+I(d_4-d_2)+I(d_2-d_4)]+128[I(d_4)+(-d_4)]\\
&+18[I(d_1+d_3)+I(-d_1-d_3)+I(d_1-d_3)+I(d_3-d_1)]-432[I(d_3)+(-d_3)]\\
&+72[I(d_2+d_3)+I(-d_2-d_3)+I(d_2-d_3)+I(d_3-d_2)]-352[I(d_2)+(-d_2)]\\
&+288[I(d_3+d_4)+I(-d_3-d_4)+I(d_3-d_4)+I(d_4-d_3)]-112[I(d_1)+(-d_1)]+8[I(d_4)-(-d_4)]
\end{aligned}
\tag{1.4}
$$

in fuzzy banach space utilizing direct and fixed point methods.

2. BASICS FACTS ABOUT FUZZY NORMED SPACE

In this partition, the authors estimate the basic definitions of fuzzy normed space. A.K. Katsaras[17] posed a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. In particular T. Bag and S.K. Samanta [9] the following S.C. Cheng and J.N. Mordeson [12] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [18]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces.
We use the definition of fuzzy normed spaces given in [9] and[20] .

Definition 2.1.*Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$(.the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$.,*

$(F1)$ $N(x,c)=0$ for $c \le 0$;

$(F2)$ $x=0$ if and only if $N(x,c)=1$ for all $c>0$;

$(F3)$ $N(cx,t)=N\left(x,t/|c|\right)$ if $c \ne 0$;

$(F4)$ $N(x+y,s+t)\ge min\{N(x,s),N(y,t)\}$;

$(F5)$ $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty}N(x,t)=1$;

$(F6)$ for $x\ne 0, N(x,\cdot)$ is (upper semi) continuous on$\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 2.2.*Let $\left(X,\|\cdot\|\right)$ be a normed linear space. Then* $N\left(x,t\right)=\begin{cases}\dfrac{t}{t+\|x\|}, & t>0,\ x\in X,\\ 0, & t\le 0,\ x\in X\end{cases}$

is a fuzzy norm on X.

Definition 2.3.*Let (X,N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then x_n is said to be convergent if there exists $x\in X$ such that $\lim_{n\to\infty} N(x_n-x,t)=1$ for all $t>0$. In that case, x is called the limit of the sequence x_n and we denote it by $N-\lim_{n\to\infty} x_n = x$.*

Definition 2.4.*A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon>0$ and each $t>0$ there exists n_0 such that for all $n\ge n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,t)>1-\varepsilon$.*

Definition 2.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

3. FUZZY STABILITY RESULTS

In this portion the generalized Ulam-Hyers stability of a quartic-adiitive mixed type functional equation (1.4). Henceforth, unless otherwise stated let us consider $\mathcal{L}$.to be a linear space. For our convenience for a given mapping $I:\mathcal{L}\to\mathcal{M}$. By

$$\begin{aligned}
CI(d_4,d_3,d_2,d_1) &= I(4d_4+3d_3+2d_2+d_1)+I(4d_4-3d_3+2d_2+d_1)+I(4d_4+3d_3-2d_2+d_1)\\
&+I(4d_4+3d_3+2d_2-d_1)+I(4d_4-3d_3-2d_2+d_1)+I(4d_4-3d_3+2d_2-d_1)\\
&+I(4d_4+3d_3-2d_2-d_1)+I(4d_4-3d_3-2d_2-d_1)-64[I(d_4+d_1)+I(d_4-d_1)]\\
&+30[I(d_4+d_1)-I(-d_4-d_1)+I(d_4-d_1)-I(d_1-d_4)]-256[I(d_4+d_2)+I(d_4-d_2)]\\
&-8[I(d_1+d_2)+I(-d_1-d_2)+I(d_1-d_2)+I(d_2-d_1)]+126[I(d_4+d_2)+I(-d_4-d_2)+I(d_4-d_2)+I(d_2-d_4)]\\
&-128[I(d_4)+(-d_4)]-18[I(d_1+d_3)+I(-d_1-d_3)+I(d_1-d_3)+I(d_3-d_1)]+432[I(d_3)+(-d_3)]\\
&-72[I(d_2+d_3)+I(-d_2-d_3)+I(d_2-d_3)+I(d_3-d_2)]+352[I(d_2)+(-d_2)]\\
&-288[I(d_3+d_4)+I(-d_3-d_4)+I(d_3-d_4)+I(d_4-d_3)]+112[I(d_1)+(-d_1)]-8[I(d_4)-(-d_4)]
\end{aligned}$$

for all $d_4,d_3,d_2,d_1\in\mathcal{L}$.

Direct Method : Odd Category.

Theorem 3.1. Let $\psi\in\{-1,1\}$be fixed and let $Y:\mathcal{L}^4\to\mathbb{Z}$ be a mapping with $0<\left(\frac{\rho}{10}\right)^{\psi}<1$

$$N'\left(Y\left(10^{\psi}d_4,10^{\psi}d_3,10^{\psi}d_2,10^{\psi}d_1\right),\zeta\right)\geq N'\left(\rho^{\psi}Y(d_4,d_3,d_2,d_1),\zeta\right) \tag{3.1}$$

for all $d_4,d_3,d_2,d_1\in\mathcal{L}$and all $\zeta>0$ and

$$\lim_{s\to\infty} N'\left(Y\left(10^{\psi s}d_4,10^{\psi s}d_3,10^{\psi s}d_2,10^{\psi s}d_1\right),10^{\psi s}\zeta\right)=1 \tag{3.2}$$

for all $d_4,d_3,d_2,d_1\in\mathcal{L}$ and all $\zeta>0$. Suppose that a function $I:\mathcal{L}\to\mathcal{M}$ fulfills the inequality

$$N'(CI(d_4,d_3,d_2,d_1),\zeta)\geq N'(Y(d_4,d_3,d_2,d_1),\zeta) \tag{3.3}$$

for all $d_4,d_3,d_2,d_1\in\mathcal{L}$and all $\zeta>0$. Then the limit

$$\mathcal{A}(d)=N-\lim_{s\to\infty}\frac{I(10^{s\psi}d)}{10^{s\psi}} \tag{3.4}$$

exists for all $d\in\mathcal{L}$and all $\zeta>0$ and the mapping $\mathcal{A}:\mathcal{L}\to\mathcal{M}$ is a idiosyncratic additive mapping satisfying (1.4) and

$$N(\mathcal{A}(d)-I(d),\zeta)\geq N'\left(\Theta_a(d),\frac{4\zeta|10-\rho|}{35}\right) \tag{3.5}$$

for all $d_4,d_3,d_2,d_1\in\mathcal{L}$ and all $\zeta>0$.

Proof. First presume $\psi=1$. Fitting (d_4,d_3,d_2,d_1) by (d,d,d,d) in (3.3) and adopting oddness the resultant becomes

$$N(I(10d)+I(8d)+I(6d)+2I(4d)-8I(2d)-16I(d),\zeta)\geq N'(Y(d,d,d,d),\zeta) \tag{3.6}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. Again plotting (d_4,d_3,d_2,d_1) by $(d,d,0,-d)$ in (3.3) and apply oddness the output becomes

$$N\left(I(8d)+I(6d)-I(2d)-12I(d),\frac{\zeta}{2}\right)\geq N'(Y(d,d,0,-d),\zeta) \tag{3.7}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. One more time plugging (d_4,d_3,d_2,d_1) by $(d_4,d_3,0,0)$ in (3.3) and using oddness the outcome becomes

$$N\left(I(4d_4+3d_3)+I(4d_4-3d_3)\,I(6d)-8I(d),\frac{\zeta}{4}\right)\geq N'(Y(d_4,d_3,0,0),\zeta)$$

for all $d\in\mathcal{L}$and all $\zeta>0$. Pasting $d_4=\frac{d}{4}$and $d_3=\frac{d}{3}$ in above inequality , we receive

$$N\left(I(2d)-8I\left(\frac{d}{4}\right),\frac{\zeta}{4}\right)\geq N'\left(Y\left(\frac{d}{4},\frac{d}{3},0,0\right),\zeta\right)$$

for all $d\in\mathcal{L}$and all $\zeta>0$. Again putting $d=4d$ in above inequality, we reach

$$N\left(I(8d)-8I(d),\frac{\zeta}{4}\right)\geq N'\left(Y\left(d,\frac{4d}{3},0,0\right),\zeta\right) \tag{3.8}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. Another time switching (d_4,d_3,d_2,d_1) by $(d,0,d,0)$ in (3.3) and the help of oddness , we access

$$N\left(I(6d)-6I(d),\frac{\zeta}{4}\right)\geq N'(Y(d,0,d,0),\zeta) \tag{3.9}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. Once more inserting (d_4,d_3,d_2,d_1) by $(d,0,0,0)$ in (3.3) and the help of oddness , we attain

$$N\left(2I(4d)-8I(d),\frac{\zeta}{4}\right)\geq N'(Y(d,0,0,0),\zeta) \tag{3.10}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. From the inequalities (3.8) and (3.9) , we acquire

$$N\left(I(8d)+I(6d)-14I(d),\frac{\zeta}{2}\right)\geq \min\left\{N'\left(Y\left(d,\frac{4d}{3},0,0\right),\zeta\right),N'(Y(d,0,d,0),\zeta)\right\} \tag{3.11}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. From the inequalities (3.7) and (3.11) , we yield

$$N(7I(2d)-14I(d),7\zeta)\geq \min\left\{N'\left(Y\left(d,\frac{4d}{3},0,0\right),\zeta\right),N'(Y(d,0,d,0),\zeta),N'(Y(d,d,0,-d),\zeta)\right\} \tag{3.12}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. From the inequalities (3.6) and (3.7) , we gain

$$N\left(I(10d)+2I(4d)-7I(2d)-4I(d),\frac{3\zeta}{2}\right)\geq \min\{N'(Y(d,d,d,d),\zeta),N'(Y(d,d,0,-d),\zeta)\} \tag{3.13}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. From the inequalities (3.10) and (3.13) , we obtain

$$N\left(I(10d)+7I(2d)-4I(d),\frac{7\zeta}{4}\right)\geq \min\{N'(Y(d,d,d,d),\zeta),N'(Y(d,d,0,-d),\zeta),N'(Y(d,0,0,0),\zeta)\} \tag{3.14}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. Finally from the inequalities (3.12) and (3.14) , we land

$$\begin{aligned}&N\left(I(10d)-10I(d),\frac{35\zeta}{4}\right)\\&\geq \min\left\{N'(Y(d,d,d,d),\zeta),N'(Y(d,d,0,-d),\zeta),N'(Y(d,0,0,0),\zeta),N'\left(Y\left(d,\frac{4d}{3},0,0\right),\zeta\right),N'(Y(d,0,d,0),\zeta)\right\}\end{aligned} \tag{3.15}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. From the inequality (3.15) can be redesigning as

$$N\left(I(10d)-10I(d),\frac{35\zeta}{4}\right)\geq N'(\Theta_a(d),\zeta) \tag{3.16}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. Using $(F3)$ in the above inequality , we access

$$N\left(\frac{I(10d)}{10}-I(d),\frac{35}{4}.\frac{\zeta}{10}\right)\geq N'(\Theta_a(d),\zeta) \tag{3.17}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. Refixing d by $10^n d$ in (3.17), we obtain

$$N\left(\frac{I(10^{n+1}d)}{10}-I(10^n d),\frac{35}{4}.\frac{\zeta}{10}\right)\geq N'(\Theta_a(10^n d),\zeta) \tag{3.18}$$

for all $d\in\mathcal{L}$and all $\zeta>0$. Using (3.1), $(F3)$ in (3.18), we arrive

$$N\left(\frac{I(10^{n+1}d)}{10}-I(10^n d),\frac{35}{4}.\frac{\zeta}{10}\right)\geq N'\left(\Theta_a(d),\frac{\zeta}{\rho^n}\right) \tag{3.19}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. It is simple to verify from (3.19) that

$$N\left(\frac{I(10^{n+1}d)}{10^{n+1}}-\frac{I(10^n d)}{10^n},\frac{35}{40}.\frac{\zeta}{10^n}\right)\geq N'\left(\Theta_a(d),\frac{\zeta}{\rho^n}\right) \tag{3.20}$$

holds for all $d\in\mathcal{L}$ and all $\zeta>0$. Resetting ζ by $\rho^n\zeta$ in (3.20), we get

$$N\left(\frac{I(10^{n+1}d)}{10^{n+1}}-\frac{I(10^n d)}{10^n},\frac{35}{40}.\frac{\rho^n\zeta}{10^n}\right)\geq N'(\Theta_a(d),\zeta) \tag{3.21}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. It is easy to visit that

$$\frac{I(10^n d)}{10^n}-I(d)=\sum_{s=0}^{n-1}\left[\frac{I(10^{s+1}d)}{10^{s+1}}-\frac{I(10^s d)}{10^s}\right] \tag{3.22}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. From the inequalities (3.21), (3.22) and the help of $(F3)$, we attain

$$\begin{aligned}N\left(\frac{I(10^n d)}{10^n}-I(d),\frac{35}{40}\sum_{s=0}^{n-1}\frac{\rho^s\zeta}{10^s}\right)&\geq min\bigcup_{s=0}^{n-1}\left\{\frac{I(10^{s+1}d)}{10^{s+1}}-\frac{I(10^s d)}{10^s},\frac{35}{40}\frac{\rho^s\zeta}{10^s}\right\}\\&\geq min\bigcup_{s=0}^{n-1}\{N'(\Theta_a(d),\zeta)\}\\&\geq N'(\Theta_a(d),\zeta)\end{aligned} \tag{3.23}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. Replacing d by $10^m d$ in (3.23) and using (3.1), $(F3)$, we observe

$$N\left(\frac{I(10^{n+m}d)}{10^{n+m}}-\frac{I(10^m d)}{10^m},\frac{35}{40}.\sum_{s=0}^{n-1}\frac{\rho^s\zeta}{10^{m+s}}\right)\geq N'\left(\Theta_a(d),\frac{\zeta}{\rho^m}\right) \tag{3.24}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$ and all $m,n\geq 0$. Replacing ζ by $\rho^m\zeta$ in (3.24), we arrive

$$N\left(\frac{I(10^{n+m}d)}{10^{n+m}}-\frac{I(10^m d)}{10^m},\frac{35}{40}.\sum_{s=m}^{m+n-1}\frac{\rho^s\zeta}{10^s}\right)\geq N'(\Theta_a(d),\zeta) \tag{3.25}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$ and all $m,n\geq 0$. Using $(F3)$ in (3.25), we procure

$$N\left(\frac{I(10^{n+m}d)}{10^{n+m}}-\frac{I(10^{m}d)}{10^{m}},\zeta\right)\geq N'\left(\Theta_a(d),\frac{\zeta}{\frac{35}{40}\cdot\sum_{s=m}^{m+n-1}\frac{\rho^s}{10^s}}\right) \tag{3.26}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$ and all $m,n\geq 0$. Since $0<\rho<10$ and $\sum_{s=0}^{n}\left(\frac{\rho}{10^s}\right)^s<\infty$, the cauchy criterion for convergence and $(F5)$ implies that $\left\{\frac{I(10^n d)}{10^n}\right\}$ is a Cauchy sequence in $(\mathcal{M},N)$. Since $(\mathcal{M},N)$ is a fuzzy banach space, this sequence converges to some point $\mathcal{A}(d)\in\mathcal{M}$. So one can define the mapping $\mathcal{A}:\mathcal{L}\to\mathcal{M}$ by

$$\mathcal{A}(d)=N-\lim_{s\to\infty}\frac{I(10^s d)}{10^s}$$

for all $d\in\mathcal{L}$. Letting $m=0$ in (3.26), we obtain

$$N\left(\frac{I(10^{n}d)}{10^{n}}-I(d),\zeta\right)\geq N'\left(\Theta_a(d),\frac{\zeta}{\frac{35}{40}\cdot\sum_{s=0}^{n-1}\frac{\rho^s}{10^s}}\right) \tag{3.27}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. Letting $n\to\infty$ in (3.27) and using $(F6)$, we arrive

$$N(\mathcal{A}(d)-I(d),\zeta)\geq N'\left(\Theta_a(d),\frac{4(10-\rho)\zeta}{35}\right)$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. Now, we show that $\mathcal{A}$ satisfies the functional equation (1.4). Reframing (d_4,d_3,d_2,d_1) by $(10^n d_4,10^n d_3,10^n d_2,10^n d_1)$ in (3.3) respectively, we gather

$$N'\left(\frac{1}{10^n}CI(10^n d_4,10^n d_3,10^n d_2,10^n d_1),\zeta\right)\geq N'(Y(10^n d_4,10^n d_3,10^n d_2,10^n d_1),10^n\zeta) \tag{3.28}$$

for all $d_4,d_3,d_2,d_1\in\mathcal{L}$ and all $\zeta>0$. Now

$$\begin{aligned}
&N(\mathcal{A}(4d_4+3d_3+2d_2+d_1)+\mathcal{A}(4d_4-3d_3+2d_2+d_1)+\mathcal{A}(4d_4+3d_3-2d_2+d_1)\\
&+\mathcal{A}(4d_4+3d_3+2d_2-d_1)+\mathcal{A}(4d_4-3d_3-2d_2+d_1)+\mathcal{A}(4d_4-3d_3+2d_2-d_1)\\
&+\mathcal{A}(4d_4+3d_3-2d_2-d_1)+\mathcal{A}(4d_4-3d_3-2d_2-d_1)-64[\mathcal{A}(d_4+d_1)+\mathcal{A}(d_4-d_1)]\\
&+30[\mathcal{A}(d_4+d_1)-\mathcal{A}(-d_4-d_1)+\mathcal{A}(d_4-d_1)-\mathcal{A}(d_1-d_4)]-256[\mathcal{A}(d_4+d_2)+\mathcal{A}(d_4-d_2)]\\
&+126[\mathcal{A}(d_4+d_2)-\mathcal{A}(-d_4-d_2)+\mathcal{A}(d_4-d_2)-\mathcal{A}(d_2-d_4)]+112[\mathcal{A}(d_1)+\mathcal{A}(-d_1)]\\
&-8[\mathcal{A}(d_4)-\mathcal{A}(-d_4)]+126[\mathcal{A}(d_4+d_2)-\mathcal{A}(-d_4-d_2)+\mathcal{A}(d_4-d_2)-\mathcal{A}(d_2-d_4)]\\
&-128[\mathcal{A}(d_4)+\mathcal{A}(-d_4)]-18[\mathcal{A}(d_1+d_3)-\mathcal{A}(-d_1-d_3)+\mathcal{A}(d_1-d_3)-\mathcal{A}(d_3-d_1)]\\
&+432[\mathcal{A}(d_3)+\mathcal{A}(-d_3)]-72[\mathcal{A}(d_2+d_3)-\mathcal{A}(-d_2-d_3)+\mathcal{A}(d_2-d_3)-\mathcal{A}(d_3-d_2)]\\
&+352[\mathcal{A}(d_2)+\mathcal{A}(-d_2)]-288[\mathcal{A}(d_3+d_4)-\mathcal{A}(-d_3-d_4)+\mathcal{A}(d_3-d_4)-\mathcal{A}(d_4-d_3)]\ ,\zeta)\\
&\geq min\left\{N\left(\mathcal{A}(4d_4-3d_3+2d_2+d_1)-\frac{I(10^n(4d_4-3d_3+2d_2+d_1))}{10^n},\frac{\zeta}{47}\right),\right.\\
&N\left(\mathcal{A}(4d_4-3d_3+2d_2+d_1)-\frac{I(10^n(4d_4-3d_3+2d_2+d_1))}{10^n},\frac{\zeta}{47}\right),\\
&N\left(\mathcal{A}(4d_4-3d_3+2d_2+d_1)-\frac{I(10^n(4d_4-3d_3+2d_2+d_1))}{10^n},\frac{\zeta}{47}\right),\\
&N\left(\mathcal{A}(4d_4-3d_3+2d_2+d_1)-\frac{I(10^n(4d_4-3d_3+2d_2+d_1))}{10^n},\frac{\zeta}{47}\right),\\
&N\left(\mathcal{A}(4d_4-3d_3+2d_2+d_1)-\frac{I(10^n(4d_4-3d_3+2d_2+d_1))}{10^n},\frac{\zeta}{47}\right),\\
&N\left(\mathcal{A}(4d_4-3d_3+2d_2+d_1)-\frac{I(10^n(4d_4-3d_3+2d_2+d_1))}{10^n},\frac{\zeta}{47}\right),\\
&N\left(\mathcal{A}(4d_4-3d_3+2d_2+d_1)-\frac{I(10^n(4d_4-3d_3+2d_2+d_1))}{10^n},\frac{\zeta}{47}\right),\\
&N\left(\mathcal{A}(4d_4-3d_3+2d_2+d_1)-\frac{I(10^n(4d_4-3d_3+2d_2+d_1))}{10^n},\frac{\zeta}{47}\right),\\
&N\left(-64\mathcal{A}(d_4+d_1)+\frac{64\,I(10^n(d_4+d_1))}{10^n},\frac{\zeta}{47}\right),N\left(-64\mathcal{A}(d_4-d_1)+\frac{64\,I(10^n(d_4-d_1))}{10^n},\frac{\zeta}{47}\right),\\
&N\left(30\mathcal{A}(d_4+d_1)-\frac{30\,I(10^n(d_4+d_1))}{10^n},\frac{\zeta}{47}\right),N\left(-30\mathcal{A}(-d_4-d_1)+\frac{30\,I(10^n(-d_4-d_1))}{10^n},\frac{\zeta}{47}\right),\\
&N\left(30\mathcal{A}(d_4-d_1)-\frac{30\,I(10^n(d_4-d_1))}{10^n},\frac{\zeta}{47}\right),N\left(-30\mathcal{A}(d_1-d_4)+\frac{30\,I(10^n(d_1-d_4))}{10^n},\frac{\zeta}{47}\right),
\end{aligned}$$

$$N\left(-256\mathcal{A}(d_4+d_2)-\frac{256\,I\left(10^n(d_4+d_2)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-256\mathcal{A}(d_4-d_2)+\frac{256\,I\left(10^n(d_1-d_4)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(126\mathcal{A}(d_4+d_2)-\frac{126\,I\left(10^n(d_4+d_2)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-126\mathcal{A}(-d_4-d_2)+\frac{126\,I\left(10^n(-d_4-d_2)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(126\mathcal{A}(d_4-d_2)-\frac{126\,I\left(10^n(d_4-d_2)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-126\mathcal{A}(d_2-d_4)+\frac{126\,I\left(10^n(d_2-d_4)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(-8\mathcal{A}(d_1+d_2)+\frac{8\,I\left(10^n(d_1+d_2)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-8\mathcal{A}(-d_1-d_2)+\frac{8\,I\left(10^n(-d_1-d_2)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(-8\mathcal{A}(d_1-d_2)+\frac{8\,I\left(10^n(d_1-d_2)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-8\mathcal{A}(d_2-d_1)+\frac{8\,I\left(10^n(d_2-d_1)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(-128\mathcal{A}(d_4)+\frac{128\,I\left(10^n(d_4)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-128\mathcal{A}(-d_4)+\frac{128\,I\left(10^n(-d_4)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(-18\mathcal{A}(d_1+d_3)+\frac{18\,I\left(10^n(d_1+d_3)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-8\mathcal{A}(-d_1-d_3)+\frac{8\,I\left(10^n(-d_1-d_3)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(-18\mathcal{A}(d_1-d_3)+\frac{18\,I\left(10^n(d_1-d_3)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-8\mathcal{A}(d_3-d_1)+\frac{8\,I\left(10^n(d_3-d_1)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(432\mathcal{A}(d_3)+\frac{432\,I\left(10^n(d_3)\right)}{10^n},\frac{\zeta}{47}\right),N\left(432\mathcal{A}(-d_3)+\frac{432\,I\left(10^n(-d_3)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(-72\mathcal{A}(d_2+d_3)+\frac{72\,I\left(10^n(d_2+d_3)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-72\mathcal{A}(-d_2-d_3)+\frac{8\,I\left(10^n(-d_2-d_3)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(-72\mathcal{A}(d_2-d_3)+\frac{72\,I\left(10^n(d_2-d_3)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-72\mathcal{A}(d_3-d_2)+\frac{8\,I\left(10^n(d_3-d_2)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(352\mathcal{A}(d_2)+\frac{352\,I\left(10^n(d_2)\right)}{10^n},\frac{\zeta}{47}\right),N\left(352\mathcal{A}(-d_2)+\frac{352\,I\left(10^n(-d_2)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(-288\mathcal{A}(d_3+d_4)+\frac{288\,I\left(10^n(d_3+d_4)\right)}{10^n},\frac{\zeta}{47}\right),\quad N\left(-288\mathcal{A}(d_3-d_4)+\frac{288\,I\left(10^n(d_3-d_4)\right)}{10^n},\frac{\zeta}{47}\right)$$

,

$$N\left(112\mathcal{A}(d_1)+\frac{112\,I\left(10^n(d_1)\right)}{10^n},\frac{\zeta}{47}\right),N\left(112\mathcal{A}(-d_1)+\frac{112\,I\left(10^n(-d_1)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(-8\mathcal{A}(d_4)+\frac{8\,I\left(10^n(d_4)\right)}{10^n},\frac{\zeta}{47}\right),N\left(-8\mathcal{A}(-d_4)+\frac{8\,I\left(10^n(-d_4)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$N\left(-288\mathcal{A}(d_4-d_3)+\frac{288\,I\left(10^n(d_4-d_3)\right)}{10^n},\frac{\zeta}{47}\right),$$

$$\begin{gathered}
N\Bigg[\frac{1}{10^n}\big(I10^n(4d_4+3d_3+2d_2+d_1)+I\,10^n(4d_4-3d_3+2d_2+d_1) \\
+I\,10^n(4d_4+3d_3+2d_2-d_1)+I10^n(4d_4-3d_3-2d_2+d_1)+I\,10^n(4d_4-3d_3+2d_2-d_1) \\
+I\,10^n(4d_4+3d_3-2d_2-d_1)+I\,10^n(4d_4-3d_3-2d_2-d_1) \\
-64[I\,10^n(d_4+d_1)+I\,10^n(d_4-d_1)]-8[I\,10^n(d_4)-I\,10^n(-d_4)] \\
+30[I\,10^n(d_4+d_1)-I\,10^n(-d_4-d_1)+I\,10^n(d_4-d_1)-I\,10^n(d_1-d_4)] \\
-256[I\,10^n(d_4+d_2)+I\,10^n(d_4-d_2)]+112[I\,10^n(d_1)+I\,10^n(-d_1)] \\
+126[I\,10^n(d_4+d_2)-I\,10^n(-d_4-d_2)+I\,10^n(d_4-d_2)-I\,10^n(d_2-d_4)] \\
+126[I\,10^n(d_4+d_2)-I\,10^n(-d_4-d_2)+I\,10^n(d_4-d_2)-I\,10^n(d_2-d_4)] \\
-18[I\,10^n(d_1+d_3)-I\,10^n(-d_1-d_3)+I\,10^n(d_1-d_3)-I\,10^n(d_3-d_1)] \\
-72[I\,10^n(d_2+d_3)-I\,10^n(-d_2-d_3)+I\,10^n(d_2-d_3)-I\,10^n(d_3-d_2)] \\
+I\,10^n(4d_4+3d_3-2d_2+d_1)+352[I\,10^n(d_2)+I\,10^n(-d_2)] \\
-288[I\,10^n(d_3+d_4)-I\,10^n(-d_3-d_4)+I\,10^n(d_3-d_4)-I\,10^n(d_4-d_3)] \\
-128[I\,10^n(d_4)+I\,10^n(-d_4)]+432[I\,10^n(d_3)+I\,10^n(-d_3)],\frac{\zeta}{47}\big)\Bigg]\Bigg\}
\end{gathered} \tag{3.29}$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Letting $n \to \infty$ in (3.29) and using (3.2), we contain

$$N(\mathcal{A}(4d_4+3d_3+2d_2+d_1)+\mathcal{A}(4d_4-3d_3+2d_2+d_1)+\mathcal{A}(4d_4+3d_3-2d_2+d_1)$$
$$+\mathcal{A}(4d_4+3d_3+2d_2-d_1)+\mathcal{A}(4d_4-3d_3-2d_2+d_1)+\mathcal{A}(4d_4-3d_3+2d_2-d_1)$$
$$+\mathcal{A}(4d_4+3d_3-2d_2-d_1)+\mathcal{A}(4d_4-3d_3-2d_2-d_1)-64[\mathcal{A}(d_4+d_1)+\mathcal{A}(d_4-d_1)]$$
$$+30[\mathcal{A}(d_4+d_1)-\mathcal{A}(-d_4-d_1)+\mathcal{A}(d_4-d_1)-\mathcal{A}(d_1-d_4)]-256[\mathcal{A}(d_4+d_2)+\mathcal{A}(d_4-d_2)]$$
$$+126[\mathcal{A}(d_4+d_2)-\mathcal{A}(-d_4-d_2)+\mathcal{A}(d_4-d_2)-\mathcal{A}(d_2-d_4)]+112[\mathcal{A}(d_1)+\mathcal{A}(-d_1)]$$
$$-8[\mathcal{A}(d_4)-\mathcal{A}(-d_4)]+126[\mathcal{A}(d_4+d_2)-\mathcal{A}(-d_4-d_2)+\mathcal{A}(d_4-d_2)-\mathcal{A}(d_2-d_4)]$$
$$-128[\mathcal{A}(d_4)+\mathcal{A}(-d_4)]-18[\mathcal{A}(d_1+d_3)-\mathcal{A}(-d_1-d_3)+\mathcal{A}(d_1-d_3)-\mathcal{A}(d_3-d_1)]$$
$$+432[\mathcal{A}(d_3)+\mathcal{A}(-d_3)]-72[\mathcal{A}(d_2+d_3)-\mathcal{A}(-d_2-d_3)+\mathcal{A}(d_2-d_3)-\mathcal{A}(d_3-d_2)]$$
$$+352[\mathcal{A}(d_2)+\mathcal{A}(-d_2)]-288[\mathcal{A}(d_3+d_4)-\mathcal{A}(-d_3-d_4)+\mathcal{A}(d_3-d_4)-\mathcal{A}(d_4-d_3)],\zeta)=1$$
(3.30)

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Using $(F2)$ in the above inequality gives

$$\mathcal{A}(4d_4+3d_3+2d_2+d_1)+\mathcal{A}(4d_4-3d_3+2d_2+d_1)+\mathcal{A}(4d_4+3d_3-2d_2+d_1)$$
$$+\mathcal{A}(4d_4+3d_3+2d_2-d_1)+\mathcal{A}(4d_4-3d_3-2d_2+d_1)+\mathcal{A}(4d_4-3d_3+2d_2-d_1)$$
$$+\mathcal{A}(4d_4+3d_3-2d_2-d_1)+\mathcal{A}(4d_4-3d_3-2d_2-d_1)=64[\mathcal{A}(d_4+d_1)+\mathcal{A}(d_4-d_1)]$$
$$-30[\mathcal{A}(d_4+d_1)-\mathcal{A}(-d_4-d_1)+\mathcal{A}(d_4-d_1)-\mathcal{A}(d_1-d_4)]+256[\mathcal{A}(d_4+d_2)+\mathcal{A}(d_4-d_2)]$$
$$-126[\mathcal{A}(d_4+d_2)-\mathcal{A}(-d_4-d_2)+\mathcal{A}(d_4-d_2)-\mathcal{A}(d_2-d_4)]-112[\mathcal{A}(d_1)+\mathcal{A}(-d_1)]$$
$$+8[\mathcal{A}(d_4)-\mathcal{A}(-d_4)]-126[\mathcal{A}(d_4+d_2)-\mathcal{A}(-d_4-d_2)+\mathcal{A}(d_4-d_2)-\mathcal{A}(d_2-d_4)]$$
$$+128[\mathcal{A}(d_4)+\mathcal{A}(-d_4)]+18[\mathcal{A}(d_1+d_3)-\mathcal{A}(-d_1-d_3)+\mathcal{A}(d_1-d_3)-\mathcal{A}(d_3-d_1)]$$
$$-432[\mathcal{A}(d_3)+\mathcal{A}(-d_3)]+72[\mathcal{A}(d_2+d_3)-\mathcal{A}(-d_2-d_3)+\mathcal{A}(d_2-d_3)-\mathcal{A}(d_3-d_2)]$$
$$-352[\mathcal{A}(d_2)+\mathcal{A}(-d_2)]+288[\mathcal{A}(d_3+d_4)-\mathcal{A}(-d_3-d_4)+\mathcal{A}(d_3-d_4)-\mathcal{A}(d_4-d_3)]$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$. Hence $\mathcal{A}$ satisfies the additive functional equation (1.4). In order to prove $\mathcal{A}(d)$ is unique, let $\mathcal{B}(d)$ be another additive functional equation satisfying (1.4) and (3.5). Hence,

$$N(\mathcal{A}(d)-\mathcal{B}(d),\zeta)=N\left(\mathcal{A}(d)-\mathcal{B}(d)+\frac{I(10^n d)}{10^n}-\frac{I(10^n d)}{10^n},\frac{\zeta}{2}+\frac{\zeta}{2}\right)$$
$$\geq min\left\{N\left(\mathcal{A}(d)-\frac{I(10^n d)}{10^n},\frac{\zeta}{2}\right),N\left(\mathcal{B}(d)-\frac{I(10^n d)}{10^n},\frac{\zeta}{2}\right)\right\}$$
$$\geq N'\left(\Theta_a(10^n d),\frac{410^n|10-\rho|}{35}\zeta\right)$$
$$\geq N'\left(\Theta_a(d),\frac{410^n|10-\rho|}{35\,\rho^n}\zeta\right)$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$.Since $\lim_{n\to\infty}\frac{410^n|10-\rho|}{35\,\rho^n}\zeta=\infty$, we achieve

$$\lim_{n\to\infty} N'\left(\Theta_a(d),\frac{410^n|10-\rho|}{35\,\rho^n}\zeta\right)=1$$

Thus, $N(\mathcal{A}(d)-\mathcal{B}(d),\zeta)=1$. Hence $\mathcal{A}(d)=\mathcal{B}(d)$. Therefore $\mathcal{A}(d)$ is unique .

For $\psi=-1$. Reframing d by $\frac{d}{10}$ in (3.16), we access

$$N\left(I(d)-10I\left(\frac{d}{10}\right),\frac{35\zeta}{4}\right)\geq N'\left(\Theta_a\left(\frac{d}{10}\right),\zeta\right) \tag{3.31}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. Refitting d by $\frac{d}{10^n}$ in (3.31), we acquire

$$N\left(I\left(\frac{d}{10^n}\right)-10I\left(\frac{d}{10^{n+1}}\right),\frac{35\zeta}{4}\right)\geq N'\left(\Theta_a\left(\frac{d}{10^{n+1}}\right),\zeta\right) \tag{3.32}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. From the help of (3.1) and $(F3)$ in (3.32), we earn

$$N\left(I\left(\frac{d}{10^n}\right)-10I\left(\frac{d}{10^{n+1}}\right),\frac{35\zeta}{4}\right)\geq N'(\Theta_a(d),\rho^n\zeta) \tag{3.33}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. It is easy to verify that

$$N\left(10^n I\left(\frac{d}{10^n}\right)-10^{n+1}I\left(\frac{d}{10^{n+1}}\right),\frac{35\zeta}{4}.10^n\right)\geq N'(\Theta_a(d),\rho^n\zeta) \tag{3.34}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. Reclicking $\zeta$by $\frac{\zeta}{\rho^n}$ in (3.34), we claim

$$N\left(10^n I\left(\frac{d}{10^n}\right)-10^{n+1}I\left(\frac{d}{10^{n+1}}\right),\frac{35\zeta}{4}.\frac{10^n}{\rho^n}\right)\geq N'(\Theta_a(d),\zeta) \tag{3.35}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. The enduring proof is analogous to the antecedent case. This finishes the proof of the theorem.

Corollary 3.2. Let v, j are constants. Suppose that a function $I: \mathcal{L} \to \mathcal{M}$ satisfies the inequality

$$N(CI(d_4, d_3, d_2, d_1), \zeta) \geq \begin{cases} N'(v, \zeta), \\ N'(v\{\| d_4 \|^j + \| d_3 \|^j + \| d_2 \|^j + \| d_1 \|^j\}, \zeta), \\ N'(v\{\| d_4 \|^j \| d_3 \|^j \| d_2 \|^j \| d_1 \|^j + (\| d_4 \|^{4j} + \| d_3 \|^{4j} + \| d_2 \|^{4j} + \| d_1 \|^{4j})\}, \zeta) \end{cases} \quad (3.36)$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Then there persist a unique additive mapping $\mathcal{A}: \mathcal{L} \to \mathcal{M}$ such that

$$N(I(d) - \mathcal{A}(d), \zeta) \geq \begin{cases} N'\left(\left|\frac{35}{36}\right| v, \zeta\right), \\ N'\left(\frac{35v\left(\|d\|^j + \frac{4^j}{3^j}\right)}{4|10 - 10^j|}, \zeta\right), \\ N'\left(\frac{35v\left(\|d\|^{4j} + \frac{4^{4j}}{3^{4j}}\right)}{4|10 - 10^{4j}|}, \zeta\right) \end{cases} \quad (3.37)$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$.

3.2 Direct Method: Even Category.

Theorem 3.3. Let $\psi \in \{-1,1\}$ be fixed and let $Y: \mathcal{L}^4 \to \mathbb{Z}$ be a mapping with $0 < \left(\frac{\rho}{10^4}\right)^{\psi} < 1$

$$N'(Y(10^{\psi} d_4, 10^{\psi} d_3, 10^{\psi} d_2, 10^{\psi} d_1), \zeta) \geq N'(\rho^{\psi} Y(d_4, d_3, d_2, d_1), \zeta) \quad (3.38)$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$ and

$$\lim_{s \to \infty} N'(Y(10^{\psi s} d_4, 10^{\psi s} d_3, 10^{\psi s} d_2, 10^{\psi s} d_1), 10^{4\psi s} \zeta) = 1 \quad (3.39)$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Suppose that a function $I: \mathcal{L} \to \mathcal{M}$ satisfies the inequality

$$N(CI(d_4, d_3, d_2, d_1), \zeta) \geq N'(Y(d_4, d_3, d_2, d_1), \zeta) \quad (3.40)$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Then the limit

$$Q(d) = N - \lim_{s \to \infty} \frac{I(10^{s\psi} d)}{10^{4s\psi}} \quad (3.41)$$

exists for all $d \in \mathcal{L}$ and all $\zeta > 0$ and the mapping $Q: \mathcal{L} \to \mathcal{M}$ is a idiosyncratic quartic mapping satisfying (1.4) and

$$N(I(d) - Q(d), \zeta) \geq N'\left(\Theta_q(d), \frac{8\zeta|10 - \rho|}{1035}\right) \quad (3.42)$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$.

Proof. First presume $\psi = 1$. Plotting (d_4, d_3, d_2, d_1) by (d, d, d, d) in (3.40) and adopting evenness the output becomes

$$N(I(10d) + I(8d) + I(6d) + 2I(4d) - 1090I(2d) - 1536I(d), \zeta) \geq N'(Y(d, d, d, d), \zeta) \quad (3.43)$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. Again fitting (d_4, d_3, d_2, d_1) by $(d, 0, d, 2d)$ in (3.40) and apply evenness the resultant becomes

$$N\left(I(8d) + 2I(4d) - 40I(3d) - 52I(d) - 536I(d), \frac{\zeta}{2}\right) \geq N'(Y(d, 0, d, 2d)), \zeta) \quad (3.44)$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. Another time plugging (d_4, d_3, d_2, d_1) by $(0,0,d,d)$ in (3.40) and using evenness the outcome becomes

$$N(40I(3d) - 160I(2d) - 680\, I(d), 10\zeta) \geq N'(Y(0,0,d,d), \zeta) \quad (3.45)$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. One more time pasting (d_4, d_3, d_2, d_1) by $(d, 0, d, 0)$ in (3.40) and the help of evenness the outcome becomes, we obtain

$$N\left(I(6d) - 63I(2d) - 288I(d), \frac{\zeta}{4}\right) \geq N'(Y(d, 0, d, 0), \zeta) \quad (3.46)$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. One more time switching (d_4, d_3, d_2, d_1) by $(0,0,d,0)$ in (3.40) and the help of evenness the outcome becomes, we obtain

$$N\left(I(2d) - 16I(d), \frac{\zeta}{8}\right) \geq N'(Y(0,0,d,0), \zeta) \quad (3.47)$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$.
From the above inequality can be reframing as

$$N\left(63I(2d)-1008I(d),\frac{63\zeta}{8}\right)\geq N'(Y(0,0,d,0),\zeta) \tag{3.48}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$ .Again from the inequality (3.47) can be recreating as

$$N\left(878I(2d)-14048I(d),\frac{63\zeta}{8}\right)\geq N'(Y(0,0,d,0),\zeta) \tag{3.49}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. From the inequalities (3.44) and (3.45), we attain

$$N\left(I(8d)+21I(4d)-212I(2d)-1216I(d),\frac{21\zeta}{2}\right)\geq \min\{N'(Y(d,0,d,2d)),\zeta),N'(Y(0,0,d,d),\zeta)\} \tag{3.50}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. From the inequalities (3.46) and (3.47), we gain

$$N\left(I(6d)-1296I(d),\frac{65\zeta}{8}\right)\geq \min\{N'(Y(d,0,d,0)),\zeta),N'(Y(0,0,d,0),\zeta)\} \tag{3.51}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. From the inequalities (3.43) and (3.50), we access

$$N\left(I(10d)+I(6d)-878I(2d)+2752I(d),\frac{23\zeta}{2}\right)\geq \min\{N'(Y(d,d,d,d),\zeta),N'(Y(d,0,d,2d)),\zeta), N'(Y(0,0,d,d),\zeta)\} \tag{3.52}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. From the inequalities (3.49) and (3.52), we obtain

$$N\left(I(10d)+I(6d)-11296I(d)-4I(d),\frac{970\zeta}{8}\right)\geq \min\{N'(Y(d,d,d,d),\zeta),N'(Y(d,0,d,2d)),\zeta), N'(Y(0,0,d,d),\zeta),N'(Y(0,0,d,0),\zeta)\} \tag{3.53}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. From the inequalities (3.51) and (3.53) , we obtain

$$N\left(I(10d)-10000I(d),\frac{1035\zeta}{8}\right)\geq \min\{N'(Y(d,d,d,d),\zeta),N'(Y(d,0,d,2d)),\zeta), N'(Y(0,0,d,d),\zeta),N'(Y(0,0,d,0),\zeta),N'(Y(d,0,d,0)),\zeta)\} \tag{3.54}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. From the inequality (3.54) can be redesigning as

$$N\left(I(10d)-10^4I(d),\frac{1035\zeta}{8}\right)\geq N'\left(\Theta_q(d),\zeta\right) \tag{3.55}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. With the help of $(F3)$ in the above inequality, we access

$$N\left(\frac{I(10d)}{10^4}-I(d),\frac{1035}{8}.\frac{\zeta}{10^4}\right)\geq N'\left(\Theta_q(d),\zeta\right) \tag{3.56}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. The endure of the proof is identical to that Theorem 3.1 for$\psi=1$.
For $\psi=-1$, Plotting$d$ by $\frac{d}{10}$ in (3.55), we achieve

$$N\left(I\left(\frac{d}{10}\right)-10^4I\left(\frac{d}{10}\right),\frac{1035\zeta}{8}\right)\geq N'\left(\Theta_q\left(\frac{d}{10}\right),\zeta\right) \tag{3.57}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. The endure of the proof is identical to that Theorem 3.1 for$\psi=-1$. The upcoming corollary is an instantaneous outcome of Theorem 3.3 regarding the stability of (1.4).

Corollary 3.4. Let v,j are constants. Suppose that a function $I:\mathcal{L}\to\mathcal{M}$ satisfies the inequality

$$N(CI(d_4,d_3,d_2,d_1),\zeta)\geq\begin{cases}N'(v,\zeta),\\ N'(v\{\|d_4\|^j+\|d_3\|^j+\|d_2\|^j+\|d_1\|^j\},\zeta),\\ N'(v\{\|d_4\|^j\|d_3\|^j\|d_2\|^j\|d_1\|^j+(\|d_4\|^{4j}+\|d_3\|^{4j}+\|d_2\|^{4j}+\|d_1\|^{4j})\},\zeta)\end{cases} \tag{3.58}$$

for all $d_4,d_3,d_2,d_1\in\mathcal{L}$ and all $\zeta>0$. Then there persist a unique quartic mapping $Q:\mathcal{L}\to\mathcal{M}$ such that

$$N(I(d)-Q(d),\zeta)\geq\begin{cases}N'\left(\left|\frac{1035}{79992}\right|v,\zeta\right),\\ N'\left(\frac{1035v(\|d\|^j+2^j)}{8|10^4-10^{4j}|},\zeta\right),\\ N'\left(\frac{35v(\|d\|^{4j}+2^{4j})}{8|10^4-10^{16j}|},\zeta\right)\end{cases} \tag{3.59}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$.

Direct Method : Mixed Category.

Theorem 3.5. Let $\psi \in \{-1,1\}$ be fixed and let $Y: \mathcal{L}^4 \to \mathbb{Z}$ be a mapping with $0 < \left(\frac{\rho}{10}\right)^{\psi} < 1$ and $0 < \left(\frac{\rho}{10^4}\right)^{\psi} < 1$ with conditions (3.1),(3.38),(3.2) and (3.39) for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all$\zeta > 0$. Suppose that a function $I: \mathcal{L} \to \mathcal{M}$ fulfills the inequality

$$N'(CI(d_4, d_3, d_2, d_1), \zeta) \geq N'(Y(d_4, d_3, d_2, d_1), \zeta) \tag{3.60}$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Then there persist a unique additive mapping $\mathcal{A}: \mathcal{L} \to \mathcal{M}$ and a unique quartic mapping $Q: \mathcal{L} \to \mathcal{M}$ such that

$$N(\ I(d) - \mathcal{A}(d) - Q(d), 2\zeta) \geq \min\left\{N'\left(\Theta_a(d), \frac{4\zeta|10-\rho|}{35}\right), N'\left(\Theta_a(-d), \frac{4\zeta|10-\rho|}{35}\right)\right.$$
$$\left. N'\left(\Theta_q(d), \frac{8\zeta|10-\rho|}{1035}\right), N'\left(\Theta_q(-d), \frac{8\zeta|10-\rho|}{1035}\right)\right\} \tag{3.61}$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$.

Proof. Let $I_A(d) = \frac{I(d)-I(-d)}{2}$ for all $d \in \mathcal{L}$. Then $I_A(0) = 0$; $I_A(-d) = -I_A(d)$ for all $d \in \mathcal{L}$. By the Theorem 3.1, we attain

$$N(I_A(d) - \mathcal{A}(d), \zeta) \geq \min\left\{N'\left(\Theta_a(d), \frac{4\zeta|10-\rho|}{35}\right), N'\left(\Theta_a(-d), \frac{4\zeta|10-\rho|}{35}\right)\right\} \tag{3.62}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. Also, let $I_Q(d) = \frac{I(d)+I(-d)}{2}$ for all$d \in \mathcal{L}$. Then$I_Q(0) = 0$; $I_Q(-d) = I_Q(d)$ for all $d \in \mathcal{L}$. By the Theorem 3.3, we acquire

$$N\left(I_Q(d) - Q(d), \zeta\right) \geq \min\left\{N'\left(\Theta_q(d), \frac{8\zeta|10-\rho|}{1035}\right), N'\left(\Theta_q(-d), \frac{8\zeta|10-\rho|}{1035}\right)\right\} \tag{3.63}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. Define$I(d) = I_A(d) + I_Q(d)$, it follows from (3.62) and (3.63) we appeared our coveted result.

Corollary 3.6. Let v, j are constants. Suppose that a function $I: \mathcal{L} \to \mathcal{M}$ satisfies the inequality

$$N(CI(d_4, d_3, d_2, d_1), \zeta) \geq \begin{cases} N'(v, \zeta), \\ N'(v\{\| d_4 \|^j + \| d_3 \|^j + \| d_2 \|^j + \| d_1 \|^j\}, \zeta), \\ N'(v\{\| d_4 \|^j \| d_3 \|^j \| d_2 \|^j \| d_1 \|^j + (\| d_4 \|^{4j} + \| d_3 \|^{4j} + \| d_2 \|^{4j} + \| d_1 \|^{4j})\}, \zeta) \end{cases} \tag{3.64}$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Then there persist a unique additive mapping $\mathcal{A}: \mathcal{L} \to \mathcal{M}$ and a unique quartic mapping$Q: \mathcal{L} \to \mathcal{M}$ such that

$$N(\ I(d) - \mathcal{A}(d) - Q(d), 2\zeta) \geq \begin{cases} N'\left(\left|\frac{1199}{1250}\right| v, \zeta\right), \\ N'\left(\left|\frac{1105}{8}\right| \left[\frac{v\left(\|d\|^j + \frac{4^j}{3^j}\right)}{|10-10^j|} + \frac{v(\|d\|^j + 2^j)}{|10^4 - 10^{4j}|}\right], \zeta\right) \\ N'\left(\left|\frac{1105}{8}\right| \left[\frac{v\left(\|d\|^{4j} + \frac{4^{4j}}{3^{4j}}\right)}{|10-10^{4j}|} + \frac{v(\|d\|^{4j} + 2^{4j})}{|10^4 - 10^{16j}|}\right], \zeta\right) \end{cases} \tag{3.65}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$.

4. FIXED POINT RESULTS

In this site, the authors inquire the generalized Ulam – Hyers stability of the functional equation (1.4) in fuzzy normed space with the help of fixed point method. Now we will revision the basic concepts of fixed point theory.

Theorem 4.1. [19] (*The alternative of fixed point) Suppose that for a complex generalized metric space* (X, d) *and a strictly contractive mapping* $T: X \to X$ *with Lipschitz constant L. Then, for each element* $x \in X$,*either*

(A1) $d(T^n x, T^{n+1} x) = \infty \ \forall n \geq 0,$ *(Or)*

(A2) there exists a natural number n_0 *such that :*

(AFP1) $d(T^n x, T^{n+1} x) < \infty$ *for all* $n \geq n_0$ *;*

(AFP2) The sequence $(T^n x)$ *is convergent to a fixed point* y^* *of* T.

(AFP3) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X : d(T^{n_0} x, y) < \infty\}$ *;*

(AFP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in Y$.

Fixed Point Method : Odd Category.

Theorem 4.2. Let $I: \mathcal{L} \to \mathcal{M}$ be a mapping for which there exist a function $Y: \mathcal{L}^4 \to \mathbb{Z}$ with the condition

$$\lim_{c\to\infty} N'(Y(\eta_t{}^c d_4, \eta_t{}^c d_3, \eta_t{}^c d_2, \eta_t{}^c d_1), \eta_t{}^c \zeta) = 1 \tag{4.1}$$

where η_t is a constant for

$$\eta_t = \begin{cases} 10 & if\ t = 0 \\ \frac{1}{10} & if\ t = 1 \end{cases},$$

such that the functional imparity

$$N(CI(d_4, d_3, d_2, d_1), \zeta) \geq N'(Y(d_4, d_3, d_2, d_1), \zeta) \tag{4.2}$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. If there exists $L = L(t)$ such that the function

$$d \to \mathcal{K}(d) = \Theta_a\left(\frac{d}{10}\right)$$

has the property

$$N'\left(L\frac{\mathcal{K}(\eta_t d)}{\eta_t}, \zeta\right) \geq N'(\mathcal{K}(d), \zeta) \tag{4.3}$$

Then there exist a idiosyncratic additive function $\mathcal{A}: \mathcal{L} \to \mathcal{M}$ satiating the functional equation (1.4) and

$$N\left(\mathcal{A}(d) - I(d), \frac{35}{4}\zeta\right) \geq N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}(d), \zeta\right) \tag{4.4}$$

holds for all $d \in \mathcal{L}$ and all $\zeta > 0$.

Proof. Let d be a general metric on Ω, such that

$$d(o,p) = inf\{W \in (0,\infty)/N(o(d) - p(d), \zeta) \geq N'(\mathcal{K}(d), W\zeta), d \in \mathcal{L}\}.$$

It is easy to visit that (Ω, d) is complete. Define $G: \Omega \to \Omega$ by $Go(d) = \frac{1}{\eta_t} o(\eta_t d)$, for all $d \in \mathcal{L}$.For $o, p \in \Omega$, we attain

$$\begin{aligned} d(o,p) = W \quad &\Rightarrow \quad N(o(d) - p(d), \zeta) \geq N'(\mathcal{K}(d), W\zeta) \\ &\Rightarrow \quad N\left(\frac{o(\eta_t d)}{\eta_t} - \frac{p(\eta_t d)}{\eta_t}, \frac{\zeta}{\eta_t}\right) \geq N'(\mathcal{K}(\eta_t d), W\zeta) \\ &\Rightarrow \quad N\left(\frac{o(\eta_t d)}{\eta_t} - \frac{p(\eta_t d)}{\eta_t}, \zeta\right) \geq N'(\mathcal{K}(\eta_t d), W\eta_t \zeta) \\ &\Rightarrow \quad N(Go(d) - Gp(d), \zeta) \geq N'(\mathcal{K}(\eta_t d), WL\zeta) \\ &\Rightarrow \quad d(Go(d), Gp(d)) \leq WL \\ &\Rightarrow \quad d(Go, Gp) \leq Ld(o,p) \forall o, p \in \Omega. \end{aligned}$$

Therefore G is strictly contractive mapping on Ω with Lipschitz constant$L$.
From the imparity (3.16), we claim

$$N\left(I(10d) - 10I(d), \frac{35\zeta}{4}\right) \geq N'(\Theta_a(d), \zeta) \tag{4.5}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. Using $(F2)$ in (4.5), we arrive

$$N\left(\frac{I(10d)}{10} - I(d), \frac{35\zeta}{4}\right) \geq N'(\Theta_a(d), 10\zeta) \tag{4.6}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. From the help of (4.3), when $t = 0$, it follows from (4.6), that

$$N\left(\frac{I(10d)}{10} - I(d), \frac{35\zeta}{4}\right) \geq N'(\mathcal{K}(d), L\zeta)$$

$$\Rightarrow \quad d(GI, I) \leq L = L^1 = \mathrm{L}^{1-\mathrm{t}} \tag{4.7}$$

Resetting d by $\frac{d}{10}$ in (4.6), we acquire

$$N\left(I(d) - 10\mathrm{I}\left(\frac{\mathrm{d}}{10}\right), \frac{35\zeta}{4}\right) \geq N'\left(\Theta_a\left(\frac{\mathrm{d}}{10}\right), \zeta\right) \tag{4.8}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. when$t = 0$, it follows from (4.8), that

$$N\left(I(d) - 10\mathrm{I}\left(\frac{\mathrm{d}}{10}\right), \frac{35\zeta}{4}\right) \geq N'(\mathcal{K}(d), \zeta)$$

$$\Rightarrow \quad d(GI, I) \leq 1 = \mathrm{L}^{1-\mathrm{t}} \tag{4.9}$$

Then from (4.7) and (4.9) we can landed

$$d(GI, I) \leq \mathrm{L}^{1-\mathrm{t}} < \infty$$

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point $\mathcal{A}$ of G in Ω such that

$$\mathcal{A}(d) = N - \lim_{c\to\infty} \frac{I({\eta_t}^c d)}{{\eta_t}^c} \qquad \forall\, d \in \mathcal{L}\,. \tag{4.10}$$

Repasting (d_4, d_3, d_2, d_1) by$({\eta_t}^c d_4, {\eta_t}^c d_3, {\eta_t}^c d_2, {\eta_t}^c d_1)$ in (4.2) , we arrive

$$N\left(\frac{1}{{\eta_t}^c} CI({\eta_t}^c d_4, {\eta_t}^c d_3, {\eta_t}^c d_2, {\eta_t}^c d_1), \zeta\right) \geq N'(Y({\eta_t}^c d_4, {\eta_t}^c d_3, {\eta_t}^c d_2, {\eta_t}^c d_1), {\eta_t}^c \zeta) \tag{4.11}$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Hence, $\mathcal{A}: \mathcal{L} \to \mathcal{M}$ is additive and it gratifies the functional equation (1.4). Since $\mathcal{A}$ is idiosyncratic fixed point of G in the set

$$\Delta = \{ I \in \Omega \,/\, d(I, \mathcal{A}) < \infty \}$$

therefore $\mathcal{A}$ is a unique function such that

$$N(\mathcal{A}(d) - I(d), \zeta) \geq N'(\mathcal{K}(d), W\zeta) \tag{4.12}$$

holds for all $d \in \mathcal{L}$ and all $\zeta > 0$.Once more time using the fixed point alternative, we obtain

$$\Rightarrow \qquad d(I, \mathcal{A}) \leq \frac{1}{1-L} d(I, GI)$$

$$\Rightarrow \qquad d(I, \mathcal{A}) \leq \frac{L^{1-t}}{1-L}$$

$$\Rightarrow \quad N(\mathcal{A}(d) - I(d), \zeta) \geq N'\left(\frac{L^{1-t}}{1-L} \mathcal{K}(d), \zeta\right) \tag{4.13}$$

This finishes the proof of the theorem.

Corollary 4.3. Let v, j are constants. Suppose that a function $I: \mathcal{L} \to \mathcal{M}$ satisfies the inequality

$$N(CI(d_4, d_3, d_2, d_1), \zeta) \quad \geq \begin{cases} N'(v, \zeta), \\ N'(v\{\| d_4 \|^j + \| d_3 \|^j + \| d_2 \|^j + \| d_1 \|^j\}, \zeta), \\ N'(v\{\| d_4 \|^j \| d_3 \|^j \| d_2 \|^j \| d_1 \|^j + (\| d_4 \|^{4j} + \| d_3 \|^{4j} + \| d_2 \|^{4j} + \| d_1 \|^{4j})\}, \zeta) \end{cases} \tag{4.14}$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Then there persist a unique additive mapping $\mathcal{A}: \mathcal{L} \to \mathcal{M}$ such that

$$N(I(d) - \mathcal{A}(d), \zeta) \;\geq \begin{cases} N'\left(\left|\frac{35}{36}\right| v, \zeta\right), \\ N'\left(\frac{35v\left(\|d\|^j + \frac{4^j}{3^j}\right)}{4|10-10^j|}, \zeta\right), \\ N'\left(\frac{35v\left(\|d\|^{4j} + \frac{4^{4j}}{3^{4j}}\right)}{4|10-10^{4j}|}, \zeta\right) \end{cases} \tag{4.15}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$.

Proof . Pasting

$$\Upsilon(d_4, d_3, d_2, d_1) = \begin{cases} v, \\ v\{\| d_4 \|^j + \| d_3 \|^j + \| d_2 \|^j + \| d_1 \|^j\} \\ v\{\| d_4 \|^j \| d_3 \|^j \| d_2 \|^j \| d_1 \|^j + [\| d_4 \|^{4j} + \| d_3 \|^{4j} + \| d_2 \|^{4j} + \| d_1 \|^{4j}]\} \end{cases}$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$. Then,

$$N'(\Upsilon({\eta_t}^c d_4, {\eta_t}^c d_3, {\eta_t}^c d_2, {\eta_t}^c d_1), {\eta_t}^c \zeta)$$

$$= \begin{cases} N'(v, {\eta_t}^c \zeta) \\ N'(v\{\| {\eta_t}^c d_4 \|^j + \| {\eta_t}^c d_3 \|^j + \| {\eta_t}^c d_2 \|^j + \| {\eta_t}^c d_1 \|^j\}, {\eta_t}^c \zeta) \\ N'(v\{\| {\eta_t}^c d_4 \|^j \| {\eta_t}^c d_3 \|^j \| {\eta_t}^c d_2 \|^j \| {\eta_t}^c d_1 \|^j + [\| {\eta_t}^c d_4 \|^{4j} + \| {\eta_t}^c d_3 \|^{4j} + \| {\eta_t}^c d_2 \|^{4j} + \| {\eta_t}^c d_1 \|^{4j}]\}, {\eta_t}^c \zeta) \end{cases}$$

$$= \begin{cases} N'(v, {\eta_t}^c \zeta) \\ N'(v\, {\eta_t}^{cj}\{\| d_4 \|^j + \| d_3 \|^j + \| d_2 \|^j + \| d_1 \|^j\}, {\eta_t}^c \zeta) \\ N'(v\, {\eta_t}^{cj}\{\| d_4 \|^j \| d_3 \|^j \| d_2 \|^j \| d_1 \|^j + [\| {\eta_t}^c d_4 \|^{4j} + \| {\eta_t}^c d_3 \|^{4j} + \| {\eta_t}^c d_2 \|^{4j} + \| {\eta_t}^c d_1 \|^{4j}]\}, {\eta_t}^c \zeta) \end{cases}$$

$$= \begin{cases} \to 1 \quad as \quad n \to \infty \\ \to 1 \quad as \quad n \to \infty \\ \to 1 \quad as \quad n \to \infty \end{cases}$$

Thus, (4.1) is holds.

But we have $\mathcal{K}(d) = \Theta_a\left(\frac{d}{10}\right)$ has the property

$$N'\left(L\frac{\mathcal{K}(\eta_t d)}{\eta_t},\zeta\right) \geq N'(\mathcal{K}(d),\zeta)$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. Now,

$$N'\left(\frac{\mathcal{K}(\eta_t d)}{\eta_t},\zeta\right) = \begin{cases} N'(\eta_t{}^{-1}v,\zeta), \\ N'\left(\eta_t{}^{j-1}\frac{v}{10^j}\left(\| d \|^j + \frac{4^j}{3^j}\right),\zeta\right), \\ N'\left(\eta_t{}^{4j-1}\frac{v}{10^{4j}}\left(\| d \|^{4j} + \frac{4^{4j}}{3^{4j}}\right),\zeta\right) \end{cases}$$

$$= \begin{cases} N'(\eta_t{}^{-1}\,\mathcal{K}(d),\zeta), \\ N'(\eta_t{}^{j-1}\,\mathcal{K}(d),\zeta), \\ N'(\eta_t{}^{4j-1}\,\mathcal{K}(d),\zeta) \end{cases}$$

Hence the imparity (4.3) holds. Now from (4.4), we prove the following cases :

Case 1: $\mathrm{L} = \eta_t{}^{-1} = 10^{-1}$ if $t = 0$

$$N\left(\mathcal{A}(d) - I(d),\frac{35}{4}\zeta\right) \geq N'\left(\frac{L^{1-0}}{1-L}\mathcal{K}(d),\zeta\right)$$

$$N\left(\mathcal{A}(d) - I(d),\frac{35}{4}\zeta\right) \geq N'\left(\frac{10^{-1}}{10^{-1}(10-1)}v,\zeta\right)$$

$$N(\mathcal{A}(d) - I(d),\zeta) = N'\left(\frac{35}{36}v,\zeta\right)$$

Case 2: $\mathrm{L} = \eta_t{}^{-1} = \left(\frac{1}{10}\right)^{-1}$ if $t = 1$

$$N\left(\mathcal{A}(d) - I(d),\frac{35}{4}\zeta\right) \geq N'\left(\frac{L^{1-1}}{1-L}\mathcal{K}(d),\zeta\right)$$

$$N\left(\mathcal{A}(d) - I(d),\frac{35}{4}\zeta\right) \geq N'\left(\frac{10}{10(1-10)}v,\zeta\right)$$

$$N(\mathcal{A}(d) - I(d),\zeta) = N'\left(-\frac{35}{36}v,\zeta\right)$$

Case 3 : $\mathrm{L} = \eta_t{}^{j-1} = 10^{\mathrm{j}-1}$ if $t = 0$

$$N\left(\mathcal{A}(d) - I(d),\frac{35}{4}\zeta\right) \geq N'\left(\frac{L^{1-0}}{1-L}\mathcal{K}(d),\zeta\right)$$

$$N\left(\mathcal{A}(d) - I(d),\frac{35}{4}\zeta\right) \geq N'\left(\frac{10^{\mathrm{j}-1}}{1-10^{\mathrm{j}-1}}\mathcal{K}(d),\zeta\right)$$

$$N\left(\mathcal{A}(d) - I(d),\frac{35}{4}\zeta\right) \geq N'\left(\frac{10^{\mathrm{j}-1}}{1-10^{\mathrm{j}-1}}\frac{v}{10^j}\left(\| d \|^j + \frac{4^j}{3^j}\right),\zeta\right)$$

$$N(\mathcal{A}(d) - I(d),\zeta) = N'\left(\frac{35\,v\left(\| d \|^j + \frac{4^j}{3^j}\right)}{4(10-10^j)},\zeta\right)$$

Case 4 : $\mathrm{L} = \eta_t{}^{1-j} = 10^{1-\mathrm{j}}$ if $t = 1$

$$N\left(\mathcal{A}(d) - I(d),\frac{35}{4}\zeta\right) \geq N'\left(\frac{L^{1-1}}{1-L}\mathcal{K}(d),\zeta\right)$$

$$N\left(\mathcal{A}(d) - I(d),\frac{35}{4}\zeta\right) \geq N'\left(\frac{10^{1-\mathrm{j}}}{1-10^{1-\mathrm{j}}}\mathcal{K}(d),\zeta\right)$$

$$N\left(\mathcal{A}(d) - I(d),\frac{35}{4}\zeta\right) \geq N'\left(\frac{10^{1-\mathrm{j}}}{1-10^{1-\mathrm{j}}}\frac{v}{10^j}\left(\| d \|^j + \frac{4^j}{3^j}\right),\zeta\right)$$

$$N(\mathcal{A}(d) - I(d),\zeta) = N'\left(\frac{35\,v\left(\| d \|^j + \frac{4^j}{3^j}\right)}{4(10^j-10)},\zeta\right)$$

Case 5 : $L = {\eta_t}^{4j-1} = 10^{4j-1}$ if $t = 0$

$$N\left(\mathcal{A}(d) - I(d), \frac{35}{4}\zeta\right) \geq N'\left(\frac{L^{1-0}}{1-L}\mathcal{K}(d), \zeta\right)$$

$$N\left(\mathcal{A}(d) - I(d), \frac{35}{4}\zeta\right) \geq N'\left(\frac{10^{4j-1}}{1-10^{4j-1}}\mathcal{K}(d), \zeta\right)$$

$$N\left(\mathcal{A}(d) - I(d), \frac{35}{4}\zeta\right) \geq N'\left(\frac{10^{4j-1}}{1-10^{4j-1}}\frac{v}{10^{4j}}\left(\| d \|^{4j} + \frac{4^{4j}}{3^{4j}}\right), \zeta\right)$$

$$N(\mathcal{A}(d) - I(d), \zeta) = N'\left(\frac{35\, v\left(\| d \|^{4j} + \frac{4^{4j}}{3^{4j}}\right)}{4(10 - 10^{4j})}, \zeta\right)$$

Case 6 : $L = {\eta_t}^{1-4j} = 10^{1-4j}$ if $t = 1$

$$N\left(\mathcal{A}(d) - I(d), \frac{35}{4}\zeta\right) \geq N'\left(\frac{L^{1-1}}{1-L}\mathcal{K}(d), \zeta\right)$$

$$N\left(\mathcal{A}(d) - I(d), \frac{35}{4}\zeta\right) \geq N'\left(\frac{10^{1-4j}}{1-10^{1-4j}}\mathcal{K}(d), \zeta\right)$$

$$N\left(\mathcal{A}(d) - I(d), \frac{35}{4}\zeta\right) \geq N'\left(\frac{10^{1-4j}}{1-10^{1-4j}}\frac{v}{10^{j}}\left(\| d \|^{4j} + \frac{4^{4j}}{3^{4j}}\right), \zeta\right)$$

$$N(\mathcal{A}(d) - I(d), \zeta) = N'\left(\frac{35\, v\left(\| d \|^{4j} + \frac{4^{4j}}{3^{4j}}\right)}{4(10^{4j} - 10)}, \zeta\right)$$

Hence the proof is complete.

Fixed Point Method : Even Category.

Theorem 4.4. Let $I: \mathcal{L} \to \mathcal{M}$ be a mapping for which there exist a function $Y : \mathcal{L}^4 \to \mathbb{Z}$ with the condition

$$\lim_{c\to\infty} N'(Y({\eta_t}^c d_4, {\eta_t}^c d_3, {\eta_t}^c d_2, {\eta_t}^c d_1), {\eta_t}^{4c}\zeta) = 1 \qquad (4.16)$$

where η_t is a constant for

$$\eta_t = \begin{cases} 10 & if\ t = 0 \\ \frac{1}{10} & if\ t = 1 \end{cases},$$

such that the functional imparity

$$N(CI(d_4, d_3, d_2, d_1), \zeta) \geq N'(Y(d_4, d_3, d_2, d_1), \zeta) \qquad (4.17)$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. If there exists $L = L(t)$ such that the function

$$d \to \mathcal{K}(d) = \Theta_q\left(\frac{d}{10}\right)$$

has the property

$$N'\left(L\frac{\mathcal{K}(\eta_t d)}{{\eta_t}^4}, \zeta\right) \geq N'(\mathcal{K}(d), \zeta) \qquad (4.18)$$

Then there exist a idiosyncratic Quartic function $Q: \mathcal{L} \to \mathcal{M}$ satiating the functional equation (1.4) and

$$N\left(I(d) - Q(d), \frac{1035}{8}\zeta\right) \geq N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}(d), \zeta\right) \qquad (4.19)$$

holds for all $d \in \mathcal{L}$ and all $\zeta > 0$.

Proof. Let d be a general metric onΩ, such that

$$d(o,p) = inf\{W \in (0,\infty)/N(o(d) - p(d), \zeta) \geq N'(\mathcal{K}(d), W\zeta), d \in \mathcal{L}\}.$$

It is easy to visit that (Ω, d) is complete. Define $G: \Omega \to \Omega$ by $Go(d) = \frac{1}{{\eta_t}^4}o(\eta_t d)$, for all $d \in \mathcal{L}$. For $o, p \in \Omega$, we attain

$$d(o,p) = W \quad \Rightarrow \quad N(o(d) - p(d), \zeta) \geq N'(\mathcal{K}(d), W\zeta)$$

$$\Rightarrow \quad N\left(\frac{o(\eta_t d)}{{\eta_t}^4} - \frac{p(\eta_t d)}{{\eta_t}^4}, \frac{\zeta}{{\eta_t}^4}\right) \geq N'(\mathcal{K}(\eta_t d), W\zeta)$$

$$\Rightarrow \quad N\left(\frac{o(\eta_t d)}{{\eta_t}^4} - \frac{p(\eta_t d)}{{\eta_t}^4}, \zeta\right) \geq N'(\mathcal{K}(\eta_t d), W{\eta_t}^4\zeta)$$

$$\Rightarrow \quad N(Go(d) - Gp(d), \zeta) \geq N'(\mathcal{K}(\eta_t d), WL\zeta)$$

$$\Rightarrow \quad d\big(Go(d), Gp(d)\big) \leq WL$$

$$\Longrightarrow \qquad d(Go,Gp) \le Ld(o,p) \forall o,p \in \Omega.$$

Therefore G is strictly contractive mapping on Ω with Lipschitz constant L.
From the impairity (3.55), we claim

$$N\left(I(10d) - 10^4 I(d), \frac{1035\zeta}{8}\right) \ge N'\left(\Theta_q(d), \zeta\right) \tag{4.20}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. Using $(F2)$ in (4.20), we arrive

$$N\left(\frac{I(10d)}{10^4} - I(d), \frac{1035\zeta}{8}\right) \ge N'\left(\Theta_q(d), 10^4\zeta\right) \tag{4.21}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. From the help of (4.18), when $t = 0$, it follows from (4.21), that

$$N\left(\frac{I(10d)}{10^4} - I(d), \frac{1035\zeta}{8}\right) \ge N'(\mathcal{K}(d), L\zeta)$$

$$\Longrightarrow \qquad d(GI,I) \le L = L^1 = \mathrm{L}^{1-\mathrm{t}} \tag{4.22}$$

Resetting d by $\frac{d}{10}$ in (4.22) , we acquire

$$N\left(I(d) - 10^4 \mathrm{I}\left(\frac{\mathrm{d}}{10}\right), \frac{1035\zeta}{8}\right) \ge N'\left(\Theta_q\left(\frac{\mathrm{d}}{10}\right), \zeta\right) \tag{4.23}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$. when $t = 0$, it follows from (4.23), that

$$N\left(I(d) - 10^4 \mathrm{I}\left(\frac{\mathrm{d}}{10}\right), \frac{1035\zeta}{8}\right) \ge N'(\mathcal{K}(d), \zeta)$$

$$\Longrightarrow \qquad d(GI,I) \le 1 = \mathrm{L}^{1-\mathrm{t}} \tag{4.24}$$

Then from (4.22) and (4.24) we can landed

$$d(GI,I) \le \mathrm{L}^{1-\mathrm{t}} < \infty$$

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point Q of G in Ω such that

$$Q(d) = N - \lim_{c\to\infty} \frac{I({\eta_t}^c d)}{{\eta_t}^{4c}} \qquad \forall\, d \in \mathcal{L} \tag{4.25}$$

Repasting (d_4, d_3, d_2, d_1) by $({\eta_t}^c d_4, {\eta_t}^c d_3, {\eta_t}^c d_2, {\eta_t}^c d_1)$ in (4.17), we arrive

$$N\left(\frac{1}{{\eta_t}^{4c}} CI({\eta_t}^c d_4, {\eta_t}^c d_3, {\eta_t}^c d_2, {\eta_t}^c d_1), \zeta\right) \ge N'(Y({\eta_t}^c d_4, {\eta_t}^c d_3, {\eta_t}^c d_2, {\eta_t}^c d_1), {\eta_t}^{4c}\zeta) \tag{4.26}$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Hence, $Q: \mathcal{L} \to \mathcal{M}$ is quartic and it gratifies the functional equation (1.4). Since Q is idiosyncratic fixed point of G in the set

$$\Delta = \{\, I \in \Omega \,/\, \mathrm{d}(\mathrm{I}, Q) < \infty \,\}$$

therefore Q is a unique function such that

$$N(Q(d) - I(d), \zeta) \ge N'(\mathcal{K}(d), W\zeta) \tag{4.27}$$

holds for all $d \in \mathcal{L}$ and all $\zeta > 0$. Once more time using the fixed point alternative , we obtain

$$\Longrightarrow \qquad d(I,Q) \le \frac{1}{1-L} d(I,GI)$$

$$\Longrightarrow \qquad d(I,Q) \le \frac{L^{1-t}}{1-L}$$

$$\Longrightarrow \qquad N(I(d) - Q(d), \zeta) \ge N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}(d), \zeta\right) \tag{4.28}$$

This finishes the proof of the theorem.

Corollary 4.5. Let v, j are constants. Suppose that a function $I: \mathcal{L} \to \mathcal{M}$ satisfies the inequality

$$N(CI(d_4, d_3, d_2, d_1), \zeta) \quad \ge \begin{cases} N'(v, \zeta), \\ N'(v\{\| d_4 \|^j + \| d_3 \|^j + \| d_2 \|^j + \| d_1 \|^j\}, \zeta), \\ N'(v\{\| d_4 \|^j \| d_3 \|^j \| d_2 \|^j \| d_1 \|^j + (\| d_4 \|^{4j} + \| d_3 \|^{4j} + \| d_2 \|^{4j} + \| d_1 \|^{4j})\}, \zeta) \end{cases} \tag{4.29}$$

for all $d_4, d_3, d_2, d_1 \in \mathcal{L}$ and all $\zeta > 0$. Then there persist a unique quartic mapping $Q: \mathcal{L} \to \mathcal{M}$ such that

$$N(I(d) - Q(d), \zeta) \ \ge \begin{cases} N'\left(\left|\frac{1035}{79992}\right| v, \zeta\right), \\ N'\left(\frac{1035v(\|d\|^j + 2^j)}{8|10^4 - 10^{4j}|}, \zeta\right), \\ N'\left(\frac{35v(\|d\|^{4j} + 2^{4j})}{8|10^4 - 10^{16j}|}, \zeta\right) \end{cases} \tag{4.30}$$

for all $d \in \mathcal{L}$ and all $\zeta > 0$.

Proof . The proof is identical to the corollary 4.3.

Direct Method : Mixed Category.

Theorem 4.6. Let $I:\mathcal{L}\to\mathcal{M}$ be a mapping for which there exist a function $Y:\ \mathcal{L}^4\to\mathbb{Z}$ be a mapping with $0<\left(\frac{\rho}{10}\right)^{\psi}<1$ and $0<\left(\frac{\rho}{10^4}\right)^{\psi}<1$ with conditions (4.1) and (4.16) for all $d\in\mathcal{L}$ and all $\zeta>0$. where η_t is a constant define

$$\eta_t=\begin{cases}10 & if\ t=0\\ \frac{1}{10} & if\ t=1\end{cases},$$

such that the functional imparity

$$N(CI(d_4,d_3,d_2,d_1),\zeta)\geq N'(Y(d_4,d_3,d_2,d_1),\zeta) \tag{4.31}$$

for all $d_4,d_3,d_2,d_1\in\mathcal{L}$ and all $\zeta>0$. If there exists $L=L(t)$ has the properties (4.3) and (4.18)then there persist a unique additive mapping $\mathcal{A}:\mathcal{L}\to\mathcal{M}$ and a unique quartic mapping $Q:\mathcal{L}\to\mathcal{M}$ such that

$$N\left(\ I(d)-\mathcal{A}(d)-Q(d),\frac{1105}{8}\zeta\right)\geq\ \min\left\{N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}_a(d),\zeta\right),N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}_a(-d),\zeta\right),\right.$$
$$\left.N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}_q(d),\zeta\right),N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}_q(-d),\zeta\right)\right\} \tag{4.32}$$

for all $d_4,d_3,d_2,d_1\in\mathcal{L}$ and all $\zeta>0$.

Proof. Let $I_A(d)=\frac{I(d)-I(-d)}{2}$ for all $d\in\mathcal{L}$. Then $I_A(0)=0$; $I_A(-d)=-I_A(d)$ for all $d\in\mathcal{L}$. By the Theorem 4.2, we attain

$$N\left(I_A(d)-\mathcal{A}(d),\frac{35}{4}\zeta\right)\geq\ \min\left\{N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}_a(d),\zeta\right),N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}_a(-d),\zeta\right)\right\} \tag{4.33}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. Also, let $I_Q(d)=\frac{I(d)+I(-d)}{2}$ for all $d\in\mathcal{L}$. Then $I_Q(0)=0$;
$I_Q(-d)=I_Q(d)$ for all $d\in\mathcal{L}$. By the Theorem 4.4, we acquire

$$N\left(I_Q(d)-Q(d),\frac{1035}{8}\zeta\right)\geq\ \min\left\{\ N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}_q(d),\zeta\right),N'\left(\frac{L^{1-t}}{1-L}\mathcal{K}_q(-d),\zeta\right)\right\} \tag{4.34}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$. Define $I(d)=I_A(d)+I_Q(d)$, it follows from (4.33) and (4.34) we appeared our coveted result.

Corollary 4.7. Let v,j are constants. Suppose that a function $I:\mathcal{L}\to\mathcal{M}$ satisfies the inequality

$$N(CI(d_4,d_3,d_2,d_1),\zeta)\quad\geq\begin{cases}N'(v,\zeta),\\ N'(v\{\|d_4\|^j+\|d_3\|^j+\|d_2\|^j+\|d_1\|^j\},\zeta),\\ N'(v\{\|d_4\|^j\|d_3\|^j\|d_2\|^j\|d_1\|^j+(\|d_4\|^{4j}+\|d_3\|^{4j}+\|d_2\|^{4j}+\|d_1\|^{4j})\},\zeta)\end{cases} \tag{4.35}$$

for all $d_4,d_3,d_2,d_1\in\mathcal{L}$ and all $\zeta>0$. Then there persist a unique additive mapping $\mathcal{A}:\mathcal{L}\to\mathcal{M}$ and a unique quartic mapping $Q:\mathcal{L}\to\mathcal{M}$ such that

$$N(\ I(d)-\mathcal{A}(d)-Q(d),2\zeta)\ \geq\begin{cases}N'\left(\left|\frac{1199}{1250}\right|v,\zeta\right),\\ N'\left(\left|\frac{1105}{8}\right|\left[\frac{v\left(\|d\|^j+\frac{4^j}{3^j}\right)}{|10-10^j|}+\frac{v(\|d\|^j+2^j)}{|10^4-10^{4j}|}\right],\zeta\right)\\ N'\left(\left|\frac{1105}{8}\right|\left[\frac{v\left(\|d\|^{4j}+\frac{4^{4j}}{3^{4j}}\right)}{|10-10^{4j}|}+\frac{v(\|d\|^{4j}+2^{4j})}{|10^4-10^{16j}|}\right],\zeta\right)\end{cases} \tag{4.36}$$

for all $d\in\mathcal{L}$ and all $\zeta>0$.

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