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Star–In–Coloring of Friendship and Double Wheel Graphs

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Abstract: - This paper discusses Star–in–Coloring of a graph and gives the exact Star–in–Chromatic number of some variants friendship graphs. We show that the path union cycle C_m and open star cycle $S(n.C_m)$ have Star–in–Coloring properties. We also obtain bounds on the Star–in–Chromatic number of double wheel.

Keywords: *Star–in–coloring, Splitting graph, Friendship graph, Double wheel.*

1. Introduction

Graph coloring is a very important topic of study in graph theory. In this field, coloring is assigning colors to graph vertices using a function. The major goal is to utilize fewer colors while making sure that neighboring vertices have different colors. Examine a straightforward, connected directed graph G with vertex set V and edge set E, where each directed edge is represented by an element in E. Assign directions to edges $e = v_s v_t \epsilon E,$ either from v_s to v_t or v_t to v_s , to determine an orientation for G. B. Grunbaum first proposed the idea of acyclic coloring in 1973 [3].

A graph's Star–Coloring means coloring the vertices in such a way that every four-vertex path uses at least three different colors. Fertin [2] and Nestril et al.[6] have analyzed the Star–Coloring of graphs. The star chromatic number $\chi_s(G)$shows how many colors are needed to do a proper star coloring of G. Sudha and Kanniga [8,9] presented the novel concept of Star–in–Coloring for graphs, derived from the in–coloring of a graph. Sugumaran and Kasirajan [10, 11, 12, 13, 14] advanced the discipline by delineating the constraints for the star-in-chromatic number of specific graphs.

First, we explain how to Star–in–Color a cycle graph, as shown in Fig.1. Think about the vertices $v_1, v_2, v_3,$ and v_4 . The number inside each circle shows what color that vertex is assigned.

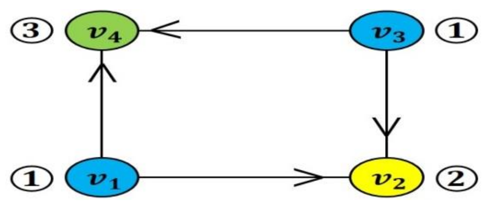

Fig.1. Star–in–Coloring of Cycle C_4

When we look at the graph, we see that adjacent vertices are given different colors, there are no paths with two colors that connect four vertices, and edges in a path of length two with the same color at both ends point toward the central vertex. As a result, the graph has a star-in-coloring with orientation, and its $\chi_{si}(C_4)$ is 3.

This paper examines the Star–in–Coloring of specific splitting graphs, as originally delineated by Sampathkumar and Walikar [7]. We begin our investigation with basic definitions and insights derived from [1,4,5,7,8,15].

2. Main Results

Theorem 2.1 The friendship graph $Fr_n^{(4)}$ allows for star–in–coloring, and its $\chi_{si}\left[Fr_n^{(4)}\right] = 3$.

Proof: The friendship graph, which is written as $Fr_n^{(4)}$, has $3n + 1$ vertices and $4n$ edges. It is made by linking $n$copies of C_4 that are shared withv_0. The inner vertices of each cycle C_4 are labeled $u_1, u_2, \dots, u_{2n}$, while the outer vertices are labeled $v_1, v_2, \dots, v_n$.

Consider the set (V, E) of $Fr_n^{(4)}$. Define a function, denoted as f, which maps vertices in V to the set $\{1, 2, 3, \dots\}$. Ensure that for any edge $v_s v_t \epsilon\, E,\ f(v_s) \neq f(v_t)$.

$$f(v_0) = 1,$$

$$f(u_s) = 2, \qquad s = 1, \dots, 2n.$$

$$f(v_t) = 3, \qquad t = 1, \dots, n.$$

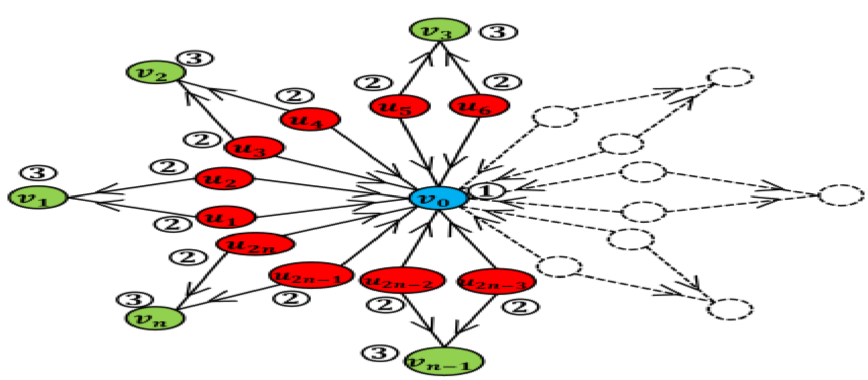

Fig.2. Star-in-coloring of $Fr_n^{(4)}$

Therefore, $\chi_{si}\left[Fr_n^{(4)}\right] = 3$.

Theorem 2.2 Splitting of the friendship graph $S\left(Fr_n^{(4)}\right)$, permits star–in–coloring with its $\chi_{si}\left[S\left(Fr_n^{(4)}\right)\right] = 5$.

Proof: The friendship graph, denoted as $Fr_n^{(4)}$ has $3n + 1$ vertices and $4n$ edges. The splitting graph$S\left(Fr_n^{(4)}\right)$ has $6n + 2$ vertices and $12n$ edges. The common vertex is deonted by v_0, the inner vertices in each cycle C_4 are denoted by $u_1, u_2, \dots, u_{2n}$ and the outer vertices are denoted by $v_1, v_2, \dots, v_n$.Let u'_s, v'_t $(1 \le s \le 2n, 0 \le t \le n)$ be the new vertices (corresponding to the vertices u_s, v_t) added in $Fr_n^{(4)}$.

Consider the set (V, E) of $S\left(Fr_n^{(4)}\right)$. Define a function, denoted as f, which maps vertices in V to the set $\{1, 2, 3, \dots\}$. Ensure that for any edge $v_s v_t \epsilon E$, $f(v_s)$ is not equal to $f(v_t)$.

$$f(v_0) = 1,$$
$$f(u_s) = f(u'_s) = 2, \ if \ s = 1, \dots, 2n.$$
$$f(v_t) = 3, \ if \ t = 1, \dots, n.$$
$$f(v'_0) = 4$$
$$f(v'_t) = 5, \ if \ t = 1, \dots n.$$

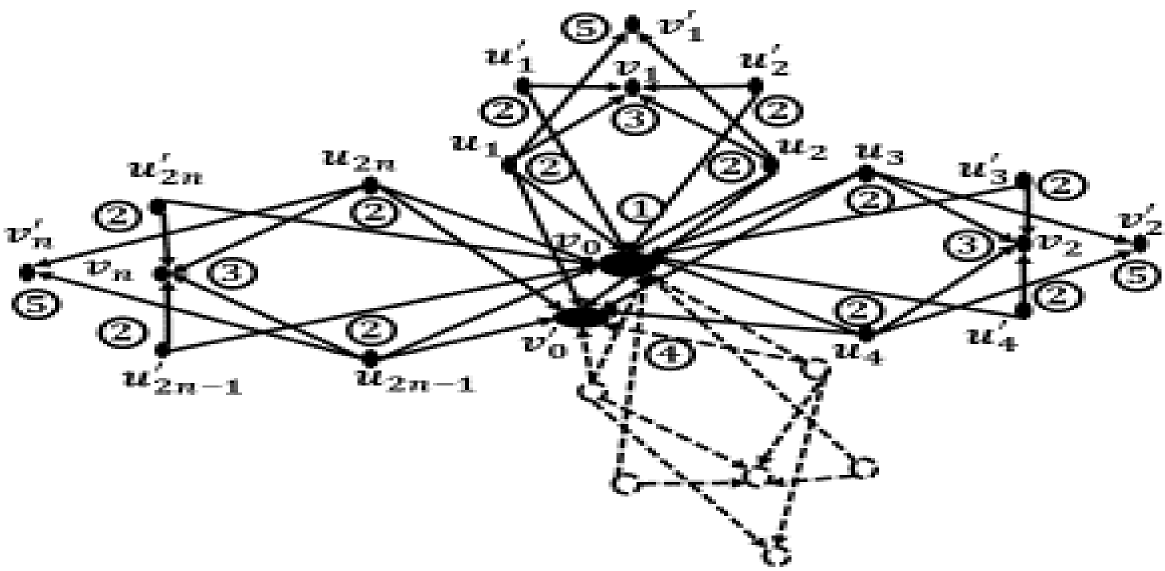

Fig.3. Star-in-coloring of $S\left(Fr_n^{(4)}\right)$

Therefore, $\chi_{si}\left[S\left(Fr_n^{(4)}\right)\right] = 5$.

Theorem 2.3 The friendship graph $\overline{F}r_n^{(3)}$ allows for star–in–coloring, and its $\chi_{si}\left[\overline{F}r_n^{(3)}\right] = 3$, where $n > 1$.

Proof: The friendship graph, denoted as $\overline{F}r_n^{(3)}$consists of vertices $3n + 1$ and edges $5n$, created by connecting n copies of F_3 with vertex v_0. Let $u_1, \dots, u_n$ be the central vertices of each fan graph F_3, the outer vertices are labeled as $v_1, v_2, \dots, v_{2n}$.

Consider the set (V, E) of $\overline{F}r_n^{(3)}$. Define a function, denoted as f, which maps vertices in V to the set $\{1, 2, 3, \dots\}$. Make sure that $f(v_s)$is not equal to $f(v_t)$ for each edge $v_s v_t \epsilon E$,

$$f(v_0) = 1,$$

$$f(v_t) = 2, \qquad t = 1, \dots, 2n.$$

$$f(u_s) = 3, \qquad s = 1, \dots, n.$$

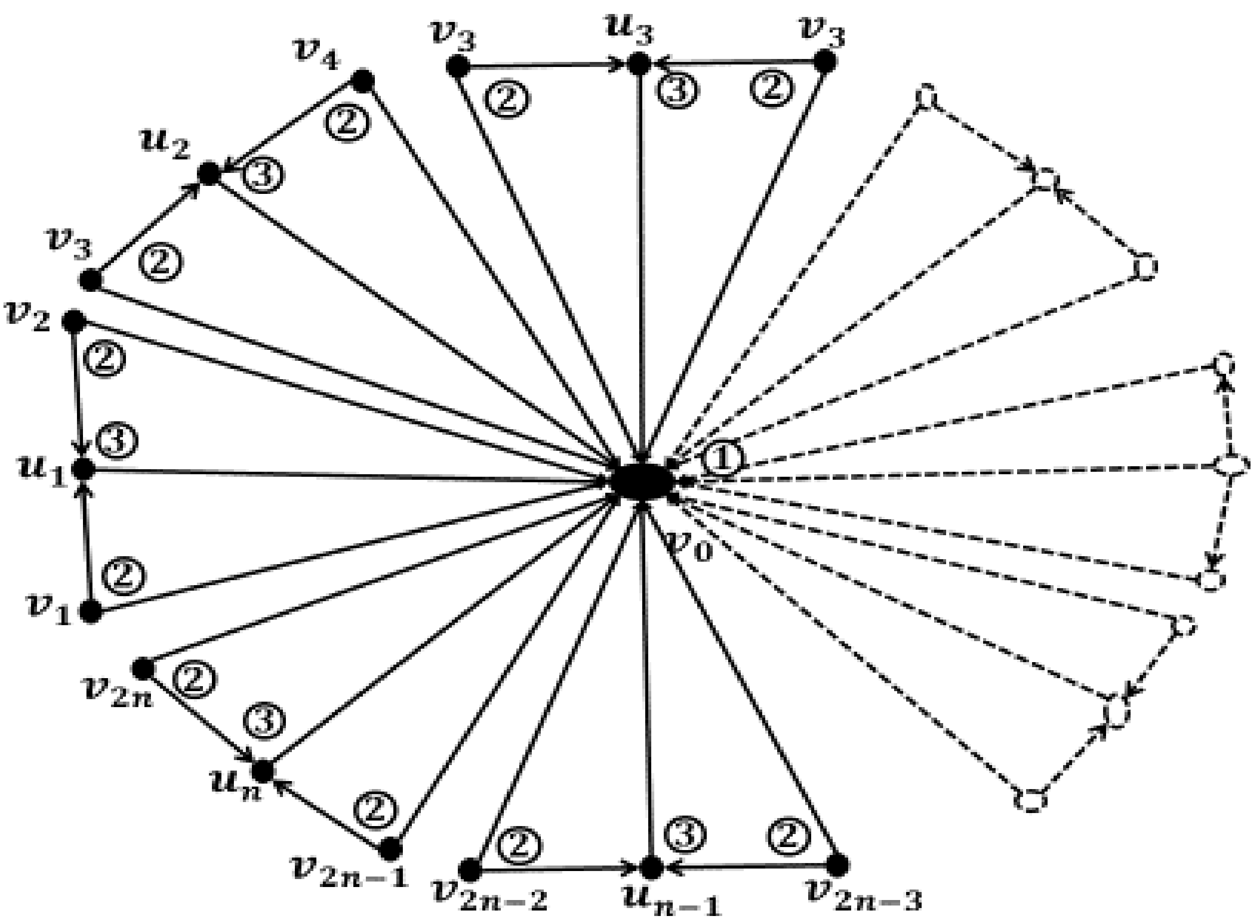

Fig.4. Star-in-coloring of $\overline{F}r_n^{(3)}$

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Therefore, $\chi_{si}\left[\overline{F}r_n^{(3)}\right] = 3$.

Theorem 2.4 The splitting graph of the friendship graph, denoted as $S\left(\overline{F}r_n^{(3)}\right)$ allows for star–in–coloring, and its $\chi_{si}\left[S\left(\overline{F}r_n^{(3)}\right)\right] = 5$, where $n > 1$.

Proof: The friendship graph, denoted as $\overline{F}r_n^{(3)}$ has $3n + 1$, vertices and $5n$ edges. The $S\left(\overline{F}r_n^{(3)}\right)$ had $6n + 2$ vertices and $15n$ edges. We call the common vertex by v_0, the inner vertex in each fan F_n is denoted by $u_1, u_2, \dots, u_n$ and the outer vertices is denoted by $v_1, v_2, \dots, v_{2n}$.The new vertices $u_i', v_j' (1 \leq s \leq n, 0 \leq t \leq 2n)$are corresponding to u_s and v_t added in $\overline{F}r_n^{(3)}$.

Take a look at the set (V, E) of $S\left(\overline{F}r_n^{(3)}\right)$. Define a function, denoted as f, which maps vertices in V to the set $\{1, 2, 3, \dots\}$. Make sure that $f(v_s)$is not equal to $f(v_t)$ for each edge $v_s v_t \epsilon E$,

$$f(v_0) = 1,$$

$$f(v_t) = f(v_t') = 2,\ if\ t = 1,2, \dots ,2n.$$

$$f(u_s) = 3,\ if\ s = 1,2, \dots , n.$$

$$f(v_0') = 4,$$

$$f(u_s') = 5,\ \ if\ s = 1,2, \dots n.$$

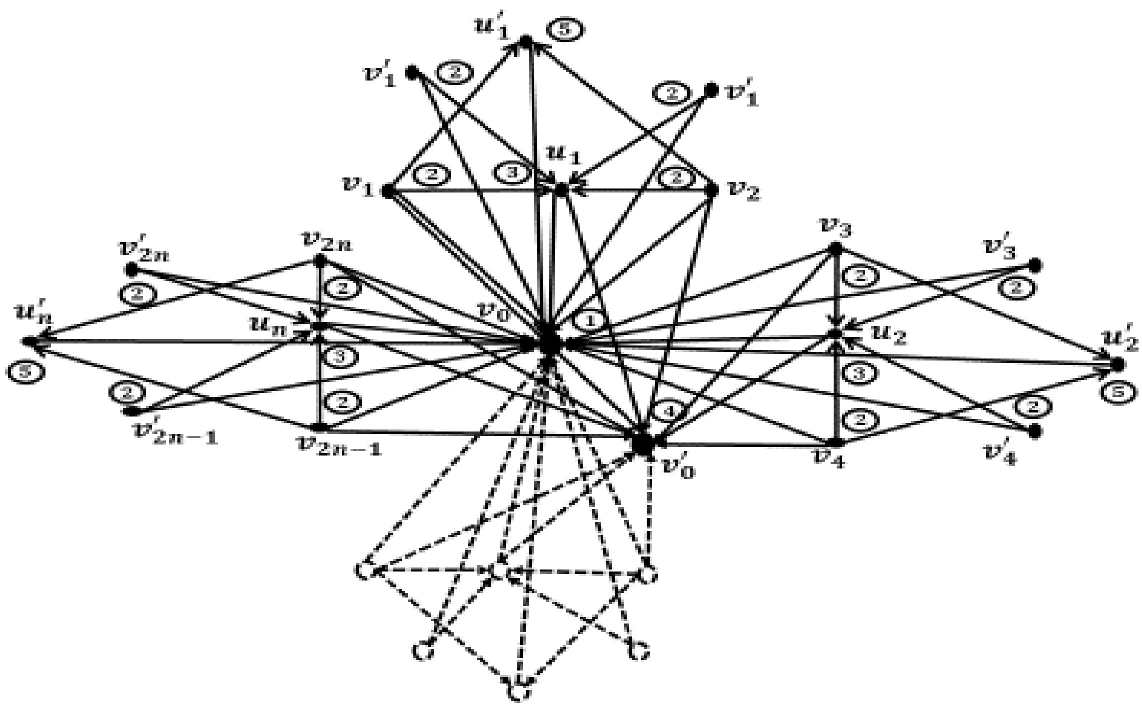

Fig.5. Star-in-coloring of $S\left(\overline{Fr}_n^{(3)}\right)$

Therefore, $\chi_{si}\left[S\left(\overline{F}r_n^{(3)}\right)\right] = 5$.

Theorem 2.5 When m is even, the path union of cycle C_m in a route P_n that allows star–in–coloring has a star–in–chromatic number of 4.

Proof: The graph G consists of mn vertices and $mn + n$ edges, where we connect n copies of cycle graph C_m with a path P_n. Let v_s^t represent t^th copy of s^th vertex in G, $0 \le s \le m$, and $1 \le t \le n$.

Case1: $For\ m \equiv 0\ (mod\ 4)$

$$f(v_s^t) = \begin{cases} 1, & if\ s + t \equiv 0\ (mod\ 2) \\ 2, & if\ (s + t) - 3 \equiv 0\ (mod\ 4)\ and\ s \neq 3 \\ 3, & if\ (s + t) - 1 \equiv 0\ (mod\ 4)\ and\ s \neq 3 \\ 4, & if\ t \equiv 0\ (mod\ 2)\ and\ s = 3 \end{cases}$$

Case 2: $For\ m \equiv 2\ (mod\ 4)$

For each $t = 1, 2, 3, \ldots, n$, we assign $f(v_0^t) = 4$ and

$$f(v_s^t) = \begin{cases} 1, & if\ s \equiv 1\ (mod\ 2) \\ 2, & if\ s \equiv 2\ (mod\ 4) \\ 3, & if\ s \equiv 0\ (mod\ 4)\ and\ s > 0 \end{cases}$$

Illustration 2.5.1

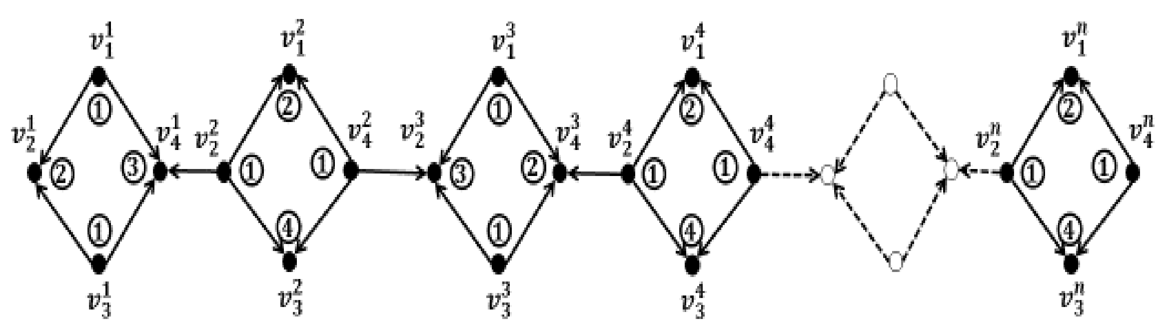

Fig.6. Cycle C_4

Illustration 2.5.2

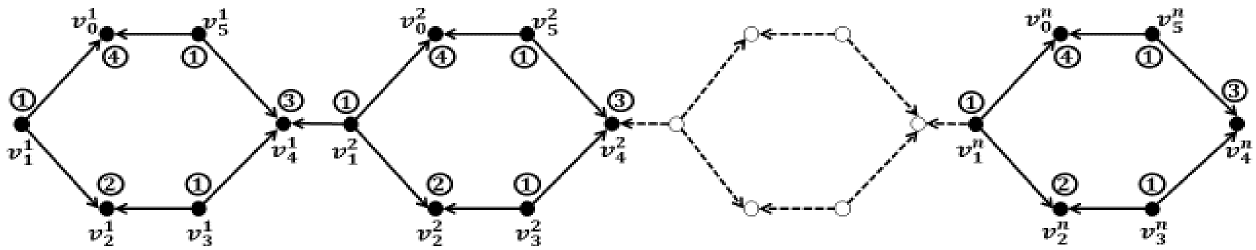

Fig.7. Cycle C_6

Based on the two scenarios above, $\chi_{si}(G) = 4$.

Remark: *The star–in–coloring of C_m is not possible, when m is odd. If we give each vertex a different color, the first and last vertices will have the same color. If you give the last vertex a new color, though, the in-coloring property is not met.*

Theorem 2.6 An open star constructed from by n copies of the cycle graph C_m admits star-in-coloring, and its star–in–chromatic number satisfies $3 \le \chi_{si}[S(n.C_m)] \le 4$, where m is even.

Proof: Consider graph G formed with replace of each vertex of $K_{1,n}$ (excluding the central vertex) with the cycle graph C_m, i.e., $G = S(n.C_m)$. This graph has $mn + 1$ vertices and $(m + 1)n$ edges.

Let v_0 denote the central vertex of the G, and denote v_s^t as the s^th^ vertex in the t^th^ copy of C_m, where $1 \le s \le m,\ 1 \le t \le n$. Then, we establish connections from the k^th^ vertex of all copies of C_m to the central vertex v_0, where k is a constant, with k being any integer from 1 to m.

Case 1: $When\ \ m \equiv 0\ (mod\ 4)$

Subcase 1.1: $For\ t \equiv 1\ (mod\ 2)$

$$f(v_s^t) = \begin{cases} 1, & if\ s \equiv 1\ (mod\ 2) \\ 2, & if\ s \equiv 2\ (mod\ 4) \\ 3, & if\ s \equiv 0\ (mod\ 4) \end{cases}$$

Subcase 1.2: $For\ t \equiv 0\ (mod\ 2)$

$$f(v_s^t) = \begin{cases} 1, & if\ s \equiv 1\ (mod\ 2) \\ 2, & if\ s \equiv 0\ (mod\ 4) \\ 3, & if\ s \equiv 2\ (mod\ 4) \end{cases}$$

and $f(v_0) = 2$

Case 2: $When\ \ m \equiv 2\ (mod\ 4)$

Subcase 2.1: $For\ t \equiv 1\ (mod\ 2)$

$$f(v_s^t) = \begin{cases} 1, & if\ s\ \equiv 1\ (mod\ 2) \\ 2, & if\ s\ \equiv 2\ (mod\ 4) \\ 3, & if\ s\ \equiv 0\ (mod\ 4)\ \ and\ s > 0 \end{cases}$$

$$f(v_0^t) = 4\ and\ f(v_0) = 2$$

Subcase 2.2: $For\ t\ \equiv 0\ (mod\ 2)$

$$f(v_s^t) = \begin{cases} 1, & if\ s\ \equiv 0\ (mod\ 2) \\ 2, & if\ s\ \equiv 3\ (mod\ 4) \\ 3, & if\ s\ \equiv 1\ (mod\ 4)\ \ and\ s < m-1 \end{cases}$$

and $f(v_{m-1}^t) = 4$

Illustration 2.6.1

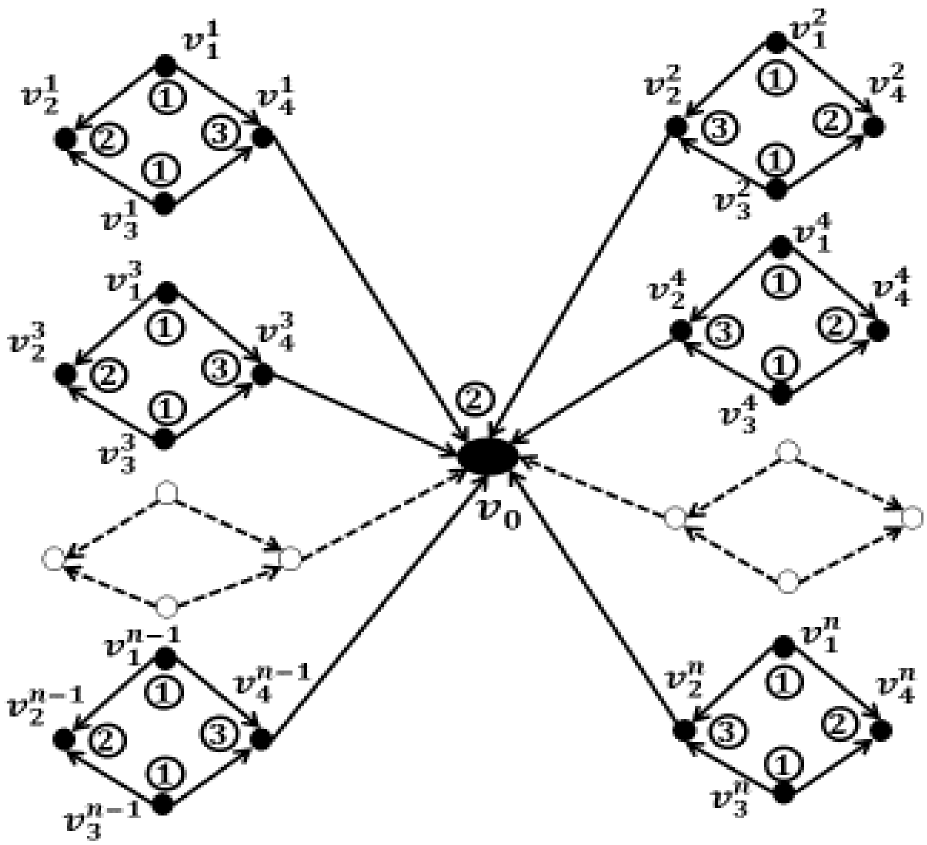

Fig.8. Star-in-coloring of $S(n.C_4)$

In this case, $\chi_{si}(S(n.C_4)) = 3$.

Illustration 2.6.2

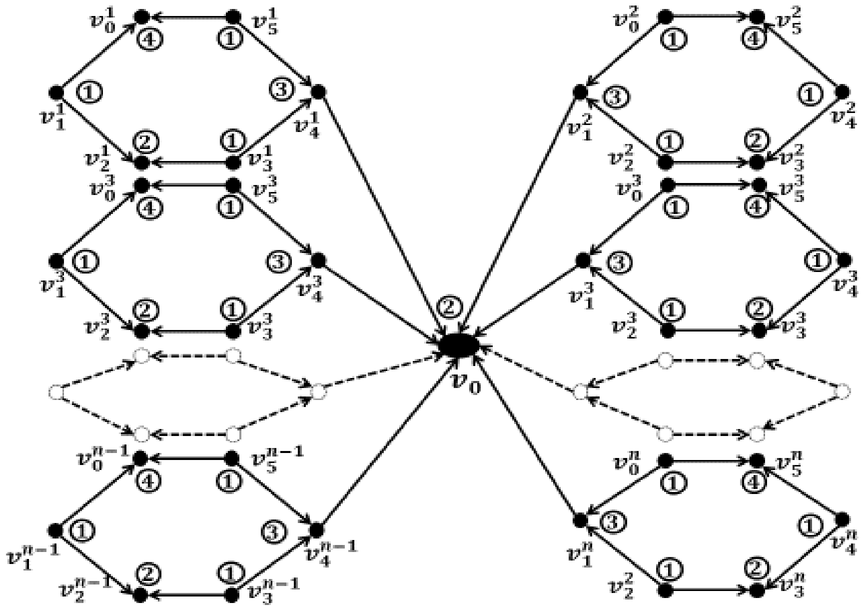

Fig.9. Star-in-coloring of $S(n.C_6)$

In this case, $\chi_{si}(S(n.C_6)) = 4$.

Based on the analysis of the above cases, we deduce the $S(n.C_m)$ falls within the range $3 \leq \chi_{si}[S(n.C_m)] \leq 4$.

Theorem 2.7 The double wheel graph, denoted as $W_{n,n}$ for all even values of n, enables star–in–coloring, with its star–in–chromatic number lying within the inequality $4 \leq \chi_{si}(W_{n,n}) \leq 5$.

Proof: The double wheel $W_{n,n}$ linked to a central hub. This structure entails a total of $2n+1$ vertices and $4n$ edges. The central vertex is denoted by v_0 the inner wheel vertices are denoted by $v_1, v_2, v_3, \dots, v_n$ and the outer wheel vertices are denoted by $v_{n+1}, v_{n+2}, v_{n+3}, \dots, v_{2n}$.

Define a mathematical function, denoted as f, which maps vertices in V to the set $\{1, 2, 3, \dots\}$. Ensure that for any edge $v_s v_t \epsilon E$, $f(v_s) \neq f(v_t)$.

Case 1: $n \equiv 0\ (mod 4)$

$$f(v_s) = \begin{cases} 1, & if\ s \equiv 1\ (mod2) \\ 2, & if\ s \equiv 2\ (mod4) \\ 3, & if\ s \equiv 0\ (mod4)\ and\ s > 0 \end{cases}$$

and $f(v_0) = 4$

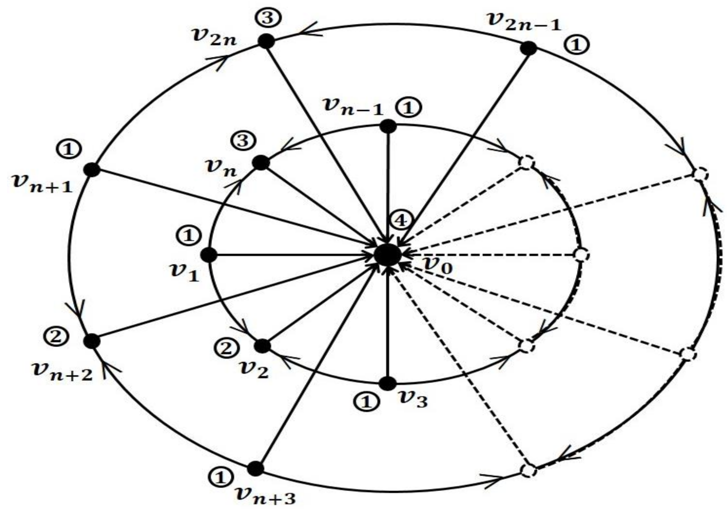

Fig.10. Star-in-coloring of $W_{n,n}, n \equiv 0\ (mod4)$

Case 2: $n \equiv 2\ (mod4)$

$$f(v_s) = \begin{cases} 1, & if\ s-1 \equiv 0\ (mod\ 2) \\ 2, & if\ s-2 \equiv 0\ (mod\ 4)\ and\ s \neq n, s \neq 2n \\ 3, & if\ s \equiv 0\ (mod\ 4)\ and\ s > 0 \\ 4, & if\ s = n\ \ and\ \ s = 2n \end{cases}$$

and $f(v_0) = 5$

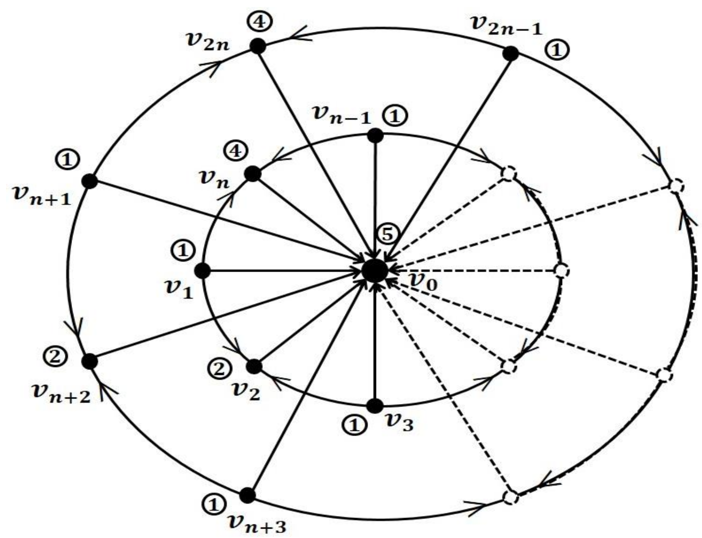

Fig.11. Star-in-coloring of $W_{n,n}, n \equiv 2\ (mod4)$

The two cases mentioned earlier, we deduce $W_{n,n}$ falls within the range $4 \leq \chi_{si}(W_{n,n}) \leq 5$.

Remark: *The Star–in–Coloring of $W_{n,n}$ is not possible, when n is odd. Since if we assign alternate colors for consecutive vertices, we obtain the initial and final vertices are of the same* *color. On the other hand assign, a new color to the final vertex, then the in – coloring property is not satisfied.*

3. Conclusion

In this study, we have conducted an analysis to determine the star–in–chromatic numbers for various graphs. The results are summarized as follows:

1. $\chi_{si}\left[Fr_n^{(4)}\right] = 3$
2. $\chi_{si}\left[S\left(Fr_n^{(4)}\right)\right] = 5$
3. $\chi_{si}\left[\overline{F}r_n^{(3)}\right] = 3$, where $n > 1$
4. $\chi_{si}\left[S\left(\overline{F}r_n^{(3)}\right)\right] = 5$, where $n > 1$.
5. The star–in–chromatic number of path union of cycle graph C_m is 4, where m is even
6. $3 \leq \chi_{si}[S(n.C_m)] \leq 4$, where m is even
7. $4 \leq \chi_{si}(W_{n,n}) \leq 5$, where n is even.

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