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Bulk Queueing Model With Bernoulli Vacation, Different Service And Breakdown Disciplines And State Dependent Arrival Rates

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Abstact: - The basic model of this article is a bulk arrival, batch service non-Markovian queue. The customers arrive in batches of variable size follows compound Poisson process. If the service is not immediate, the customer waits in a waiting line of infinite capacity, while waiting in the line, the first in first out queue discipline is applied for service. The services are given in batches of fixed size. Two types of services are given by the server for each group. The first type is of essential kind of service with two modes, based on Bernoulli schedule. The second type is of optional kind of service. The second type is based on Bernoulli schedule. After completion of each service, the server takes vacation of Bernoulli type. During the busy mode of the server, the server may breakdown, immediately send for repair. There are two types of repairs, called essential and optional. The optional repairs follows Bernoulli schedule. The service periods, vacation period and repair periods, all are random time periods follows different general distributions. The number of breakdowns follows Poisson process. In addition, the arrival rates are state dependent, based on the server state. This model is analysed in time independent domain, using supplementary variable technique by obtaining probability generating functions at various server states. Some performance measures and particular models are derived. A numerical study is also carried out.

Keywords: *Non-Markovian queue - Bulk arrival queue - Fluctuating mode - Fixed batch size - Bernoulli vacation - Essential and Optional service - Unreliable server - State dependent rates.*

2000 Mathematics Subject Classification Number: *90B22,60K25,60K30.*

1. Introduction

Queueing theory refers, the mathematical study of the formation, function, and congestion of waiting lines, or queues. Queueing theory scrutinizes the entire system of waiting in line, including elements like the customer arrival rate, nature of arrival, number of servers, server's behavior, nature of service, number of customers, customer's behavior, capacity of the waiting area, average service completion time, and queuing discipline. Based on the above characters different models have been defined and analyzed by the researchers.

Queueing theory has countless and diverse applications. Some areas of applications are Telecommunications, Transportation, Logistics, Finance, Emergency services, Computing, Industrial engineering, Project management and Operation research. Queueing theory is helpful in explaining the math behind how queues run. But when queues involve humans, queue psychology is important to understand the queue experience as well. Queue psychology research shows it's not the length of the wait that determines how positive or negative the queue experience is, but rather how people feel while waiting.

During busy the server may need break for refreshment or the server may do some different work, if the situation require. In queueing context this time period is called vacation period. Different vacation policies are designed and applied to the queueing model by researchers. One such vacation policy is Bernoulli vacation policy. After each service completion, the server decides whether to take a vacation with probability p or stay and serve the next customer with probability $1-p$. The vacation duration is a random variable. In a contrary situation, during busy the server may breakdown. The server is sent for repair. Here also the number of break downs and repair period are random variables. Many models with breakdown have been defined and analysed by researchers. In the literature, there are models with both vacation and breakdown.

A bulk queue (batch service queue/batch arrival queue/both) is a type of queueing system where customers arrive and are served in groups (batches) rather than individually. In the literature many models are designed with bulk arrival or bulk service or both the arrivals and service are bulk. Queues with different service pattern are very useful applications in practical situations. In similar way, repair with different pattern is also useful in practical situations. With all these factors in mind we define and analyse the following model.

A bulk arrival, batch service non-Markovian queue is the basic model of the analysis. The customers arrive in batches of variable size follows compound Poisson process. If the service is not immediate, the customer waits in a waiting line of infinite capacity, while waiting in the line, the first in first out queue discipline is applied for service. The services are given in batches of fixed size. Two types of services are given by the server for each group. The first type is of essential kind of service with two modes, based on Bernoulli schedule. The second type is of optional kind of service. The second type is based on Bernoulli schedule. After completion of each service, the server takes vacation of Bernoulli type. During the busy mode of the server, the server may breakdown, immediately send for repair. There are two types of repairs, called essential and optional. The optional repairs follow Bernoulli schedule. The service periods, vacation period and repair periods, all are random time periods follows different general distributions. The number of breakdowns follows Poisson process. In addition, the arrival rates are state dependent, based on the server state.

The design of the article is outlined here. Section 2 presents a related literature survey. The complete definition of the model is given in the section 3, Section 4, demonstrate the steady state analysis of the model in terms of probability generating functions. Section 5, presents some related system measures. Section 6, shows some particular models. Section 7, exhibit some numerical illustrations and Section 8 ends with a conclusion.

2. Literature review

Queue with vacation is one of the fertile areas in queueing theory. Many researchers are working in vacation queues. For more details, interested researchers may refer the survey papers by Doshi (1986) and Takagi (1991). Lee (1989) investigates $M/G/1$ bulk arrival queues with server vacations, focusing on two vacation policies: one where the server immediately takes another vacation if no customers are present upon return, and another where the server waits for the first customer group. Yinghui Tang and Xiaowo Tang (2000) investigates the bulk-arrival $M^X/G/1$ queue with a single server vacation, deriving the recursion expression for the Laplace transform of the transient queue-length distribution. Tamrakar and Banerjee (2020) investigate a single server, infinite buffer, bulk service Poisson queue with single and multiple vacations, focusing on the steady-state joint distributions of queue and server content. Tamrakar and Banerjee (2021) focus on an infinite buffer single server queue model that incorporates batch-size dependent service and queue length dependent vacations, utilizing an embedded Markov chain technique for analysis. Pradhan et al. (2024) analyzes a versatile bulk-service queueing system with group arrivals and batch-size-dependent service times, incorporating queue-length-dependent vacation policies. The study derives steady-state system equations and the bivariate probability generating function, providing insights through numerical examples and graphical representations of the model's behavior.

Gautam Choudhury, Mitali Deka (2012) investigates a single $M/G/1$ queue with two phases of service, focusing on the effects of server breakdowns and Bernoulli vacations. Kalyanaraman and Nagarajan (2016) derives steady state solution for a bulk arrival queue with fixed batch service, considering an unreliable server that takes compulsory vacations after each service completion. Jeyakumar and Rameshkumar (2019) investigates a single server queue model that incorporates server breakdowns and controllable customer arrivals during multiple adaptive vacations. Ayyappan and Karpagam (2019) analyzes a non-Markovian bulk service queueing model featuring an unreliable main server, a stand-by server, and various operational policies including N-policy and Bernoulli scheduling for multiple vacations. Key performance measures and the probability generating function (PGF) of queue size are derived, with extensive numerical results provided to illustrate the findings. Kalyanaraman and Nagarajan (2019) studies an $M^X/G^K/1$ queue with Bernoulli vacation. In addition, the server may breakdown, but for the repair the server may have to wait for a random period of time. Kalyanaraman and Nagarajan (2023) considered a Poisson arrival queue with batch arrival, service in a batch of variable size with a minimum of 1 and a maximum of K customers is considered. After completion of service to the customers the server takes on Bernoulli vacation, also during the service of customers the server may breakdown.

Madhu Jain et al. (2021) investigates a non-Markovian queue system featuring bulk arrivals, server breakdowns, and a randomized vacation policy, incorporating phenomena such as balking and optional service. Anjana Begum and Gautam Choudhury (2022) analyzes a bulk arrival single server queue with two types of optional repeated service and delayed repair, utilizing a Bernoulli vacation schedule and N-policy. Shweta Upadhyaya et al., (2024) analyzes the $M^X/G/1$ queue with retrials and an optional service facility, focusing on the effects of active and passive breakdowns of the service provider.

3. Model definitions

The model analyzed in this paper has the following statistical characters:

- Arrival follows compound Poisson process with state dependent arrival rates. The arriving customers wait in a queue, if service is not immediate, of infinite capacity. The customers arrive in batches of variable size X, whose probability is $\Pr\{X = j\} = C_j$, $j = 1, 2, 3, \ldots$
- The system has single server. The Service times are generally distributed and the services are given in batches of fixed size K. Server provides two types of services called, essential Service (ES) and optional Service (OS). The essential service is given in two modes (Mode 1 and Mode 2), called service in fluctuating mode. The probability of providing service in mode 1 is δ_1 and mode 2 is δ_2 ($\delta_1 + \delta_2$=1). After completing ES (Mode 1 or Mode 2), the batch may request OS with probability q $(0 \leq q \leq 1)$. The duration of Mode 1 (Mode 2) service period is S_{11} (S_{12}), a random variable with distribution function $G_{11}(x)$ ($G_{12}(x)$). The duration of OS is S_2, with distribution function $G_2(x)$.The total service time of a customer is $S_0 = \begin{cases} \delta_1(1-q)(S_{11}+S_{12}) + qS_2 \\ \delta_2(1-q)(S_{11}+S_{12}) + qS_2 \end{cases}$
- During busy period, the server may breakdown, the number of breakdowns follows a Poisson process with parameter α. The server is immediately undergoing repair process. There are two types, called essential repair (ER) and optional repair (OR). The essential repair period R_1 is generally distributed with distribution function $H_1(x)$. In addition, after completion of the repair period, the server undergoes another repair process called optional repair, with probability r or the server enter into the system with $1 - r$ $(0 \leq r \leq 1)$. The optional repair period R_2 follows general distribution with distribution function $H_2(x)$. The total repair period of the customer is $R = (1-r)R_1 + rR_2$.
- After completion of each service, the server takes vacation of random length V with Bernoulli probability p $(0 \leq p \leq 1)$. The vacation period follows general distribution whose distribution function is $B(x)$.
- The arrival rate is

$$\lambda = \begin{cases} \lambda_0 & , \ \textit{the arrival rate is during idle period} \\ \lambda_1 & , \ \textit{the arrival rate is during ES period in fluctuating mode } 1 \\ \lambda_2 & , \ \textit{the arrival rate is during ES period in fluctuating mode } 2 \\ \lambda_3 & , \ \textit{the arrival rate is during OS period} \\ \lambda_4 & , \ \textit{the arrival rate is during vacation period} \\ \lambda_5 & , \ \textit{the arrival rate is during ER period} \\ \lambda_6 & , \ \textit{the arrival rate is during OR period} \end{cases}$$

- The mean batch size, the mean total ES period in mode 1 and 2, the mean vacation period and mean ER and OR period are $E(X), E(S_{11}), E(S_{12}), E(S_2), E(R_1)$ and $E(R_2)$.

The following notations are used for the analysis:

The hazard function for the distribution function of the essential service in fluctuating mode 1 time distribution $G_{11}(x)$ is $\mu_{11}(x) = \frac{g_{11}(x)}{1-G_{11}(x)}$, where $\mu_{11}(x) = \Pr$ {essential service in fluctuating mode 1 will be completed in $(x, x+dx)$ given that the service time exceeds x } and the hazard function for the distribution function of the essential service in fluctuating mode 2 time distribution $G_{12}(x)$ is $\mu_{12}(x) = \frac{g_{12}(x)}{1-G_{12}(x)}$, where $\mu_{12}(x) =$ Pr {essential service in fluctuating mode 2 will be completed in $(x, x+dx)$ given that the service time exceeds x}.

The hazard function for the distribution function of the optional service time distribution $G_2(x)$ is $\mu_2(x) = \frac{g_2(x)}{1-G_2(x)}$ where $\mu_2(x)$dx = Pr {optional service will be completed in $(x, x+dx)$ given that the service time exceeds x}. The hazard function for the distribution function of the vacation time distribution $B(x)$ is $\beta(x) = \frac{b(x)}{1-B(x)}$ where $\beta(x)$dx = Pr {Vacation will be completed in $(x, x+dx)$ given that the vacation time exceeds x }.

The hazard function for the distribution function of the ER time distribution $H_1(x)$ is $\gamma_1(x) = \frac{h_1(x)}{1-H_1(x)}$ where $\gamma_1(x)$dx = Pr{ER will be completed in $(x, x+dx)$ given that the repair time exceeds x} and the hazard function for the distribution function of the OR time distribution $H_2(x)$ is $\gamma_2(x) = \frac{h_2(x)}{1-H_2(x)}$ where $\gamma_2(x)$dx = Pr{ER will be completed in $(x, x+dx)$ given that the repair time exceeds x}.

At time t, let $K(t)$ be the number of customers in the waiting line and $\eta(t)$ be the supplementary variable at time t. The $\eta(t)$ have the following random identifications.

$$\eta(t) = \begin{cases} \eta_0(t) & , \ the\ elapsed\ ES\ period\ in\ fluctuating\ mode\ 1 \\ \eta_1(t) & , \ the\ elapsed\ ES\ period\ in\ fluctuating\ mode\ 2 \\ \eta_2(t) & , \ the\ elapsed\ OS\ period \\ \eta_3(t) & , \ the\ elapsed\ vacation\ period \\ \eta_4(t) & , \ the\ elapsed\ ER\ period \\ \eta_5(t) & , \ the\ elapsed\ OR\ period \end{cases}$$

The two-dimensional process $\{(K(t), \eta(t)) : t \geq 0\}$ is a Markov process.

The following probabilities and probability generating functions are introduced for the analysis:

$Q_n(t) = \Pr\{K(t) = n, \text{the server is idle}\}, n = 0,1, \dots, K-1.$

$P_{n1}^{(1)}(x;t) = \Pr\{K(t) = n, \eta_0(t) \in (x, x+\Delta t)\}, n = 0,1, \dots.$

$P_{n1}^{(2)}(x;t) = \Pr\{K(t) = n, \eta_1(t) \in (x, x+\Delta t)\}, n = 0,1, \dots.$

$P_{n2}(x;t) = \Pr\{K(t) = n, \eta_2(t) \in (x, x+\Delta t)\}, n = 0,1, \dots.$

$V_n(x;t) = \Pr\{K(t) = n, \eta_3(t) \in (x, x+\Delta t)\}, n = 0,1, \dots.$

$R_{n1}(x;t) = \Pr\{K(t) = n, \eta_4(t) \in (x, x+\Delta t)\}, n = 0,1, \dots.$

$R_{n2}(x;t) = \Pr\{K(t) = n, \eta_5(t) \in (x, x+\Delta t)\}, n = 0,1, \dots.$

In steady state,

$P_{n1}^{(i)}(x) = \lim_{n\to\infty} P_{n1}^{(i)}(x;t)\,; i = 1,2\ ; P_{n2}(x) = \lim_{n\to\infty} P_{n2}(x;t)\ ; V_n(x) = \lim_{n\to\infty} V_n(x;t),\ R_n(x) = \lim_{n\to\infty} R_n(x;t), C(z) = \sum_{j=1}^{\infty} C_j z^j\ , P_1^{(i)}(x,z) = \sum_{n=0}^{\infty} P_1^{(i)}(x) z^n\,; i = 1,2\,; P_2(x,z) = \sum_{n=0}^{\infty} P_2(x) z^n$

$V(x,z) = \sum_{n=0}^{\infty} V_n(x) z^n, R(x,z) = \sum_{n=0}^{\infty} R_n(x) z^n, Q(z) = \sum_{n=0}^{K-1} Q_n z^n$; where $|z| \le 1$

4. Steady state analysis

The system discussed is a non-Markovian queueing system, the following differential-difference equations are obtained using the supplementary variable technique as outlined in Cox (1955).

$$\frac{dP_{01}^{(1)}(x)}{dx} = -(\lambda_1 + \mu_{11}(x) + \alpha)P_{01}^{(1)}(x) \tag{4.1a}$$

$$\frac{dP_{n1}^{(1)}(x)}{dx} = -(\lambda_1 + \mu_{11}(x) + \alpha)P_{n1}^{(1)}(x) + \lambda_1 \sum_{j=1}^{n} C_j P_{n-j1}^{(1)}(x)\,, n \ge 1 \tag{4.1b}$$

$$\frac{dP_{01}^{(2)}(x)}{dx} = -(\lambda_2 + \mu_{12}(x) + \alpha)P_{02}(x) \tag{4.2a}$$

$$\frac{dP_{01}^{(2)}(x)}{dx} = -(\lambda_2 + \mu_{12}(x) + \alpha)P_{01}^{(2)}(x) + \lambda_2 \sum_{j=1}^{n} C_j P_{n-j1}^{(2)}(x)\,, n \ge 1 \tag{4.2b}$$

$$\frac{dP_{02}(x)}{dx} = -(\lambda_3 + \mu_2(x) + \alpha)P_{02}(x) \tag{4.3a}$$

$$\frac{dP_{n2}(x)}{dx} = -(\lambda_3 + \mu_2(x) + \alpha)P_{n2}(x) + \lambda_2 \sum_{j=1}^{n} C_j P_{n-j2}(x)\,, n \ge 1 \tag{4.3b}$$

$$\frac{dV_0(x)}{dx} = -\big(\lambda_4 + \beta(x)\big)V_0(x) \tag{4.4a}$$

$$\frac{dV_n(x)}{dx} = -\big(\lambda_4 + \beta(x)\big)V_n(x) + \lambda_4 \sum_{j=1}^{n} C_j V_{n-j}(x)\,, n \ge 1 \tag{4.4b}$$

$$\frac{dR_{01}(x)}{dx} = -\big(\lambda_5 + \gamma_1(x)\big)R_{01}(x) \tag{4.5a}$$

$$\frac{dR_{n1}(x)}{dx} = -\big(\lambda_5 + \gamma_1(x)\big)R_{n1}(x) + \lambda_5 \sum_{j=1}^{n} C_j R_{n-j1}(x)\,, n \ge 1 \tag{4.5b}$$

$$\frac{dR_{02}(x)}{dx} = -\big(\lambda_6 + \gamma_2(x)\big)R_{02}(x) \tag{4.6a}$$

$$\frac{dR_{n2}(x)}{dx} = -\big(\lambda_6 + \gamma_2(x)\big)R_{n2}(x) + \lambda_6 \sum_{j=1}^{n} C_j R_{n-j2}(x)\,, n \ge 1 \tag{4.6b}$$

$$0 = -\lambda_0 Q_n + \lambda_0\big(1 - \delta_{n,0}\big) \sum_{j=1}^{n} C_j Q_{n-j} \ + \int_0^{\infty} \beta(x)\ V_n(x)dx + (1 - r)$$

$$\left\{\int_0^\infty \gamma_1(x)R_{n1}(x)dx + \int_0^\infty \gamma_2(x)R_{n2}(x)dx\right\} + (1-p)$$
$$\left\{(1-q)\left[\int_0^\infty \mu_{11}(x)P_{n1}^{(1)}(x)dx + \int_0^\infty \mu_{12}(x)P_{n1}^{(2)}(x)dx\right] + \int_0^\infty \mu_2(x)P_{n2}(x)dx\right\}, n = 0,..,K-1 \quad (4.7)$$

The conditions at the initial time 0, called the boundary conditions are

$$P_{n1}^{(1)}(0) = \delta_1 \left\{(1-q)\left[\int_0^\infty \mu_{11}(x)P_{n+K1}^{(1)}(x)dx + \int_0^\infty \mu_{12}(x)P_{n+K1}^{(2)}(x)dx\right] + \int_0^\infty \mu_2(x)P_{n+K2}(x)dx\right\}$$
$$(1-p) + \delta_1 \int_0^\infty \beta(x)\ V_{n+k}(x)dx + \delta_1(1-r)\left\{\int_0^\infty \gamma_1(x)R_{n+K1}(x)dx + \int_0^\infty \gamma_2(x)R_{n+K2}(x)dx\right\}$$
$$+\delta_1\,\lambda_0 \sum_{j=0}^{K-1} C_{n+K-j}Q_J\ ; n = 0,1,2,\dots,K-1 \quad (4.8a)$$

$$P_{n1}^{(2)}(0) = \delta_2 \int_0^\infty \mu_{11}(x)P_{n+K1}^{(1)}(x)dx, n \geq 0 \quad (4.8b)$$

$$P_{n2}(0) = \delta_1 \int_0^\infty \mu_{11}(x)P_{n+K1}^{(1)}(x)dx + \delta_2 \int_0^\infty \mu_{12}(x)P_{n+K1}^{(2)}(x)dx, n \geq 0 \quad (4.8c)$$

$$V_n(0) = p\left\{(1-q)\left[\int_0^\infty \mu_{11}(x)P_{n1}^{(1)}(x)dx + \int_0^\infty \mu_{12}(x)P_{n1}^{(2)}(x)dx\right] + \int_0^\infty \mu_2(x)P_{n2}(x)dx\right\}, n \geq 0 \quad (4.9)$$

$$R_{n1}(0) = (1-r)\alpha\left\{(1-q)\left[\int_0^\infty P_{n-K1}^{(1)}(x)dx + \int_0^\infty P_{n-K1}^{(2)}(x)dx\right] + \int_0^\infty P_{n-K2}(x)dx\right\}, n \geq K \quad (4.10a)$$

$$R_{n2}(0) = r\left\{(1-q)\left[\int_0^\infty P_{n-K1}^{(1)}(x)dx + \int_0^\infty P_{n-K1}^{(2)}(x)dx\right] + \int_0^\infty P_{n-K2}(x)dx\right\}, n \geq K \quad (4.10b)$$

$$R_{n1}(0) = R_{n2}(0) = 0, n \geq 0 \quad (4.10c)$$

and the normalization condition is

$$\sum_{n=0}^{K-1} Q_n + \int_0^\infty \sum_{n=0}^{\infty}\left[P_{n1}^{(1)}(x) + P_{n1}^{(2)}(x) + V_n(x) + R_{n1}(x) + R_{n2}(x)\right]dx = 1 \quad (4.11)$$

The above derived results are presented in the following theorem:

Theorem 4.1: Under steady state condition, the model has the following probability generating functions.

$$P_1^{(1)}(z) = \frac{\delta_1\lambda_0[1-C(z)]Q(z)a_2a_3z^{2K}[1-G_{11}^*(a_1)]}{J}$$

$$P_1^{(2)}(z)=\frac{\delta_1\delta_2\lambda_0[1-C(z)]Q(z)a_1a_3z^K G_{11}^*(a_1)[1-G_{12}^*(a_2)]}{J}$$
$$P_2(z)=\frac{\delta_1\lambda_0[1-C(z)]Q(z)a_1a_2z^K G_{11}^*(a_1)[1-G_2^*(a_3)]\{\delta_1z^K+\delta_2^2G_{12}^*(a_2)\}}{J}$$
$$V(z)=\frac{p\delta_1\lambda_0[1-C(z)]Q(z)a_1a_2a_3[1-B^*(m_1)]d_2}{Jm_1}$$
$$R_1(z)=\frac{(1-r)\delta_1\lambda_0[1-C(z)]Q(z)\alpha z^K e_2[1-H_1^*(m_2)]}{m_2J}$$
$$R_2(z)=\frac{r\delta_1\lambda_0[1-C(z)]Q(z)z^K e_2[1-H_2^*(m_3)]}{m_3J}$$
where, $d_2=p\{G_{11}^*(a_1)\{(1-q)[z^{2K}+\delta_2G_{12}^*(a_2)]z^K+G_2^*(a_3)[\delta_1z^K+\delta_2^2G_{12}^*(a_2)]\}$

$e_2=\{z^K(1-q)a_3\delta_1\{z^K[1-G_{11}^*(a_1)]a_2+\delta_2G_{11}^*(a_1)a_1[1-G_{12}^*(a_2)]\}$
$+\delta_1a_1a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1z^K+\delta_2^2G_{12}^*(a_2)]\}$

$J=(1-r)\alpha z^K H_1^*(m_2)\{z^K(1-q)a_3\delta_1\{z^K[1-G_{11}^*(a_1)]a_2+\delta_2G_{11}^*(a_1)a_1$
$[1-G_{12}^*(a_2)]\}+\delta_1a_1a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1z^K+\delta_2^2G_{12}^*(a_2)]\}+rz^K$
$H_2^*(m_3)\{z^K(1-q)a_3\delta_1\{z^K[1-G_{11}^*(a_1)]a_2+\delta_2G_{11}^*(a_1)a_1[1-G_{12}^*(a_2)]\}\delta_1a_1a_2$
$[1-G_2^*(a_3)]G_{11}^*(a_1)+[\delta_1z^K+\delta_2^2G_{12}^*(a_2)]\}-a_1a_2a_3[z^{3K}-[1-p+pB^*(m_1)]$
$G_{11}^*(a_1)\{\delta_1z^K(1-q)[z^K+\delta_2G_{12}^*(a_2)]+\delta_1G_2^*(a_3)[\delta_1z^K+\delta_2^2G_{12}^*(a_2)]\}\}]$

respectively, the probability generating function of number of customers in queue when the server provides essential service in fluctuating mode 1 and mode 2, when the server provides optional service, when the server is on vacation, when the server is in essential repair and when the server is in optional repair.

Proof:

Multiplication of equations $(4.1a)$ and $(4.1b)$ by appropriate powers of z and adding the resultant equations for $n=0,1,\dots,\infty$, leads to

$$\frac{\partial}{\partial x}\left(P_1^{(1)}(x,z)\right)+(\lambda_1-\lambda_1C(z)+\mu_{11}(x)+\alpha)P_1^{(1)}(x,z)=0 \quad (4.12)$$

Multiplication of equations $(4.2a)$ and $(4.2b)$ by appropriate powers of z and adding the resultant equations for $n=0,1,\dots,\infty$, leads to

$$\frac{\partial}{\partial x}\left(P_1^{(2)}(x,z)\right)+(\lambda_2-\lambda_2C(z)+\mu_{12}(x)+\alpha)P_1^{(2)}(x,z)=0 \quad (4.13)$$

Multiplication of equations $(4.3a)$ and $(4.3b)$ by appropriate powers of z and adding the resultant equations for $n=0,1,\dots,\infty$, leads to

$$\frac{\partial}{\partial x}\left(P_2(x,z)\right)+(\lambda_3-\lambda_3C(z)+\mu_2(x)+\alpha)P_2(x,z)=0 \quad (4.14)$$

Multiplication of equations $(4.4a)$ and $(4.4b)$ by appropriate powers of z and adding the resultant equations for $n=0,1,\dots,\infty$, leads to

$$\frac{\partial}{\partial x}\left(V(x,z)\right)+\left(\lambda_4-\lambda_4C(z)+\beta(x)\right)V(x,z)=0 \quad (4.15)$$

Multiplication of equations $(4.5a)$ and $(4.5b)$ by appropriate powers of z and adding the resultant equations for $n=0,1,\dots,\infty$, leads to

$$\frac{\partial}{\partial x}\left(R_1(x,z)\right)+\left(\lambda_5-\lambda_5C(z)+\gamma_1(x)\right)R_1(x,z)=0 \quad (4.16)$$

Multiplication of equations $(4.6a)$ and $(4.6b)$ by appropriate powers of z and adding the resultant equations for $n=0,1,\dots,\infty$, leads to

$$\frac{\partial}{\partial x}\left(R_2(x,z)\right)+\left(\lambda_6-\lambda_6 C(z)+\gamma_2(x)\right)R_2(x,z)=0 \quad (4.17)$$

Multiplication of equation (4.7) by z^n and summation over $n=0,1,\dots,K-1$,leads to

$$0=-\lambda_0\sum_{n=0}^{K-1}Q_n z^n+(1-r)\{\int_0^\infty\gamma_1(x)\sum_{n=0}^{K-1}R_{n1}(x)z^n dx+\int_0^\infty\gamma_2(x)\sum_{n=0}^{K-1}R_{n2}(x)z^n dx\}+L(z)+\int_0^\infty\beta(x)\sum_{n=0}^{K-1}V_n(x)z^n dx+(1-p)\{(1-q)\{\int_0^\infty\mu_{11}(x)\sum_{n=0}^{K-1}P_{n1}^{(1)}(x)z^n dx+\int_0^\infty\mu_{12}(x)\sum_{n=0}^{K-1}P_{n1}^{(2)}(x)z^n dx\}+\int_0^\infty\mu_2(x)\sum_{n=0}^{K-1}P_{n2}(x)z^n dx \quad (4.18)$$

where, $L(z)=\lambda_0\sum_{n=0}^{K-1}\left(1-\delta_{n,0}\right)\sum_{j=1}^{n}C_j Q_{n-j}z^n$

Multiplication of equation (4.8a) by z^{n+K} and summation over $n=0,1,\dots,\infty$, leads to

$$z^K P_1^{(1)}(0,z)=\delta_1\{(1-p)\{(1-q)\{\int_0^\infty\mu_{11}(x)\sum_{n=K}^{\infty}P_{n1}^{(1)}(x)z^n dx+\int_0^\infty\mu_{12}(x)\sum_{n=K}^{\infty}P_{n1}^{(2)}(x)\,z^n\,dx\}+\int_0^\infty\mu_2(x)\sum_{n=K}^{\infty}P_{n2}(x)z^n dx\}+\int_0^\infty\beta(x)\sum_{n=K}^{\infty}V_n(x)z^n dx+K(z)+(1-r)\{\int_0^\infty\gamma_1(x)\sum_{n=K}^{\infty}R_{n1}(x)z^n+\int_0^\infty\gamma_2(x)\sum_{n=K}^{\infty}R_{n2}(x)z^n dx\}\} \quad (4.19)$$

where,$K(z)=\lambda_0\sum_{j=0}^{K-1}\sum_{n=0}^{\infty}C_{n+K-j}\,Q_j z^{n+K}$

Multiplication of equation (4.8b) by z^{n+K} and summation over $n=0,1,\dots,\infty$, leads to

$$P_1^{(2)}(0,z)=\frac{\delta_2\int_0^\infty\mu_{11}(x)P_1^{(1)}(x,z)dx}{z^K} \quad (4.20)$$

Multiplication of equation (4.8c) by z^{n+K} and summation over $n=0,1,\dots,\infty$, leads to

$$P_2(0,z)=\frac{\delta_1\int_0^\infty\mu_{11}(x)P_1^{(1)}(x,z)dx+\delta_2\int_0^\infty\mu_{12}(x)P_1^{(2)}(x,z)dx}{z^K}$$

(4.21)

Now, addition of equations (4.18) and (4.19),we have

$$z^K P_1^{(1)}(0,z)=\delta_1\{(1-p)\{(1-q)\{\int_0^\infty P_1^{(1)}(x,z)\mu_{11}(x)dx+\int_0^\infty P_1^{(2)}(x,z)\;\mu_{12}(x)dx\}+\int_0^\infty P_2(x,z)\mu_2(x)dx\}+(1-r)\{\int_0^\infty R_1(x,z)\gamma_1(x)dx+\int_0^\infty R_2(x,z)\gamma_2(x)dx\}+\int_0^\infty V(x,z)\beta(x)dx+\lambda_0[C(z)-1]Q(z)\}$$

(4.22)

Multiplication of equation (4.9) by z^n and summation over $n=0,1,\dots,\infty$, leads to

$$V(x,z)=p\{(1-q)\{\int_0^\infty P_1^{(1)}(x,z)\mu_{11}(x)dx+\int_0^\infty P_1^{(2)}(x,z)\mu_{12}(x)dx\}+\int_0^\infty P_2(x,z)\mu_2(x)dx\}$$

(4.23)

Multiplication of equations (4.10a) and (4.10c) by appropriate powers of z and adding the resultant equations for $n=0,1,\dots,\infty$, leads to

$$R_1(x,z)=(1-r)\alpha z^K\left[(1-q)\left\{P_1^{(1)}(z)+P_1^{(2)}(z)\right\}+P_2(z)\right]$$

(4.24)

Multiplication of equations $(4.10b)$ and $(4.10c)$ by appropriate powers of z and adding the resultant equations for $n = 0,1,\dots,\infty$, leads to

$$R_2(x,z) = rz^K\left[(1-q)\left\{P_1^{(1)}(z) + P_1^{(2)}(z)\right\} + P_2(z)\right]$$
(4.25)

Integration of equation (4.12) leads to

$$P_1^{(1)}(x,z) = P_1^{(1)}(0,z)e^{-a_1x-\int_0^\infty \mu_{11}(x)dx}$$
(4.26)

where,$a_1 = \lambda_1 - \lambda_1 C(z) + \alpha$

Integration of equation (4.26) leads to

$$P_1^{(1)}(z) = \int_0^\infty P_1^{(1)}(x,z)dx = \frac{P_1^{(1)}(0,z)[1-G_{11}^*(a_1)]}{a_1}$$
(4.27)

Multiplication of equation (4.26) by $\mu_{11}(x)$ and integration of equation leads to

$$\int_0^\infty P_1^{(1)}(x,z)\mu_{11}(x)dx = P_1^{(1)}(0,z)G_{11}^*(a_1)$$
(4.28)

Integration of equation (4.13) leads to

$$P_1^{(2)}(x,z) = P_1^{(2)}(0,z)e^{-a_2x-\int_0^\infty \mu_{12}(x)dx}$$
(4.29)

where,$a_2 = \lambda_2 - \lambda_2 C(z) + \alpha$

Integration of equation (4.29) leads to

$$P_1^{(2)}(z) = \int_0^\infty P_1^{(2)}(x,z)dx = \frac{P_1^{(2)}(0,z)[1-G_{12}^*(a_2)]}{a_2}$$
(4.30)

Multiplication of equation (4.29) by $\mu_{12}(x)$ and integration of equation leads to

$$\int_0^\infty P_1^{(2)}(x,z)\mu_{12}(x)dx = P_1^{(2)}(0,z)G_{12}^*(a_2)$$
(4.31)

Integration of equation (4.14) leads to

$$P_2(x,z) = P_2(0,z)e^{-a_3x-\int_0^\infty \mu_2(x)dx}$$
(4.32)

where,$a_3 = \lambda_3 - \lambda_3 C(z) + \alpha$

Integration of equation (4.32) leads to

$$P_2(z) = \int_0^\infty P_2(x,z)dx = \frac{P_2(0,z)[1-G_2^*(a_3)]}{a_3}$$
(4.33)

Multiplication of equation (4.30) by $\mu_2(x)$ and integration of equation leads to

$\int_0^\infty P_2(x,z)\mu_2(x)dx = P_2(0,z)G_2^*(a_3)$

(4.34)

Integration of equation (4.15) leads to

$V(x,z) = V(0,z)e^{-m_1x-\int_0^\infty \beta(x)dx}$

(4.35)

where, $m_1 = \lambda_4 - \lambda_4 C(z)$

Substituting the value of (4.28),(4.31),(4.34) in(4.23), we've

$V(0,z) = p\left\{(1-q)\left\{P_1^{(1)}(0,z)G_{11}^*(a_1) + P_1^{(2)}(0,z)G_{12}^*(a_2)\right\} + P_2(0,z)G_2^*(a_3)\right\}$

(4.36)

Substituting the value of (4.36) in (4.35), we've

$V(x,z) = p\left\{(1-q)\left\{P_1^{(1)}(0,z)G_{11}^*(a_1) + P_1^{(2)}(0,z)G_{12}^*(a_2)\right\} + P_2(0,z)G_2^*(a_3)\right\}Y$

(4.37)

where, $Y = e^{-m_1x-\int_0^\infty \beta(x)dx}$

Integration of equation (4.37) leads to

$V(z) = \int_0^\infty V(x,z)dx = \frac{d_1[1-B^*(m_1)]}{m_1}$

(4.38)

where, $d_1 = p\{(1-q)\{P_1(0,z)^{(1)}G_{11}^*(a_1) + P_1(0,z)^{(2)}G_{12}^*(a_2)\} + P_2(0,z)G_2^*(a_3)\}$

Multiplication of equation (4.37) by $\beta(x)$ and integration of equation leads to

$\int_0^\infty V(x,z)\beta(x)dx = d_1B^*(m_1)$

(4.39)

Integration of equation (4.16) leads to

$R_1(x,z) = R_1(0,z)e^{-m_2x-\int_0^\infty \gamma_1(x)dx}$

(4.40)

where, $m_2 = \lambda_5 - \lambda_5 C(z)$

Substituting the value of (4.24), (4.27), (4.30), (4.33) in (4.40), we have

$R_1(x,z) = \left[\frac{(1-r)\alpha z^K e^{-m_2x-\int_0^\infty \gamma_1(x)dx} e_1}{a_1a_2a_3}\right]$

(4.41)

where, $e_1 = (1-q)\{P_1^{(1)}(0,z)[1-G_{11}^*(a_1)]a_1a_2 + P_1^{(2)}(0,z)[1-G_{12}^*(a_2)]\,a_3a_1\} + P_2(0,z)$

$[1-G_2^*(a_3)]a_1a_2$

Integration of equation (4.41) leads to

$$\int_0^{\infty} R_1(x,z)dx = R_1(z) = \left[\frac{(1-r)\alpha z^K e_1[1-H_1^*(m_2)]}{a_1a_2a_3m_2}\right]$$

(4.42)

Multiplication of equation (4.41) by $\gamma_1(x)$ and integration of equation leads to

$$\int_0^{\infty} R_1(x,z)\gamma_1(x)dx = \left[\frac{(1-r)\alpha z^K e_1 H_1^*(m_2)}{a_1a_2a_3}\right]$$

(4.43)

Integration of equation (4.17) leads to

$$R_2(x,z) = R_2(0,z)e^{-m_3x-\int_0^{\infty}\gamma_2(x)dx}$$

(4.44)

where, $m_3 = \lambda_6 - \lambda_6 C(z)$

Substituting the value of (4.25), (4.27), (4.30), (4.33) in (4.44), we have

$$R_2(x,z) = \left[\frac{rz^K e^{-m_3x-\int_0^{\infty}\gamma_2(x)dx} e_1}{a_1a_2a_3}\right]$$

(4.45)

Integration of equation (4.45) leads to

$$\int_0^{\infty} R_2(x,z)dx = R_2(z) = \left[\frac{rz^K e_1[1-H_2^*(m_3)]}{a_1a_2a_3m_3}\right]$$

(4.46)

Multiplication of equation (4.45) by $\gamma_2(x)$ and integration of equation leads to

$$\int_0^{\infty} R_2(x,z)\gamma_2(x)dx = \left[\frac{rz^K e_1 H_2^*(m_3)}{a_1a_2a_3}\right] \quad (4.47)$$

Substituting the value of (4.28) in (4.20), we have

$$P_1^{(2)}(0,z) = \frac{\delta_2 P_1^{(1)}(0,z)G_{11}^*(a_1)}{z^k}$$

(4.48)

Substituting the value of (4.28) and (4.31) in (4.21), we have

$$P_2(0,z) = \frac{\delta_1 P_1^{(1)}(0,z)G_{11}^*(a_1)+\delta_2 P_1^{(2)}(0,z)G_{12}^*(a_2)}{z^k} \quad (4.49)$$

Substituting the value of (4.28), (4.31), (4.34), (4.39), (4.43), (4.47), (4.48), (4.49) in (4.22), we have

$$P_1^{(1)}(0,z) = \frac{\delta_1\lambda_0[C(z)-1]Q(z)a_1a_2a_3z^{2K}}{O}$$

(4.50)

where, $O = a_1a_2a_3[z^{3K} - [1-p+pB^*(m_1)]G_{11}^*(a_1)\{\delta_1 z^K(1-q)\ [z^K + \delta_2 G_{12}^*(a_2)] + \delta_1 G_2^*(a_3)[\delta_1 z^K + \delta_2^2 G_{12}^*(a_2)]\}] - (1-r)\ \alpha z^K H_1^*(m_2)\{z^K(1-q)a_3\delta_1\{z^K[1-G_{11}^*(a_1)]a_2 + \delta_2 G_{11}^*(a_1)\ a_1[1-G_{12}^*(a_2)]\} + \delta_1 a_1 a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1 z^K + \delta_2^2 G_{12}^*(a_2)]$
$-rz^K H_2^*(m_3)\{z^K(1-q)a_3\delta_1\{z^K[1-G_{11}^*(a_1)]a_2 + \delta_2 G_{11}^*(a_1)\ a_1[1-G_{12}^*(a_2)]\} + \delta_1 a_1 a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1 z^K + \delta_2^2 G_{12}^*(a_2)]\}$

Substituting the value of (4.50) in (4.48), we have

$$P_1^{(2)}(0,z) = \frac{\delta_1\delta_2\lambda_0[C(z)-1]Q(z)a_1a_2a_3z^K G_{11}^*(a_1)}{O}$$

(4.51)

Substituting the value of (4.48) in (4.49), we have

$$P_2(0,z) = \frac{\delta_1 P_1^{(1)}(0,z)G_{11}^*(a_1)+\delta_1\delta_2 P_1^{(1)}(0,z)G_{11}^*(a_1)G_{12}^*(a_2)}{z^k}$$

(4.52)

Substituting the value of (4.50) in (4.27), we have

$$P_1^{(1)}(z) = \frac{\delta_1\lambda_0[1-C(z)]Q(z)a_2a_3z^{2K}[1-G_{11}^*(a_1)]}{J}$$

(4.53)

where, $J = (1-r)\alpha z^K H_1^*(m_2)\{z^K(1-q)a_3\delta_1\{z^K[1-G_{11}^*(a_1)]a_2 + \delta_2 G_{11}^*(a_1)a_1[1-G_{12}^*(a_2)]\} + \delta_1 a_1 a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1 z^K + \delta_2^2 G_{12}^*(a_2)]\ + rz^K H_2^*(m_3)\{z^K(1-q)a_3\delta_1\{z^K[1-G_{11}^*(a_1)]a_2 + \delta_2 G_{11}^*(a_1)\ \ a_1[1-G_{12}^*(a_2)]\} + \delta_1 a_1 a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1 z^K + \delta_2^2 G_{12}^*(a_2)]\} - a_1a_2a_3[z^{3K} - [1-p+pB^*(m_1)]G_{11}^*(a_1)\{\delta_1 z^K(1-q)\ \ [z^K + \delta_2 G_{12}^*(a_2)] + \delta_1 G_2^*(a_3)[\delta_1 z^K + \delta_2^2 G_{12}^*(a_2)]\}]$

Substituting the value of (4.51) in (4.30), we have

$$P_1^{(2)}(z) = \frac{\delta_1\delta_2\lambda_0[1-C(z)]Q(z)a_1a_3z^K G_{11}^*(a_1)[1-G_{12}^*(a_2)]}{J} \quad (4.54)$$

Substituting the value of (4.51) in (4.30), we have

$$P_2(z) = \frac{\delta_1\lambda_0[1-C(z)]Q(z)a_1a_2z^K G_{11}^*(a_1)[1-G_2^*(a_3)]\{\delta_1 z^K+\delta_2^2 G_{12}^*(a_2)\}}{J} \quad (4.55)$$

Substituting the value of (4.51) in (4.30), we have

$$V(z) = \frac{p\delta_1\lambda_0[1-C(z)]Q(z)a_1a_2a_3[1-B^*(m_1)]d_2}{Jm_1} \quad (4.56)$$

where, $d_2 = p\{G_{11}^*(a_1)\{(1-q)[z^{2K} + \delta_2 G_{12}^*(a_2)]z^K + G_2^*(a_3)[\delta_1 z^K + \delta_2^2 G_{12}^*(a_2)]\}$

Substituting the value of (4.48), (4.49) and (4.50) in (4.42), we have

$$R_1(z) = \left[\frac{(1-r)\delta_1\lambda_0[1-C(z)]Q(z)\alpha z^K e_2[1-H_1^*(m_2)]}{m_2 J}\right] \quad (4.57)$$

where, $e_2 = \{z^K(1-q)a_3\delta_1\{z^K[1-G_{11}^*(a_1)]a_2 + \delta_2 G_{11}^*(a_1)a_1[1-G_{12}^*(a_2)]\}$
$+\delta_1 a_1 a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1 z^K + \delta_2^2 G_{12}^*(a_2)]\}$

Substituting the value of (4.48), (4.49) and (4.50) in (4.46), we have

$$R_2(z) = \left[\frac{r\delta_1\lambda_0[1-C(z)]Q(z)z^K e_2[1-H_2^*(m_3)]}{m_3 J}\right] \quad (4.58)$$

Theorem 4.2:

Under the steady state condition, the probability generating function for number of customers in the queue is $S(z)$=$Q(z) + N(z)$, where

$N(z) = P_1^{(1)}(z) + P_1^{(2)}(z) + P_2(z) + V(z) + R_1(z) + R_2(z)$

$S(z) = \frac{Q(z)A}{Jm_1m_2m_3} = \frac{n_1(z)}{n_2(z)}$

where, $n_1(z) = mm_1m_2m_3\delta_1a_3z^K(z^Ka_2[1-G_{11}^*(a_1)] + \delta_2a_1G_{11}^*(a_1)[1-G_{12}^*(a_2)])$ $+mm_1m_2m_3a_1a_2G_{11}^*(a_1)[1-G_2^*(a_3)]\{\delta_1z^K + \delta_1\delta_2G_{12}^*(a_2)\} + \delta_1mm_2m_3a_1a_2a_3pz^K[1-B^*(m_1)]\{G_{11}^*(a_1)\{(1-q)[z^{2K} + \delta_2G_{12}^*(a_2)z^K]\} + G_2^*(a_3)[\delta_1z^K + \delta_2^2G_{12}^*(a_2)]\} + \delta_1mm_1m_3(1-r)\alpha z^K[1-H_1^*(m_2)]\ \{[(1-q)a_3z^K\{[1-G_{11}^*(a_1)]a_2z^K + \delta_2G_{11}^*(a_1)[1-G_{12}^*(a_2)]a_1\}] + a_1a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1z^K + \delta_2^2G_{12}^*(a_2)]\} + \delta_1mm_1m_2rz^K\ [1-H_2^*(m_3)]\{[(1-q)a_3z^K\{[1-G_{11}^*(a_1)]a_2z^K + \delta_2[1-G_{12}^*(a_2)]\ G_{11}^*(a_1)a_1\}] + a_1a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1z^K + \delta_2^2G_{12}^*(a_2)]\} + Jm_1m_2m_3$

$n_2(z) = (1-r)\alpha z^KH_1^*(m_2)\{z^K(1-q)a_3\delta_1\{z^K[1-G_{11}^*(a_1)]a_2 + \delta_2G_{11}^*(a_1)a_1$ $[1-G_{12}^*(a_2)]\} + \delta_1a_1a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1z^K + \delta_2^2G_{12}^*(a_2)]\} + rz^KH_2^*(m_3)\{z^K(1-q)a_3\delta_1\{z^K[1-G_{11}^*(a_1)]a_2 + \delta_2G_{11}^*(a_1)a_1[1-G_{12}^*(a_2)]\} + \delta_1a_1a_2[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1z^K + \delta_2^2G_{12}^*(a_2)]\} - a_1a_2a_3[z^{3K} - [1-p+pB^*(m_1)]G_{11}^*(a_1)\{\delta_1z^K(1-q)[z^K + \delta_2G_{12}^*(a_2)] + \delta_1G_2^*(a_3)[\delta_1z^K + \delta_2^2G_{12}^*(a_2)]\}]$

Proof:

Let $S(z)$=$Q(z) + P_1^{(1)}(z) + P_1^{(2)}(z) + P_2(z) + V(z) + R_1(z) + R_2(z)$
(4.59)

Substituting the value of (4.53), (4.54), (4.55), (4.56), (4.57) and (4.58) in (4.59) we have

$S(z) = \frac{Q(z)A}{Jm_1m_2m_3}$
(4.60)

where, $A = mm_1m_2m_3\delta_1a_3z^K(z^Ka_2[1-G_{11}^*(a_1)] + \delta_2a_1G_{11}^*(a_1)[1-G_{12}^*(a_2)]) + m$ $m_1m_2m_3a_1a_2G_{11}^*(a_1)[1-G_2^*(a_3)]\{\delta_1z^K + \delta_1\delta_2G_{12}^*(a_2)\} + \delta_1mm_2m_3a_1a_2a_3pz^K[1-B^*(m_1)]\{G_{11}^*(a_1)\{(1-q)[z^{2K} + \delta_2G_{12}^*(a_2)z^K]\} + G_2^*(a_3)[\delta_1z^K + \delta_2^2G_{12}^*(a_2)]\} + \delta_1mm_1m_3(1-r)\alpha z^K[1-H_1^*(m_2)]\ \{[(1-q)a_3z^K\{[1-G_{11}^*(a_1)]a_2z^K + \delta_2G_{11}^*(a_1)[1-G_{12}^*(a_2)]a_1\}] + a_1a_2$ $[1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1z^K + \delta_2^2G_{12}^*(a_2)]\} + \delta_1mm_1m_2rz^K[1-H_2^*(m_3)]\{[(1-q)a_3z^K\{[1-G_{11}^*(a_1)]a_2z^K + \delta_2G_{11}^*(a_1)[1-G_{12}^*(a_2)]a_1\}] + a_1a_2\ [1-G_2^*(a_3)]G_{11}^*(a_1)[\delta_1z^K + \delta_2^2G_{12}^*(a_2)]\} + Jm_1m_2m_3$

$S(z) = \frac{A}{z-z_0}$ by substituting $z = 1$, we get

$A = (1-z_0)S(1)$

$S(1) = \frac{Qf_1}{f_2}$
(4.61)

$f_1 = \lambda_0E(X)\{[[1-G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2[1-G_{12}^*(\alpha)]G_{11}^*(\alpha) + G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1^2 +\right.$

$\delta_1\delta_2^2G_{12}^*(\alpha)\Big) + \alpha pE(V)G_{11}^*(\alpha)[(1-q)(\delta_1 + \delta_1\delta_2G_{12}^*(\alpha)) + G_2^*(\alpha)\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big)] + (1-r)\alpha E(R_1)\{(1-q)([1-G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2G_{11}^*(\alpha)[1-G_{12}^*(\alpha)]) + G_{11}^*(\alpha)[1-G_2^*(\alpha)]\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big)\} + rE(R_2)\{(1-q)([1-G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2G_{11}^*(\alpha)[1-G_{12}^*(\alpha)]) + G_{11}^*(\alpha)[1-G_2^*(\alpha)]\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big)\}\} + f_2$

$f_2 = \alpha G_{11}^*(\alpha)\{(1-q)\big(\delta_1 + \delta_1\delta_2G_{12}^*(\alpha)\big) + G_2^*(\alpha)\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big)\}\ [K - p\lambda_4E(X)E(V)] - \lambda_1E(X)\delta_1[1-G_{11}^*(\alpha)](1-q) - \lambda_2E(X)[1-G_{12}^*(\alpha)]G_{11}^*(\alpha)(1-q)\delta_1\delta_2 - \lambda_3E(X)\ [1-G_{11}^*(\alpha)G_2^*(\alpha)]\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big) - (1-r)\alpha\lambda_5E(X)\ E(R_1)\{(1-q)\{[1-G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2G_{11}^*(\alpha)[1-G_{12}^*(\alpha)]\} + G_{11}^*(\alpha)[1-G_2^*(\alpha)]\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big)\}\} - r\lambda_6E(X)E(R_2)\ \{(1-q)\{[1-G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2G_{11}^*(\alpha)[1-G_{12}^*(\alpha)]\} + G_{11}^*(\alpha)[1-G_2^*(\alpha)]\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big)\}\}$

By applying Rouche's theorem,

Substituting the value of $S(1)$ in the above equation, we have

$$A = \frac{(1-z_0)Qf_1}{f_2} \tag{4.62}$$

$$S(z) = \frac{(z_0-1)Qf_1}{z_0f_2}\sum_{n=0}^{\infty}\left(\frac{z}{z_0}\right)^n \tag{4.63}$$

which is the probability generating function of number of customers in the queue.

5. Some performance measures

In this section, the following system measures which shows the performance of the model are obtained:

1. The idle probability is $Q = \sum_{n=0}^{K-1} Q_n$ which leads to

$$Q = \frac{l_1}{l_2}$$

(5.1)

where, $l_1 = \alpha G_{11}^*(\alpha)\Big\{(1-q)\big(\delta_1 + \delta_1\delta_2G_{12}^*(\alpha)\big) + G_2^*(\alpha)\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big)\Big\}[K - p\lambda_4E(X)E(V)] - \lambda_1E(X)\delta_1[1-G_{11}^*(\alpha)](1-q) - \lambda_2E(X)[1-G_{12}^*(\alpha)]G_{11}^*(\alpha)(1-q)\delta_1\delta_2 - \lambda_3E(X)\ [1-G_{11}^*(\alpha)G_2^*(\alpha)]\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big) - (1-r)\alpha\lambda_5E(X)E(R_1)\{(1-q)\{[1-G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2G_{11}^*(\alpha)[1-G_{12}^*(\alpha)] + G_{11}^*(\alpha)[1-G_2^*(\alpha)]\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big)\}\} - r\lambda_6E(X)E(R_2)\ \{(1-q)\{[1-G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2G_{11}^*(\alpha)[1-G_{12}^*(\alpha)]\} + G_{11}^*(\alpha)[1-G_2^*(\alpha)]\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big)\}\}$

$l_2 = \alpha G_{11}^*(\alpha)\Big\{(1-q)\big(\delta_1 + \delta_1\delta_2G_{12}^*(\alpha)\big) + G_2^*(\alpha)\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big)\Big\}\ [K - p\lambda_4E(X)E(V)] - \lambda_1E(X)\delta_1[1-G_{11}^*(\alpha)](1-q)\ - \lambda_2E(X)[1-G_{12}^*(\alpha)]G_{11}^*(\alpha)(1-q)\delta_1\delta_2 - \lambda_3E(X)\ \ [1-G_{11}^*(\alpha)G_2^*(\alpha)]\Big(\delta_1^2 + \delta_1\delta_2^2G_{12}^*(\alpha)\Big) - (1-r)\alpha\lambda_5E(X)E(R_1)\{(1-q)\{[1-$

$G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2 G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)]\} + G_{11}^*(\alpha)[1 - G_2^*(\alpha)]\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\}\} -$
$r\lambda_6 E(X)E(R_2)$
$\{(1-q)\{[1 - G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2 G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)]\} + G_{11}^*(\alpha)[1 - G_2^*(\alpha)]\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\}\}$
$+\lambda_0 E(X)\{[[1 - G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2[1 - G_{12}^*(\alpha)]G_{11}^*(\alpha) + G_{11}^*(\alpha)\ [1 - G_2^*(\alpha)]\left(\delta_1^2 +\right.$
$\left.\delta_1\delta_2^2 G_{12}^*(\alpha)\right) + \alpha p E(V) G_{11}^*(\alpha)[(1-q)(\delta_1 + \delta_1\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)] +$
$(1-r)\alpha E(R_1)\ \{(1-q)([1 - G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2 G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)]) + G_{11}^*(\alpha)[1 - G_2^*(\alpha)]\left(\delta_1^2 +\right.$
$\left.\delta_1\delta_2^2 G_{12}^*(\alpha)\right)\} + rE(R_2)\ \{(1-q)([1 - G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2 G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)]) + G_{11}^*(\alpha)[1 -$
$G_2^*(\alpha)]\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\}\}$

This is obtained using $Q + N(1) = 1$

2. The average number of customers in the queue when the server provides essential service in fluctuating mode 1.

$N_{11} = P_1^{\prime(1)}(1) = \frac{f_3}{4l_2\alpha^2}$

(5.2)

where, $f_3 = \delta_1\{\lambda_0 E(X)^2 Q[1 - G_{11}^*(\alpha)] - 4\lambda_0 E(X)\lambda_2 E(X) Q\alpha[1 - G_{11}^*(\alpha)]$
$-4\lambda_0 E(X)\lambda_3 E(X) Q\alpha[1 - G_{11}^*(\alpha)] + 4KQ[1 - G_{11}^*(\alpha)]\lambda_0 E(X) + 4\lambda_0 E(X) Q[G'^*_{11}(\alpha)\lambda_1 E(X)]\alpha^2\}$

3. The average number of customers in the queue when the server provides essential service in fluctuating mode 2.

$N_{12} = P_1^{\prime(2)}(1) = \frac{f_4}{4l_2\alpha^2}$

(5.3)

where, $f_4 = 4Q\delta_1\delta_2\{\lambda_0 E(X)\alpha^2 K G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)] - \lambda_0 E(X)\lambda_1 E(X)\quad \alpha G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)] -$
$\lambda_0 E(X)\lambda_3 E(X)\alpha G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)] + \lambda_0 E(X)\alpha^2 G_{11}^{\prime *}(\alpha)[1 - G_{12}^*(\alpha)] + \lambda_0 E(X)\alpha^2 G_{11}^*(\alpha)$
$\left[G_{12}^{*\prime}(\alpha)\lambda_2 E(X)\right] + \lambda_0 E(X^2)\alpha^2 G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)]\}$

4. The average number of customers in the queue when the server provides optional service.

$N_2 = P'_2(1) = \frac{f_5}{l_2\alpha^2}$

(5.4)

where, $f_5 = 4Q\{\lambda_0 E(X)\lambda_1 E(X)\alpha G_{11}^*(\alpha)[1 - G_2^*(\alpha)]\{\delta_1 + \delta_2^2 G_{12}^*(\alpha)\} + \lambda_0 E(X)\lambda_2 E(X)\alpha G_{11}^*(\alpha)$
$[1 - G_2^*(\alpha)]\{\delta_1 + \delta_2^2 G_{12}^*(\alpha)\}\ - \lambda_0 E(X)\alpha^2 G_{11}^{*\prime}(\alpha)\lambda_1 E(X)[1 - G_2^*(\alpha)]\{\delta_1 + \delta_2^2 G_{12}^*(\alpha)\} -$
$\lambda_0 E(X)\alpha^2 G_{11}^*(\alpha)\lambda_3 E(X) G_2^{*\prime}(\alpha)\{\delta_1 + \delta_2^2 G_{12}^*(\alpha)\} - \lambda_0 E(X)\alpha^2 G_{11}^*(\alpha)[1 - G_2^*(\alpha)]\{\delta_1 K + \delta_2^2 G_{12}^*(\alpha)\}$
$-\lambda_0 E(X)\alpha^2 G_{11}^*(\alpha)[1 - G_2^*(\alpha)]\{\delta_1 + \delta_2^2 G_{12}^{*\prime}(\alpha)\left(-\lambda_2 E(X)\right)\}\ \ -\lambda_0 E(X^2)\alpha^2 G_{11}^*(\alpha)[1 -$
$G_2^*(\alpha)]\{\delta_1 + \delta_2^2 G_{12}^*(\alpha)\}\}$

5. The average number of customers in the queue when the server is on vacation.

$N_3 = V'(1) = \frac{f_6}{6l_2\alpha^2}$

(5.5)

where, $f_6 = 4Q\delta_1 p\{\alpha\lambda_0 E(X)E(V)G_{11}^*(\alpha)\{(1-q)(1+\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\} - \alpha\lambda_0 E(X^2)E(V)G_{11}^*(\alpha)\left\{(1-q)(1+\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\right\} - \alpha\lambda_0 E(X)E(V^2) G_{11}^*(\alpha)\{(1-q)(1+\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\} - \alpha\lambda_0 E(X)E(V)KG_{11}^*(\alpha) \{(1-q)(1+\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\} - \alpha\lambda_0 E(X)E(V)G_{11}^{*\prime}(\alpha)\lambda_1 E(X)\{(1-q)(1+\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\} + \lambda_1 E(X)\lambda_0 E(X)E(V)G_{11}^*(\alpha)\{(1-q)(1+\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)(\delta_1 + \delta_2^2 G_{12}^*(\alpha))\} + \lambda_2 E(X)\lambda_0 E(X)E(V)G_{11}^*(\alpha)\{(1-q)\left(1+\delta_2 G_{12}^*(\alpha)\right) + G_2^*(\alpha)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\} + \lambda_3 E(X)\lambda_0 E(X)E(V)G_{11}^*(\alpha)\{(1-q)(1+\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(V)G_{11}^*(\alpha)\{(1-q)\left(2K + \delta_2 G_{12}^*(\alpha)\right) + G_2^*(\alpha)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X) E(V)G_{11}^*(\alpha)\{(1-q)(1+K\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(V)G_{11}^*(\alpha) \{(1-q)\left(1+\delta_2 G_{12}^{*\prime}(\alpha)(-\lambda_2 E(X))\right)\} + G_2^*(\alpha)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(V)G_{11}^*(\alpha)\{(1-q) - G_2^{*\prime}(\alpha)\lambda_3 E(X)\left(\delta_1 + \delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(V)G_{11}^*(\alpha)\{(1-q)(1+\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)\left(\delta_1 K + \delta_2^2 G_{12}^*(\alpha)\right) - \lambda_0 E(X)E(V)G_{11}^*(\alpha)\{(1-q)(1+\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha) \left(\delta_1 K + \delta_2^2 G_{12}^{*\prime}(\alpha)(-\lambda_2 E(X))\right)\}\}$

6. The average number of customers in the queue when the server is on essential repair.

$$N_4 = R_1'(1) = \frac{f_7}{6l_2\alpha^2} \qquad (5.6)$$

where, $f_7 = 4Q(1-r)\alpha\delta_1\{\lambda_0(E(X)-E(X)^2)E(R_1)\{(1-q)\alpha^2([1-G_{11}^*(\alpha)] + \delta_2 G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)]) + \alpha^2 G_{11}^*(\alpha)[1-G_2^*(\alpha)](\delta_1 + \delta_1\delta_2^2\ G_{12}^*(\alpha))\} - \lambda_0 E(X)E(R_1^2)\{(1-q)\alpha^2([1-G_{11}^*(\alpha)] + \delta_2 G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)]) + \alpha^2 G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(R_1)K\{(1-q)\alpha^2([1-G_{11}^*(\alpha)] + \delta_2 G_{11}^*(\alpha)\,\alpha[1-G_{12}^*(\alpha)]) + \alpha^2 G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(R_1)\{(1-q)\alpha\lambda_3 E(X)([1-G_{11}^*(\alpha)] + \delta_2 G_{11}^*(\alpha)\ \alpha[1-G_{12}^*(\alpha)]) + \alpha^2 G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(R_1)\{(1-q)\alpha K([1-G_{11}^*(\alpha)] + \delta_2 G_{11}^*(\alpha)\,\alpha[1-G_{12}^*(\alpha)]) + \alpha^2 G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(R_1)\{(1-q)\alpha(K[1-G_{11}^*(\alpha)] + \delta_2 G_{11}^*(\alpha)\,\alpha[1-G_{12}^*(\alpha)]) + \alpha^2 G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right) - \lambda_0 E(X)E(R_1)\{(1-q)\alpha([1-G_{11}^{*\prime}(\alpha)(-\lambda_1 E(X))] + \delta_2 G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)]) + \alpha^2 G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(R_1)\{(1-q)\alpha([1-G_{11}^*(\alpha)\,(-\lambda_2 E(X))] + \delta_2 G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)]) + \alpha^2 G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(R_1)\{(1-q)\alpha([1-G_{11}^*(\alpha)] + \delta_2 G_{11}^{*\prime}(\alpha)(-\lambda_1 E(X))\alpha[1-G_{12}^*(\alpha)]) + \alpha^2 G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\} - \lambda_0 E(X)E(R_1)\{(1-q)\alpha([1-G_{11}^*(\alpha)]\ + \delta_2 G_{11}^*(\alpha)(-\lambda_1 E(X))[1-G_{12}^*(\alpha)]) + \alpha^2 G_{11}^*(\alpha)[1-G_2^*(\alpha)]$

$\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_1)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\left(-\lambda_2E(X)\right)\alpha G_{12}^{*\prime}(\alpha))+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_1)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)\left(-\lambda_1E(X)\right)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_1)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)\left(-\lambda_2E(X)\right)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_1)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)](\delta_1K+\delta_1\delta_2^2\,G_{12}^*(\alpha))\}-\lambda_0E(X)E(R_1)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\,\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^{*\prime}(\alpha)[1-G_2^*(\alpha)](\delta_1+\delta_1\delta_2^2G_{12}^{*\prime}(\alpha)\,\left(-\lambda_2E(X)\right))\}-\lambda_0E(X)E(R_1)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[(\lambda_3E(X)G_2^{*\prime}(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}\}$

7. The average number of customers in the queue when the server is on optional repair.

$$N_5=R_2'(1)=\frac{f_8}{6l_2\alpha^2} \tag{5.6}$$

where, $f_7=4Qr\delta_1\{\lambda_0(E(X)-E(X)^2)E(R_2)\{(1-q)\alpha^2([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)](\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha))\}-\lambda_0E(X)E(R_2^2)\{(1-q)\alpha^2([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_2)K\{(1-q)\alpha^2([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\,\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha\lambda_3E(X)([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)-\lambda_0E(X)E(R_2)\{(1-q)\alpha K([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\,\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha(K[1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)-\lambda_0E(X)E(R_2)\{(1-q)\alpha([1-G_{11}^{*\prime}(\alpha)\left(-\lambda_1E(X)\right)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha([1-G_{11}^*(\alpha)\left(-\lambda_2E(X)\right)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^{*\prime}(\alpha)\left(-\lambda_1E(X)\right)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\left(-\lambda_1E(X)\right)[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\left(-\lambda_2E(X)\right)\alpha G_{12}^{*\prime}(\alpha))+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)\left(-\lambda_1E(X)\right)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)\left(-\lambda_2E(X)\right)[1-G_2^*(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[1-G_2^*(\alpha)](\delta_1K+\delta_1\delta_2^2\,G_{12}^*(\alpha))\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^{*\prime}(\alpha)[1-G_2^*(\alpha)](\delta_1+\delta_1\delta_2^2G_{12}^{*\prime}(\alpha)\left(-\lambda_2E(X)\right))\}-\lambda_0E(X)E(R_2)\{(1-q)\alpha([1-G_{11}^*(\alpha)]+\delta_2G_{11}^*(\alpha)\alpha[1-G_{12}^*(\alpha)])+\alpha^2G_{11}^*(\alpha)[(\lambda_3E(X)G_2^{*\prime}(\alpha)]\left(\delta_1+\delta_1\delta_2^2G_{12}^*(\alpha)\right)\}\}$

8. The average number of customers in the queue

$S = S'(1) = \frac{Qy_1}{(1-z_0)y_2}$

(5.8)

where, $y_1 = \lambda_0 E(X)\{[[1 - G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2[1 - G_{12}^*(\alpha)]G_{11}^*(\alpha) + G_{11}^*(\alpha) \ [1 - G_2^*(\alpha)]\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right) + \alpha p E(V) G_{11}^*(\alpha)[(1-q)(\delta_1 + \delta_1\delta_2 G_{12}^*(\alpha)) + G_2^*(\alpha)\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)] + (1-r)\alpha E(R_1)\ \{(1-q)([1 - G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2 G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)]) + G_{11}^*(\alpha)[1 - G_2^*(\alpha)]\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\} + rE(R_2)\ \{(1-q)([1 - G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2 G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)]) + G_{11}^*(\alpha)[1 - G_2^*(\alpha)]\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\}\} + y_2$

$y_2 = \alpha G_{11}^*(\alpha)\{(1-q)\left(\delta_1 + \delta_1\delta_2 G_{12}^*(\alpha)\right) + G_2^*(\alpha)\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\}[K - p\lambda_4 E(X)E(V)] - \lambda_1 E(X)\delta_1[1 - G_{11}^*(\alpha)](1-q) \ - \lambda_2 E(X)[1 - G_{12}^*(\alpha)]G_{11}^*(\alpha)(1-q)\delta_1\delta_2 - \lambda_3 E(X) [1 - G_{11}^*(\alpha)G_2^*(\alpha)]\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right) - (1-r)\alpha\lambda_5 E(X)\ \ E(R_1)\{(1-q)\{[1 - G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2 G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)]\} + G_{11}^*(\alpha)[1 - G_2^*(\alpha)]\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\}\} - r\lambda_6 E(X)E(R_2) \{(1-q)\{[1 - G_{11}^*(\alpha)]\delta_1 + \delta_1\delta_2 G_{11}^*(\alpha)[1 - G_{12}^*(\alpha)]\} + G_{11}^*(\alpha)[1 - G_2^*(\alpha)]\left(\delta_1^2 + \delta_1\delta_2^2 G_{12}^*(\alpha)\right)\}\}$

9. The server's utilization factor is $\rho = 1 - Q = \frac{l_1}{l_2}$

(5.9)

10. Mean waiting time of a customer $W = \frac{S}{\lambda^*}$

(5.10)

where, λ^* is the effective arrival rate and is

$\lambda^* = \lambda_0 Q + \lambda_1 P_1^{(1)}(1) + \lambda_2 P_1^{(2)}(1) + \lambda_3 P_2(1) + \lambda_4 V(1) + \lambda_5 R_1(1) + \lambda_6 R_2(1)$

6. Particular systems

In this section, some particular models are dervied by taking particular form to the distributions functions and particular values to the parameters.

Particular system 1:

First model is model without optional service ($q = 0$) and taking state independent arrival rates ($\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda$). The probability generating function of numbers of customers in the queue is $S(z) = \frac{D_1}{D_2}$

Where, $D_1 = Q(z)\{m\{\delta_1\{z^k[1 - G_{11}^*(a)] + G_{11}^*(a)[1 - G_{12}^*(a)]\delta_2\} + \delta_1 a p G_{11}^*(a)\ [1 - B^*(m)](1-q)[z^{2k} + \delta_2 G_{12}^*(a)] + \delta_1\alpha(1-r)z^K[1 - H_1^*(m)]\ (1-q)\{[1 - G_{11}^*(a)]z^k + G_{11}^*(a)[1 - G_{12}^*(a)]\delta_2\} + [1 - H_2^*(m)]\ \delta_1 r z^K(1-q)\{[1 - G_{11}^*(a)]z^k + G_{11}^*(a)[1 - G_{12}^*(a)]\delta_2\}\} + D_2\}$

$D_2 = \alpha(1-r)z^K H_1^*(m)(1-q)\{[1 - G_{11}^*(a)]z^k + G_{11}^*(a)[1 - G_{12}^*(a)]\delta_2\}\ \ r z^K H_2^*(m)(1-q)\{[1 - G_{11}^*(a)]z^k + G_{11}^*(a)[1 - G_{12}^*(a)]\delta_2\} - a\left[z^K - [1 - p + pB^*(m)]G_{11}^*(a)(1-q)[z^{2k} + \delta_2 G_{12}^*(a)]\right]$

Particular system 2:

The second model is model with independent arrival rates ($\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda$). The probability generating function of numbers of customers in the queue is $S(z) = \frac{D_3}{D_4}$

Where, $D_3 = Q(z)\{m\{\delta_1[z^k[1 - G_{11}^*(a)] + G_{11}^*(a)[1 - G_{12}^*(a)]\delta_2] + G_{11}^*(a)[1 - G_2^*(a)]\left(\delta_1 z^k + \delta_2^2 G_{12}^*(a)\right) + \delta_1 ap[1 - B^*(m)]G_{11}^*(a)(1 - q)\{[z^{2k} + \delta_2 G_{12}^*(a)] + G_2^*(a)[\delta_1 z^k + \delta_2^2 G_{12}^*(a)]\} + \delta_1(1 - r)\alpha z^K[1 - H_1^*(m)](1 - q)\{[1 - G_{11}^*(a)]z^k + G_{11}^*(a)\ [1 - G_{12}^*(a)]\delta_2\} + G_{11}^*(a)[1 - G_2^*(a)]\left(\delta_1 z^k + \delta_2^2 G_{12}^*(a)\right)\} + \delta_1 rz^K[1 - H_2^*(m)](1 - q)\{[1 - G_{11}^*(a)]z^k + G_{11}^*(a)$
$[1 - G_{12}^*(a)]\delta_2\} + G_{11}^*(a)[1 - G_2^*(a)]\left(\delta_1 z^k + \delta_2^2 G_{12}^*(a)\right)\}\} + D_4\}$

$D_4 = (1 - r)\delta_1 \alpha z^K H_1^*(m)(1 - q)\{[1 - G_{11}^*(a)]z^k + G_{11}^*(a)[1 - G_{12}^*(a)]\ \delta_2\} + G_{11}^*(a)[1 - G_2^*(a)]\left(\delta_1 z^k + \delta_2^2 G_{12}^*(a)\right)\}\} - r\delta_1 z^K H_2^*(m)(1 - q)\{[1 - G_{11}^*(a)]z^k + G_{11}^*(a)[1 - G_{12}^*(a)]\delta_2\} + G_{11}^*(a)\ [1 - G_2^*(a)]\left(\delta_1 z^k + \delta_2^2 G_{12}^*(a)\right)\}\} - a[z^{2K} - [1 - p + pB^*(m)]$
$G_{11}^*(a)\delta_1(1 - q)\{[z^{2k} + \delta_2 G_{12}^*(a)] + G_2^*(a)[\delta_1 z^k + \delta_2^2 G_{12}^*(a)]]\}$

Particular system 3:

The third model is model with single mode of essential service ($\delta_1 = 1, \delta_2 = 0$). The probability generating function of numbers of customers in the queue is $S(z) = \frac{D_5}{D_6}$

Where, $D_5 = Q(z)\{mm_1m_2m_3a_3z^{2K}[1 - G_{11}^*(a_1)] + mm_1m_2m_3a_1G_{11}^*(a_1)[1 - G_2^*(a_3)]z^K mm_2m_3a_1a_3pz^K[1 - B^*(m_1)] + G_{11}^*(a_1)\{(1 - q)z^{2K} + G_2^*(a_3)\delta_1 z^K\} + mm_1m_3(1 - r)\alpha z^K[1 - H_1^*(m_2)]\{(1 - q)a_3z^{2K}\ [1 - G_{11}^*(a_1)] + a_1[1 - G_2^*(a_3)]G_{11}^*(a_1)z^K\} + mm_1m_2rz^K[1 - H_2^*(m_3)]\ \{[(1 - q)a_3z^{2K}\{[1 - G_{11}^*(a_1)] + a_1[1 - G_2^*(a_3)]G_{11}^*(a_1)z^K\} + D_6\}$

$D_6 = m_1m_2m_3\{(1 - r)\alpha z^K H_1^*(m_2)\{z^K(1 - q)a_3\{z^{2K}[1 - G_{11}^*(a_1)] + a_1[1 - G_2^*(a_3)]$
$G_{11}^*(a_1)z^K\} + rz^K H_2^*(m_3)\{z^{2K}(1 - q)a_3[1 - G_{11}^*(a_1)] + a_1\ [1 - G_2^*(a_3)]G_{11}^*(a_1)z^K\} - a_1a_3[z^{3K} - [1 - p + pB^*(m_1)]\ G_{11}^*(a_1)\{z^{2K}(1 - q) + G_2^*(a_3)z^K\}]\}$

Particular system 4:

The fourth model is model with essential repair and ($r = 0$). The probability generating function of numbers of customers in the queue is $S(z) = \frac{D_7}{D_8}$

Where, $D_7 = Q(z)\{mm_1m_2\delta_1a_3z^K(z^K a_2[1 - G_{11}^*(a_1)] + \delta_2a_1G_{11}^*(a_1)[1 - G_{12}^*(a_2)]) + mm_1m_2a_1a_2G_{11}^*(a_1)[1 - G_2^*(a_3)]\{\delta_1 z^K + \delta_1\delta_2G_{12}^*(a_2)\} + \delta_1 mm_2a_1a_2a_3pz^K[1 - B^*(m_1)]\{G_{11}^*(a_1)\{(1 - q)[z^{2K} + \delta_2G_{12}^*(a_2)z^K]\} + G_2^*(a_3)[\delta_1 z^K + \delta_2^2G_{12}^*(a_2)]\} + \delta_1 mm_1\alpha z^K[1 - H^*(m_2)]\{[(1 - q)a_3z^K\{[1 - G_{11}^*(a_1)]a_2z^K + \delta_2G_{11}^*(a_1)[1 - G_{12}^*(a_2)]a_1\}] + a_1a_2[1 - G_2^*(a_3)]G_{11}^*(a_1)[\delta_1 z^K + \delta_2^2G_{12}^*(a_2)]\} + D_8\}$

$D_8 = m_1m_2\{\alpha z^K H^*(m_2)\{z^K(1 - q)a_3\delta_1\{z^K[1 - G_{11}^*(a_1)]a_2 + \delta_2G_{11}^*(a_1)a_1[1 - G_{12}^*(a_2)]\} + \delta_1a_1a_2[1 - G_2^*(a_3)]G_{11}^*(a_1)\ [\delta_1 z^K + \delta_2^2G_{12}^*(a_2)]\} - a_1a_2a_3[z^{3K} - [1 - p + pB^*(m_1)]G_{11}^*(a_1)$
$\{\delta_1 z^K(1 - q)[z^K + \delta_2G_{12}^*(a_2)] + \delta_1G_2^*(a_3)[\delta_1 z^K + \delta_2^2G_{12}^*(a_2)]\}]\}$

7. Numerical illustrations

In this section, the model analyzed in this paper is numerically analyzed by assuming service times, vacation times and repair time are negative exponential distributions and batch size is geometric distribution. The parameter values are $C_j = \delta(1-\delta)^{j-1}, j = 1,2,3,\ldots,0 < \delta < 1,$

$E(X) = \frac{(1-\delta)}{\delta}, E(V) = \frac{1}{\beta}, E(R_1) = \frac{1}{\gamma_1}, E(R_2) = \frac{1}{\gamma_2}, G_{11}^*(\alpha) = \frac{\mu_{11}}{\alpha+\mu_{11}},$

$G_{12}^*(\alpha) = \frac{\mu_{12}}{\alpha+\mu_{12}}, G_2^*(\alpha) = \frac{\mu_2}{\alpha+\mu_2}, E(S_1) = \frac{1}{\mu_1}, E(S_2) = \frac{1}{\mu_2}, E(S_3) = \frac{1}{\mu_3},$

$B^*(m_1) = \frac{\beta}{m_1+\beta}, H_1^*(m_2) = \frac{\gamma_1}{m_2+\gamma_1}, H_2^*(m_3) = \frac{\gamma_2}{m_3+\gamma_2}$

The performance measures are calculated using the formulas in section 5.

1.The idle probability is $Q = \frac{f_9}{f_{10}}$

where, $f_9 = \alpha\mu_{11}(1-q)\delta_1(\alpha+\mu_2)[(\alpha+\mu_{12})+\mu_{12}\delta_2] + \alpha\mu_{11}\mu_2\delta_1[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2]$ $\{K\beta\delta - p(1-\delta)\lambda_4\}\gamma_1\gamma_2 - \lambda_1(1-\delta)\delta_1(1-q)\alpha(\alpha+\mu_2)\gamma_1\gamma_2(\alpha+\mu_{12})\beta - \lambda_2(1-\delta)\delta_1\delta_2(1-q)\mu_{11}\alpha\beta\gamma_1\gamma_2(\alpha+\mu_2) - \lambda_3(1-\delta)[(\alpha+\mu_2)(\alpha+\mu_{11})-\mu_{11}\mu_2][(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}] - \gamma_2\lambda_5\alpha(1-r)(1-\delta)\beta\{(\alpha+\mu_2)\delta_1\alpha(1-q)[(\alpha+\mu_{12})+\mu_{11}\delta_2] + \mu_{11}\alpha[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}]\}$ $-\gamma_1\lambda_6 r(1-\delta)\beta\{(\alpha+\mu_2)\delta_1\alpha(1-q)[(\alpha+\mu_{12})+\mu_{11}\delta_2] + \mu_{11}\alpha[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}]\}$

$f_{10} = \alpha\mu_{11}(1-q)\delta_1(\alpha+\mu_2)[(\alpha+\mu_{12})+\mu_{12}\delta_2] + \alpha\mu_{11}\mu_2\delta_1[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2]$ $\{K\beta\delta - p(1-\delta)\lambda_4\}\gamma_1\gamma_2 - \lambda_1(1-\delta)\delta_1(1-q)\alpha(\alpha+\mu_2)\gamma_1\gamma_2(\alpha+\mu_{12})\beta - \lambda_2(1-\delta)\delta_1\delta_2(1-q)\mu_{11}\alpha\beta\gamma_1\gamma_2(\alpha+\mu_2) - \lambda_3(1-\delta)[(\alpha+\mu_2)(\alpha+\mu_{11})-\mu_{11}\mu_2][(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}] - \gamma_2\lambda_5\alpha(1-r)(1-\delta)\beta\{(\alpha+\mu_2)\delta_1\alpha(1-q)[(\alpha+\mu_{12})+\mu_{11}\delta_2] + \mu_{11}\alpha[(\alpha+\mu_{12})\delta_1^2 + \delta_2^2\delta_1\mu_{12}]\} - \gamma_1\lambda_6 r(1-\delta)\beta\{(\alpha+\mu_2)\delta_1\alpha(1-q)[(\alpha+\mu_{12})+\mu_{11}\delta_2] + \mu_{11}\alpha[(\alpha+\mu_{12})\delta_1^2 + \delta_2^2\delta_1\mu_{12}]\} + \lambda_0(1-\delta)\{\delta_1\alpha\beta\gamma_1\gamma_2(\alpha+\mu_{12})(\alpha+\mu_2) + \alpha\mu_{11}\gamma_1\gamma_2\beta\delta_1\delta_2(\alpha+\mu_2) + \alpha\mu_{11}\gamma_1\gamma_2\beta[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}] + \alpha(1-r)\beta\gamma_2[\alpha\delta_1(1-q)(\alpha+\mu_2)((\alpha+\mu_{12})+\delta_2\mu_{11}) + \mu_{11}\alpha[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}]] + \gamma_1\gamma_2 p[\mu_{11}(1-q)\alpha\delta_1(\alpha+\mu_2)[(\alpha+\mu_{12})+\mu_{12}\delta_2]] + \alpha\mu_{11}\mu_2\delta_1[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2] + r\beta\gamma_1[\alpha\delta_1(1-q)(\alpha+\mu_2)((\alpha+\mu_{12})+\delta_2\mu_{11}) + \mu_{11}\alpha[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}]]\}$

2. The average number of customers is $S = S'(1) = \frac{Qy_3}{(1-z_0)y_4}$

where, $y_3 = \lambda_0(1-\delta)\{\delta_1\alpha\beta\gamma_1\gamma_2(\alpha+\mu_{12})(\alpha+\mu_2) + \alpha\mu_{11}\gamma_1\gamma_2\beta\delta_1\delta_2(\alpha+\mu_2) + \alpha\mu_{11}\gamma_1\gamma_2\beta[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}] + \alpha(1-r)\beta\gamma_2[\alpha\delta_1(1-q)(\alpha+\mu_2)((\alpha+\mu_{12})+\delta_2\mu_{11}) + \mu_{11}\alpha[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}]] + \gamma_1\gamma_2 p[\mu_{11}(1-q)\alpha\delta_1(\alpha+\mu_2)[(\alpha+\mu_{12})+\mu_{12}\delta_2]]$ $+\alpha\mu_{11}\mu_2\delta_1[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2] + r\beta\gamma_1[\alpha\delta_1(1-q)(\alpha+\mu_2)((\alpha+\mu_{12})+\delta_2\mu_{11}) + \mu_{11}\alpha[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}]]\} + y_4$

$y_4 = \alpha\mu_{11}(1-q)\delta_1(\alpha+\mu_2)[(\alpha+\mu_{12})+\mu_{12}\delta_2] + \alpha\mu_{11}\mu_2\delta_1[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2]\{K\beta\delta - p(1-\delta)\lambda_4\}\gamma_1\gamma_2 - \lambda_1(1-\delta)\delta_1(1-q)\alpha(\alpha+\mu_2)\gamma_1\gamma_2(\alpha+\mu_{12})\beta - \lambda_2(1-\delta)\delta_1\delta_2(1-q)\mu_{11}\alpha\beta\gamma_1\gamma_2(\alpha+\mu_2) - \lambda_3(1-\delta)[(\alpha+\mu_2)(\alpha+\mu_{11})-\mu_{11}\mu_2][(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}] - \gamma_2\lambda_5\alpha(1-r)(1-\delta)\beta\{(\alpha+\mu_2)\delta_1\alpha(1-q)[(\alpha+\mu_{12})+\mu_{11}\delta_2] + \mu_{11}\alpha[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}]\}$ $-\gamma_1\lambda_6 r(1-\delta)\beta\{(\alpha+\mu_2)\delta_1\alpha(1-q)[(\alpha+\mu_{12})+\mu_{11}\delta_2] + \mu_{11}\alpha[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}]\}$

3. The average number of customers in the queue when the server provides essential service in fluctuating mode 1 is $N_{11} = P_1'^{(1)}(1) = \frac{f_{11}}{\alpha f_{10}}$

where, $f_{11} = Q\beta\gamma_1\gamma_2\delta_1(\alpha+\mu_2)(\alpha+\mu_{12})\lambda_0\{\alpha^2K(1-\delta)\delta\mu_{11} + \lambda_1(1-\delta)^2(\alpha+\mu_{11}) - \lambda_2(1-\delta)^2\alpha\mu_{11} - \lambda_3(1-\delta)^2\alpha\mu_{11} + \alpha^2\mu_{11}(2-3\delta+\delta^2)\}$

4. The average number of customers in the queue when the server provides essential service in fluctuating mode 2 is $N_{12} = P_1'^{(2)}(1) = \frac{f_{12}}{\delta\mu_{11}\mu_{12}f_{10}}$

where, $f_{12} = Q\beta\gamma_1\gamma_2\delta_1\delta_2(\alpha+\mu_2)\lambda_0\{\alpha K\mu_{12}\mu_{11}^2(1-\delta)\delta - \lambda_2(1-\delta)^2\mu_{12}\mu_{11}^2 - \lambda_3(1-\delta)^2\mu_{12}\mu_{11}^2 + \alpha\lambda_1\lambda_0(1-\delta)^2\mu_{12}(\alpha+\mu_{11}) + \alpha\mu_{12}\mu_{11}^2(2-3\delta+\delta^2) + \alpha\lambda_2\lambda_0(1-\delta)^2\mu_{11}^2(\alpha+\mu_2)\}$

5. The average number of customers in the queue when the server provides optional service is $N_2 = P_2'(1) = \frac{f_{13}}{\delta_1\mu_{11}\mu_{12}\mu_2 f_{10}}$

where, $f_{13} = Q\beta\gamma_1\gamma_2\lambda_0\{\mu_{12}\lambda_1(1-\delta)^2\mu_{11}^2\mu_2[(\alpha+\mu_{12})\delta_1^2 + \delta_2^2\delta_1\mu_{12}] + \lambda_2(1-\delta)^2\mu_{11}^2\mu_2[(\alpha+\mu_{12})\delta_1^2 + \delta_2^2\delta_1\mu_{12}] - [(\alpha+\mu_{12})\delta_1 + \delta_2^2\delta_1\mu_{12}]\lambda_1(1-\delta)^2\mu_2\alpha(\alpha+\mu_{11})\mu_{12} - \lambda_1(1-\delta)^2\mu_{12}\mu_{11}^2(\alpha+\mu_2)[(\alpha+\mu_{12})\delta_1 + \delta_2^2\mu_{12}] - (1-\delta)\delta\alpha\mu_{11}^2\mu_2\mu_{12}[(\alpha+\mu_{12})K\delta_1 + \delta_2^2\mu_{12}] - (1-\delta)\alpha\mu_{11}^2\mu_2(\alpha+\mu_{12})\{\delta\mu_{12}\delta_1 - \delta_2^2(1-\delta)\lambda_2\} - \alpha(2-3\delta+\delta^2)\mu_{11}^2\mu_2\mu_{12}[(\alpha+\mu_{12})\delta_1 + \delta_2^2\mu_{12}]\}$

6. The average number of customers in the queue when the server is on vacation

$$N_3 = V(1) = \frac{f_{14}2pQ\lambda_0\beta\gamma_1\gamma_2}{3\alpha^2\mu_{11}\mu_{12}\mu_2 f_{10}}$$

where, $f_{14} = \{\mu_{11}(1-q)\delta_1(\alpha+\mu_2)[(\alpha+\mu_{12}) + \mu_{12}\delta_2] + \mu_{11}\mu_2\delta_1[\delta_1(\alpha+\mu_{12}) + \mu_{12}\delta_2^2]\}\{\alpha\beta\delta - \alpha\delta(1-K) - \alpha\beta(2-3\delta+\delta^2) + (1-\delta)^2\beta[\lambda_1+\lambda_2+\lambda_3]\,\mu_{11}\mu_2\mu_{12} + \lambda_1\alpha(1-\delta)^2\beta(\alpha+\mu_{11})\mu_2\mu_{12}\{\mu_{11}(1-q)[(\alpha+\mu_{12}) + \mu_{12}\delta_2]\,(\alpha+\mu_2) + \mu_2[\delta_1(\alpha+\mu_{12}) + \mu_{12}\delta_2^2]\} + \mu_{11}\mu_{12}\mu_2\beta\delta\{\mu_{11}(1-q) - \delta_1(\alpha+\mu_2)[(\alpha+\mu_{12})K + \mu_{12}\delta_2] + \mu_{11}\mu_2\delta_1[\delta_1(\alpha+\mu_{12}) + \mu_{12}\delta_2^2]\} - \mu_{11}\,\mu_2\mu_{12}\beta\delta\{(\alpha+\mu_{12})(1-q)\delta_1[(\delta_1\mu_{12}) + K\mu_{12}\delta_2] + \mu_{11}\mu_2\delta_1[\delta_1(\alpha+\mu_{12}) + \mu_{12}\delta_2^2]\} - \mu_{11}\mu_2^2\beta(\alpha+\mu_{11})\{(1-q)(\alpha+\mu_2)[(\alpha+\mu_{12})[\delta_1\mu_{12} - \delta_2\lambda_2(1-\delta)] + \mu_{11}\mu_2\mu_{12}\delta_1\delta_2[\delta_1(\alpha+\mu_{12}) + \mu_{12}\delta_2^2]\} - \mu_{11}\mu_2\beta(\alpha+\mu_{11})\{\mu_{11}(1-q)\mu_2\delta\delta_1[(\alpha+\mu_{12}) + \mu_{12}\delta_2](\alpha+\mu_2) - \lambda_3(1-\delta)[\delta_1(\alpha+\mu_{12})K + \mu_{12}\delta_2^2]\} - \mu_{11}\mu_{12}\delta\mu_{12}\beta\{\mu_{11}(1-q)(\alpha+\mu_2)\delta_1[(\alpha+\mu_{12}) + \mu_{12}\delta_2] + \mu_{11}\,\mu_2\delta_1[\delta_1(\alpha+\mu_{12})K + \mu_{12}\delta_2^2]\} - \mu_{11}\mu_{12}\beta(\alpha+\mu_2)\{\mu_{11}(1-q)\mu_2\,\delta\delta_1[(\alpha+\mu_{12}) + \mu_{12}\delta_2](\alpha+\mu_2) + \mu_2\delta_1[\delta_1\delta\mu_{12} - \mu_{12}\delta_2^2\lambda_2\delta^2]\}$

7. The average number of customers in the queue when the server is on essential repair

$$N_4 = R_1(1) = \frac{f_{15}2Q\lambda_0\beta\gamma_2(1-r)}{3\alpha^2\mu_{11}\mu_{12}\mu_2\delta f_{10}}$$

where, $f_{15} = \alpha^3\mu_{11}\mu_{12}\mu_2\{\delta_1\alpha(1-q)(\alpha+\mu_2)[(\alpha+\mu_{12}) + \mu_{12}\delta_2] + \mu_{11}\alpha[\delta_1^2(\alpha+\mu_{12}) + \mu_{12}\delta_1\delta_2^2]\}\{\delta(1-\delta)\gamma_1 - \delta(1-\delta) - K\gamma_1\delta(1-\delta) - \gamma_1(2-3\delta+\delta^2)\} + \alpha\gamma_1(1-\delta)\{\delta_1\alpha(1-q)(1-\delta)\lambda_3[(\alpha+\mu_{12}) + \mu_{12}\delta_2] + \mu_{11}\alpha^2[\delta_1^2(\alpha+\mu_{12}) + \mu_{12}\delta_1\delta_2^2]\}\mu_{11}\mu_{12}\mu_2 - (1-\delta)\delta\gamma_1\alpha^3\{\delta_1\alpha(1-q)(\alpha+\mu_2)K[(\alpha+\mu_{12}) + \mu_{12}\delta_2] + \alpha\mu_{11}[\delta_1^2(\alpha+\mu_{12}) + \mu_{12}\delta_1\delta_2^2]\}\mu_{11}\mu_2\mu_{12} - (1-\delta)\delta\gamma_1\alpha^3\{\delta_1\alpha(1-q)(\alpha+\mu_2)[K(\alpha+\mu_{12}) + \mu_{12}\delta_2] + \alpha\mu_{11}[\delta_1^2(\alpha+\mu_{12}) + \mu_{12}\delta_1\delta_2^2]\}\mu_{11}\,\mu_2\mu_{12} - (1-\delta)\gamma_1\alpha^3\{\delta_1(1-q)[\lambda_1(\alpha+\mu_{12}) + \alpha\delta\delta_2\mu_{11}](\alpha+\mu_2) + \mu_{11}^2\alpha\delta[\delta_1^2(\alpha+\mu_{12}) + \mu_{12}\delta_1\delta_2^2]\}\mu_2\mu_{12} - (1-\delta)\gamma_1\alpha\{\delta_1(1-q)(\alpha+\mu_2)[\delta_2\alpha^3\mu_{11}\delta - \alpha^2\lambda_2(1-\delta)] + \mu_{11}\alpha^3\delta[\delta_1^2(\alpha+\mu_{12}) + \mu_{12}\delta_1\delta_2^2]\}\mu_{11}\mu_2\mu_{12} - (1-\delta)\gamma_1\alpha^3\{\delta_1(1-q)(\alpha+\mu_2)[(\alpha\mu_{11} + \delta_2(\alpha+\mu_{11}))(\alpha+\mu_{12})\delta - \lambda_1(1-\delta)\alpha\mu_{11}(\alpha+\mu_{11})] + \mu_{11}\alpha\delta[\delta_1^2(\alpha+\mu_{12}) + \mu_{12}\delta_1\delta_2^2]\delta\mu_{11}\}\mu_2\mu_{12} - (1-\delta)\gamma_1\alpha\{\delta_1(1-q)(\alpha+\mu_2)[\alpha^2\delta(\alpha+\mu_{12}) + \delta^2\mu_{11}\delta(\alpha+\mu_{12}) - \lambda_1(1-\delta)\alpha(\alpha+\mu_{11})] + \mu_{11}\alpha[\delta_1^2(\alpha+\mu_{12}) + \mu_{12}\delta_1\delta_2^2]\delta\}\mu_{11}\mu_{12}\mu_2 - (1-\delta)\gamma_1\alpha^3\mu_{11}\mu_2\{\delta_1(1-q)(\alpha+\mu_2)[\alpha\mu_{12}\delta + \mu_{12}\alpha\mu_{11}\lambda_2(1-\delta)] +$

$\mu_{11}\alpha[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\delta\mu_{12}\}-(1-\delta)\gamma_1\alpha\{\delta_1(1-q)(\alpha+\mu_2)\delta\left[\alpha^2\left((\alpha+\mu_{12})+\mu_{11}\delta_2\right)+\delta\delta_1\alpha(\alpha+\mu_{11})(\alpha+\mu_{12})(\alpha+\mu_2)\right]-\lambda_1(1-\delta)\alpha\mu_{11}(\alpha+\mu_{12})[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2]\}-(1-\delta)\gamma_1\alpha\{\delta_1(1-q)(\alpha+\mu_2)\,\delta\left[\alpha^2\left((\alpha+\mu_{12})+\mu_{11}\delta_2\right)+\delta\delta_1\alpha(\alpha+\mu_{11})(\alpha+\mu_{12})(\alpha+\mu_2)\right]-\lambda_2(1-\delta)\alpha\mu_{11}(\alpha+\mu_{12})[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2]\}-(1-\delta)\gamma_1\alpha^3\delta\{(\alpha+\mu_2)(1-q)\alpha\delta_1\left((\alpha+\mu_{12})+\mu_{11}\delta_2\right)+\alpha\mu_{11}[\delta_1^2(\alpha+\mu_{12})K+\mu_{12}\delta_1\delta_2^2]\}\mu_{11}\mu_2\mu_{12}-(1-\delta)\gamma_1\alpha^3\{(\alpha+\mu_2)(1-q)\alpha\delta_1\left((\alpha+\mu_{12})+\mu_{11}\delta_2\right)+\alpha\mu_{11}\ [\delta_1^2(\alpha+\mu_{12})\delta\mu_{12}-\lambda_2\delta_1\delta_2^2(1-\delta)]\}\mu_{11}\mu_2-(1-\delta)\gamma_1\alpha^3\mu_{12}\mu_2\ \{(\alpha+\mu_2)(1-q)\alpha\delta_1\left((\alpha+\mu_{12})+\mu_{11}\delta_2\right)\mu_2+\delta_1\alpha^2\lambda_3(1-\delta)\alpha[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\}$

8. The average number of customers in the queue when the server is on optional repair

$$N_5=R_2(1)=\frac{f_{16}2Q\lambda_0\beta\gamma_1 r}{3\alpha^2\mu_{11}\mu_{12}\mu_2\delta f_{10}}$$

where, $f_{16}=\alpha^3\mu_{11}\mu_{12}\mu_2\{\delta_1\alpha(1-q)(\alpha+\mu_2)[(\alpha+\mu_{12})+\mu_{12}\delta_2]+\mu_{11}\alpha[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\}\{\delta(1-\delta)\gamma_2-\delta(1-\delta)-K\gamma_2\delta(1-\delta)-\gamma_1(2-3\delta+\delta^2)\}+\alpha\gamma_2(1-\delta)\{\delta_1\alpha(1-q)(1-\delta)\lambda_3[(\alpha+\mu_{12})+\mu_{12}\delta_2]+\mu_{11}\alpha^2[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\}\mu_{11}\mu_2\mu_{12}-(1-\delta)\delta\gamma_2\alpha^3\{\delta_1\alpha(1-q)(\alpha+\mu_2)K[(\alpha+\mu_{12})+\mu_2\delta_2]+\alpha\mu_{11}[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\}\mu_{11}\mu_2\mu_{12}-(1-\delta)\delta\gamma_2\alpha^3\{\delta_1\alpha\,(1-q)(\alpha+\mu_2)[K(\alpha+\mu_{12})+\mu_{12}\delta_2]+\alpha\mu_{11}[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\}\mu_{11}\mu_2\mu_{12}-(1-\delta)\gamma_2\alpha^3\{\delta_1(1-q)[\lambda_1(\alpha+\mu_{12})+\alpha\delta\delta_2\mu_{11}](\alpha+\mu_2)+\mu_{11}^2\alpha\delta[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\}\mu_2\mu_{12}-(1-\delta)\gamma_2\alpha\{\delta_1(1-q)(\alpha+\mu_2)[\delta_2\alpha^3\mu_{11}\delta-\alpha^2\lambda_2(1-\delta)]+\mu_{11}\alpha^3\delta[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\}\mu_{11}\,\mu_2\mu_{12}-(1-\delta)\gamma_2\alpha^3\{\delta_1(1-q)(\alpha+\mu_2)[\left(\alpha\mu_{11}+\delta_2(\alpha+\mu_{11})\right)(\alpha+\mu_{12})\delta-\lambda_1(1-\delta)\alpha\mu_{11}(\alpha+\mu_{11})]+\mu_{11}\alpha\delta[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\delta\mu_{11}\}\mu_2\mu_{12}-(1-\delta)\gamma_2\alpha\{\delta_1(1-q)(\alpha+\mu_2)[\alpha^2\delta(\alpha+\mu_{12})+\delta^2\mu_{11}\delta(\alpha+\mu_{12})-\lambda_1(1-\delta)\alpha(\alpha+\mu_{11})]+\mu_{11}\alpha[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\delta\}\mu_{11}\mu_{12}\mu_2-(1-\delta)\gamma_2\alpha^3\mu_{11}\mu_2\{\delta_1(1-q)(\alpha+\mu_2)[\alpha\mu_{12}\delta+\mu_{12}\alpha\mu_{11}\lambda_2(1-\delta)]+\mu_{11}\alpha[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\delta\mu_{12}\}-(1-\delta)\gamma_2\alpha\{\delta_1(1-q)(\alpha+\mu_2)\delta\left[\alpha^2\left((\alpha+\mu_{12})+\mu_{11}\delta_2\right)+\delta\delta_1\alpha(\alpha+\mu_{11})(\alpha+\mu_{12})(\alpha+\mu_2)\right]-\lambda_1(1-\delta)\,\alpha\mu_{11}(\alpha+\mu_{12})[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2]\}-(1-\delta)\gamma_2\alpha\{\delta_1(1-q)(\alpha+\mu_2)\,\delta\left[\alpha^2\left((\alpha+\mu_{12})+\mu_{11}\delta_2\right)+\delta\delta_1\alpha(\alpha+\mu_{11})(\alpha+\mu_{12})(\alpha+\mu_2)\right]-\lambda_2(1-\delta)\,\alpha\mu_{11}(\alpha+\mu_{12})[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2]\}-(1-\delta)\gamma_2\alpha^3\delta\{(\alpha+\mu_2)(1-q)\alpha\delta_1\left((\alpha+\mu_{12})+\mu_{11}\delta_2\right)+\alpha\mu_{11}[\delta_1^2(\alpha+\mu_{12})K+\mu_{12}\delta_1\delta_2^2]\}\mu_{11}\mu_2\mu_{12}-(1-\delta)\gamma_2\alpha^3\{(\alpha+\mu_2)(1-q)\alpha\delta_1\left((\alpha+\mu_{12})+\mu_{11}\delta_2\right)+\alpha\mu_{11}[\delta_1^2(\alpha+\mu_{12})\delta\mu_{12}-\lambda_2\delta_1\delta_2^2(1-\delta)]\}\mu_{11}\mu_2-(1-\delta)\gamma_2\alpha^3\mu_2\mu_{12}$
$\{(\alpha+\mu_2)(1-q)\alpha\delta_1\left((\alpha+\mu_{12})+\mu_{11}\delta_2\right)\mu_2+\delta_1\alpha^2\lambda_3(1-\delta)\alpha\ [\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\}$

9. The server's utilization factor is $\rho=1-Q=\frac{f_{17}}{f_{18}}$

where, $f_{17}=(1-\delta)\delta_1\beta\gamma_1\gamma_2\alpha(\alpha+\mu_{12})(\alpha+\mu_2)\delta_1[\lambda_1(1-q)+\lambda_0]+(1-\delta)\beta\gamma_1\gamma_2\alpha\mu_{11}\,\delta_1\delta_2(\alpha+\mu_2)[\lambda_2(1-q)+\lambda_0]+(1-\delta)[(\alpha+\mu_2)(\alpha+\mu_{11})-\mu_{11}\mu_2]\beta\gamma_1\gamma_2\,[\lambda_3+\lambda_0][\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]+(1-\delta)\alpha[\lambda_5+\lambda_0](1-r)\beta\gamma_2\{(\alpha+\mu_2)\alpha\delta_1\,(1-q)[(\alpha+\mu_{12})+\delta_2\mu_{11}]+\mu_{11}\alpha[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\}-p\alpha\gamma_1\gamma_2\lambda_0(1-\delta)\{(\alpha+\mu_2)(1-q)\mu_{11}\delta_1[(\alpha+\mu_{12})+\delta_2\mu_{11}]+\mu_{11}\mu_2\delta_1[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2]\}+(1-\delta)[\lambda_6+\lambda_0]\beta\gamma_1 r\{(\alpha+\mu_2)\alpha\delta_1(1-q)[(\alpha+\mu_{12})+\delta_2\mu_{11}]+\mu_{11}\alpha[\delta_1^2(\alpha+\mu_{12})+\mu_{12}\delta_1\delta_2^2]\}$

$f_{18}=\{(\alpha+\mu_2)(1-q)\mu_{11}\delta_1[(\alpha+\mu_{12})+\delta_2\mu_{11}]+\mu_{11}\mu_2\delta_1[\delta_1(\alpha+\mu_{12})+\mu_{12}\delta_2^2]\}\,\alpha\gamma_1\gamma_2[\beta\delta K-p(1-\delta)\lambda_4]$

10. Mean waiting time of a customer

$W = \frac{S}{\lambda^*}$(Little's formula)

where, $\lambda^* = \lambda_0 Q + \lambda_1 P_1^{(1)}(1) + \lambda_2 P_1^{(2)}(1) + \lambda_3 P_2(1) + \lambda_4 V(1) + \lambda_5 R_1(1) + \lambda_6 R_2(1)$

$$P_1^{(1)}(1) = \frac{\delta_1 \lambda_0 Q(1-\delta)\alpha\beta\gamma(\alpha+\mu_2)(\alpha+\mu_{12})}{f_{10}}$$

$$P_1^{(2)}(1) = \frac{\delta_1 \delta_2 \lambda_0 Q(1-\delta)\alpha\beta\gamma_1\gamma_2\mu_{11}(\alpha+\mu_2)}{f_{10}}$$

$$P_2(1) = \frac{\lambda_0 Q(1-\delta)\alpha\beta\gamma_1\gamma_2\mu_{11}[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}]}{f_{10}}$$

$$V(1) = \frac{\lambda_0 Q(1-\delta)\gamma_1\gamma_2 p f_{19}}{f_{10}}$$

$$f_{19} = \{\alpha\mu_{11}(1-q)\delta_1(\alpha+\mu_2)[(\alpha+\mu_{12})+\delta_2\mu_{12}] + \alpha\mu_{11}\mu_2\delta_1[(\alpha+\mu_{12})\delta_1+\delta_2^2\mu_{12}]\}$$

$$R_1(1) = \frac{\lambda_0 Q(1-\delta)\alpha\beta\gamma_2(1-r)f_{20}}{f_{10}}$$

$$R_2(1) = \frac{\lambda_0 Q(1-\delta)\beta\gamma_1 r f_{20}}{f_{10}}$$

$$f_{20} = \{\alpha(1-q)\delta_1(\alpha+\mu_2)[(\alpha+\mu_{12})+\delta_2\mu_{11}] + \alpha\mu_{11}[(\alpha+\mu_{12})\delta_1^2+\delta_2^2\delta_1\mu_{12}]\}$$

By varying the parameter values, and fixing some parameters, the performance values are calculated and are presented in the tables (7.1 to 7.6). The mean waiting time presented as graphs in the figures (7.7 to 7.12).

Table 7.1: Performance measures $\lambda_1 = 2.0, \lambda_2 = 1.8, \lambda_3 = 1.3, \lambda_4 = 1.4, \lambda_5 = 1.9, \lambda_6 = 1.8,$ $\alpha = 1.4, \beta = 0.8, \gamma_1 = 1.7, \gamma_2 = 1.3, \delta = 0.32, \delta_1 = 0.06, \delta_2 = 0.74, q = 0.92, p = 0.39,$ $r = 0.85, K = 18, z_0 = 0.89, \mu_{11} = 2.2, \mu_{12} = 1.3, \mu_2 = 1.8$

λ_0	ρ	Q	N_{11}	N_{12}	N_2	N_3	N_4	N_5	S
1.1	0.6137	0.5856	4.2298	2.8164	43.3124	0.0067	2.1909	12.5744	6.9315
1.2	0.6463	0.5643	4.5819	2.9609	45.5345	0.0071	2.3033	13.2195	7.1927
1.3	0.6789	0.5446	4.9310	3.0953	47.6009	0.0074	2.4078	13.8195	7.4538
1.4	0.7114	0.5261	5.2775	3.2206	49.5274	0.0077	2.5053	14.3787	7.7149
1.5	0.7440	0.5089	5.6216	3.3376	51.3277	0.0080	2.5964	14.9014	7.9761
1.6	0.7766	0.4928	5.9635	3.4473	53.0139	0.0082	2.6816	15.3910	8.2372
1.7	0.8091	0.4776	6.3035	3.5502	54.5965	0.0085	2.7617	15.8504	8.4983
1.8	0.8417	0.4634	6.6417	3.6470	56.0847	0.0087	2.8370	16.2825	8.7595
1.9	0.8743	0.4500	6.9782	3.7381	57.4868	0.0089	2.9079	16.6895	9.0206
2.0	0.9068	0.4373	7.3133	3.8242	58.8100	0.0091	2.9748	17.0737	9.2817

Table 7.2: Performance measures $\lambda_0 = 2.0, \lambda_1 = 1.8, \lambda_2 = 1.3, \lambda_3 = 1.4,$

$\lambda_4 = 1.9, \lambda_5 = 1.8, \lambda_6 = 2.2, \alpha = 1.4, \beta = 0.8, \gamma_1 = 1.7, \gamma_2 = 1.3, \delta = 0.5, \delta_1 = 0.06,$
$\delta_2 = 0.74, q = 0.92, p = 0.39, r = 0.85, K = 18, z_0 = 0.89, \mu_{12} = 1.3, \mu_2 = 1.8$

μ_{11}	ρ	Q	N_{11}	N_{12}	N_2	N_3	N_4	N_5	S
1.1	0.8295	0.5901	3.6383	1.7580	47.5370	0.0123	1.5410	8.8012	4.2264
1.2	0.7722	0.6042	3.7575	1.7286	48.2873	0.0251	1.5363	8.7760	4.4782
1.3	0.7236	0.6165	3.8726	1.7083	48.9450	0.0358	1.5334	8.7614	4.7175
1.4	0.6820	0.6273	3.9845	1.6940	49.5260	0.0449	1.5319	8.7540	4.9476
1.5	0.6459	0.6369	4.0937	1.6838	50.0431	0.0527	1.5313	8.7515	5.1703
1.6	0.6144	0.6455	4.2006	1.6766	50.5061	0.0596	1.5313	8.7525	5.3873
1.7	0.5865	0.6532	4.3056	1.6715	50.9231	0.0656	1.5317	8.7558	5.5996
1.8	0.5618	0.6602	4.4089	1.6680	51.3007	0.0710	1.5325	8.7608	5.8083
1.9	0.5396	0.6665	4.5108	1.6656	51.6440	0.0758	1.5334	8.7668	6.0138
2.0	0.5197	0.6723	4.6115	1.6640	51.9577	0.0802	1.5345	8.7737	6.2167

Table 7.3: Performance measures $\lambda_0 = 2.0, \lambda_1 = 1.8, \lambda_2 = 1.3, \lambda_3 = 1.4,$
$\lambda_4 = 1.9, \lambda_5 = 1.8, \lambda_6 = 2.2, \alpha = 1.4, \beta = 0.8, \gamma_1 = 1.7, \gamma_2 = 1.3, \delta = 0.5, \delta_1 = 0.06,$
$\delta_2 = 0.74, q = 0.92, p = 0.39, r = 0.85, K = 18, z_0 = 0.89, \mu_{11} = 1.3, \mu_2 = 1.8$

μ_{12}	ρ	Q	N_{11}	N_{12}	N_2	N_3	N_4	N_5	S
1.1	0.8288	0.6013	3.8570	1.9697	44.3046	0.0330	1.5125	8.6432	4.4114
1.2	0.7728	0.6093	3.8625	1.8285	46.8569	0.0345	1.5233	8.7040	4.6135
1.3	0.7236	0.6165	3.8726	1.7083	48.9450	0.0358	1.5334	8.7614	4.8129
1.4	0.6801	0.6229	3.8869	1.6044	50.6782	0.0368	1.5430	8.8154	5.0102
1.5	0.6412	0.6288	3.9046	1.5135	52.1352	0.0376	1.5519	8.8658	5.2057
1.6	0.6065	0.6341	3.9253	1.4331	53.3734	0.0384	1.5603	8.9131	5.3997
1.7	0.5751	0.6389	3.9486	1.3614	54.4360	0.0390	1.5681	8.9572	5.5924
1.8	0.5467	0.6434	3.9742	1.2970	55.3558	0.0395	1.5754	8.9985	5.7841
1.9	0.5209	0.6475	4.0018	1.2387	56.1581	0.0400	1.5823	9.0372	5.9749
2.0	0.4974	0.6512	4.0311	1.1856	56.8629	0.0404	1.5887	9.0735	6.1649

Table 7.4: Performance measures $\lambda_0 = 2.0, \lambda_1 = 1.8, \lambda_2 = 1.3, \lambda_3 = 1.4,$
$\lambda_4 = 1.9, \lambda_5 = 1.8, \lambda_6 = 2.2, \alpha = 1.4, \beta = 0.8, \gamma_1 = 1.7, \gamma_2 = 1.3, \delta = 0.5, \delta_1 = 0.06,$
$\delta_2 = 0.74, q = 0.92, p = 0.39, r = 0.85, K = 18, z_0 = 0.89, \mu_{11} = 1.3, \mu_{12} = 1.8$

μ_2	ρ	Q	N_{11}	N_{12}	N_2	N_3	N_4	N_5	S
1.1	0.4652	0.6073	4.4470	1.4501	52.7870	0.0075	2.1734	12.4171	4.2404
1.2	0.4774	0.6143	4.3560	1.4206	53.3703	0.0154	2.0598	11.7675	4.4658
1.3	0.4894	0.6205	4.2750	1.3944	53.8615	0.0217	1.9579	11.1850	4.6887
1.4	0.5012	0.6260	4.2023	1.3709	54.2751	0.0268	1.8661	10.6602	4.9097
1.5	0.5129	0.6310	4.1368	1.3497	54.6227	0.0309	1.7830	10.1852	5.1290
1.6	0.5243	0.6355	4.0775	1.3305	54.9137	0.0343	1.7075	9.7534	5.3469
1.7	0.5356	0.6396	4.0235	1.3130	55.1560	0.0371	1.6385	9.3594	5.5638
1.8	0.5467	0.6434	3.9742	1.2970	55.3558	0.0395	1.5754	8.9985	5.7796
1.9	0.5577	0.6468	3.9289	1.2823	55.5184	0.0415	1.5174	8.6669	5.9946
2.0	0.5685	0.6500	3.8872	1.2688	55.6483	0.0432	1.4639	8.3612	6.2089

Table 7.5: Performance measures $\lambda_0 = 2.0, \lambda_1 = 1.8, \lambda_2 = 1.3, \lambda_3 = 1.4, \lambda_4 = 1.9, \lambda_5 = 1.8, \lambda_6 = 2.2, \alpha = 1.4, \beta = 0.8, \gamma_2 = 1.3, \delta = 0.5, \delta_1 = 0.06, \delta_2 = 0.74, q = 0.92, p = 0.39, r = 0.85, K = 18, z_0 = 0.89, \mu_{11} = 1.3, \mu_{12} = 1.8, \mu_2 = 1.7$

γ_1	ρ	Q	N_{11}	N_{12}	N_2	N_3	N_4	N_5	S
1.1	0.5834	0.6148	4.2005	1.4174	57.9953	0.0285	1.7767	10.0082	3.5891
1.2	0.5808	0.6153	4.2016	1.4176	57.9981	0.0285	1.7709	10.0087	3.9440
1.3	0.5787	0.6157	4.2025	1.4178	58.0005	0.0285	1.7660	10.0091	4.2988
1.4	0.5769	0.6160	4.2033	1.4180	58.0025	0.0285	1.7618	10.0094	4.6536
1.5	0.5753	0.6163	4.2040	1.4181	58.0043	0.0285	1.7582	10.0097	5.0082
1.6	0.5739	0.6166	4.2046	1.4183	58.0058	0.0285	1.7550	10.0100	5.3628
1.7	0.5727	0.6168	4.2051	1.4184	58.0072	0.0285	1.7522	10.0103	5.7174
1.8	0.5716	0.6170	4.2056	1.4185	58.0084	0.0285	1.7497	10.0105	6.0719
1.9	0.5706	0.6172	4.2060	1.4186	58.0095	0.0285	1.7474	10.0106	6.4264
2.0	0.5698	0.6173	4.2064	1.4186	58.0105	0.0285	1.7454	10.0108	6.7808

Table 7.6: Performance measures $\lambda_0 = 2.0, \lambda_1 = 1.8, \lambda_2 = 1.3, \lambda_3 = 1.4, \lambda_4 = 1.9, \lambda_5 = 1.8, \lambda_6 = 2.2, \alpha = 1.4, \beta = 0.8, \gamma_1 = 1.3, \delta = 0.5, \delta_1 = 0.06, \delta_2 = 0.74, q = 0.92, p = 0.39, r = 0.85, K = 18, z_0 = 0.89, \mu_{11} = 1.3, \mu_{12} = 1.8, \mu_2 = 1.7$

γ_2	ρ	Q	N_{11}	N_{12}	N_2	N_3	N_4	N_5	S
1.1	0.6017	0.6119	4.1925	1.4169	58.0215	0.0285	1.7666	10.0742	3.5967
1.2	0.5893	0.6140	4.1980	1.4174	58.0101	0.0285	1.7663	10.0389	3.9429
1.3	0.5787	0.6157	4.2025	1.4178	58.0005	0.0285	1.7660	10.0091	4.2887
1.4	0.5697	0.6172	4.2065	1.4182	57.9922	0.0285	1.7657	9.9835	4.6340
1.5	0.5618	0.6184	4.2098	1.4185	57.9850	0.0285	1.7655	9.9614	4.9790
1.6	0.5550	0.6196	4.2128	1.4188	57.9788	0.0284	1.7653	9.9420	5.3238
1.7	0.5489	0.6206	4.2154	1.4190	57.9733	0.0284	1.7652	9.9250	5.6684
1.8	0.5436	0.6214	4.2177	1.4192	57.9683	0.0284	1.7650	9.9098	6.0128
1.9	0.5387	0.6222	4.2198	1.4194	57.9640	0.0284	1.7649	9.8962	6.3570
2.0	0.5344	0.6229	4.2216	1.4196	57.9600	0.0284	1.7648	9.8840	6.7011

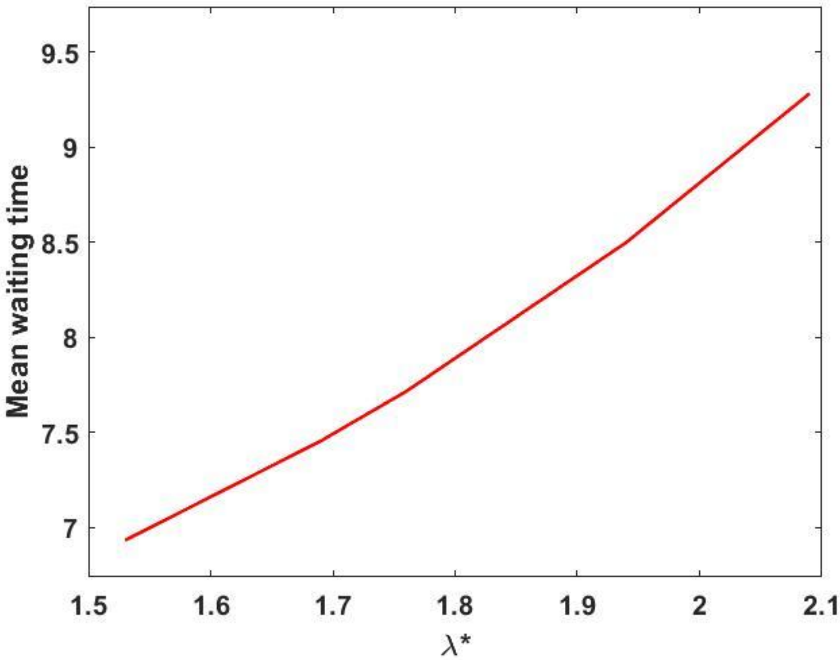

Figure 7.7: Mean waiting time: $\lambda_1 = 1.9, \lambda_2 = 2.9, \lambda_3 = 1.5, \lambda_4 = 1.4,$ $\lambda_5 = 1.9, \lambda_6 = 1.8, \alpha = 1.4, \beta = 0.8, \gamma_1 = 1.7, \gamma_2 = 1.3, \delta = 0.32, \delta_1 = 0.06, \delta_2 = 0.74,$ $r = 0.85, q = 0.92, p = 0.39, K = 18, z_0 = 0.89, \mu_{11} = 2.2, \mu_{12} = 1.3, \mu_2 = 1.8$

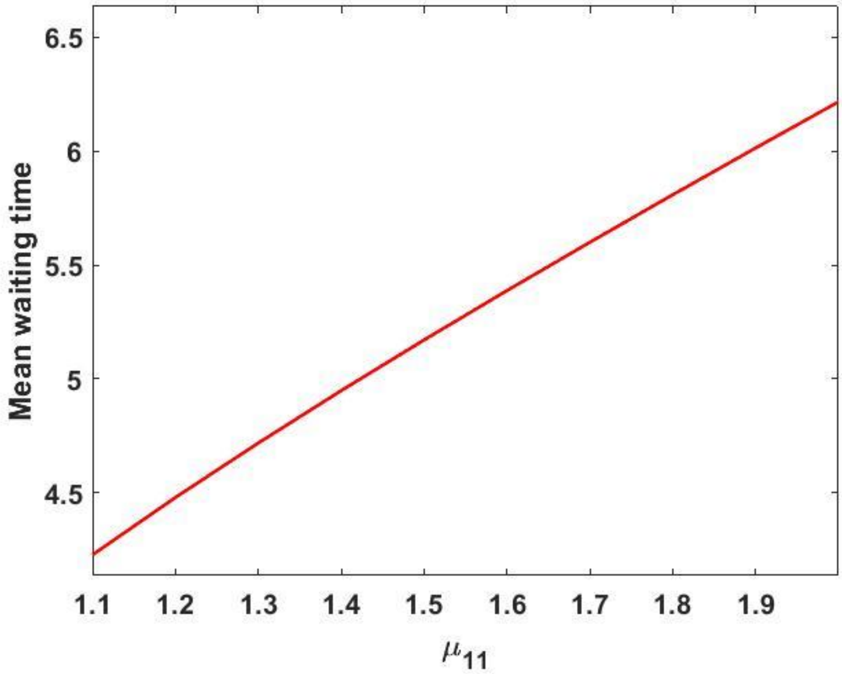

Figure 7.8: Mean waiting time: $\lambda_0 = 1.9, \lambda_1 = 2.9, \lambda_2 = 1.5, \lambda_3 = 1.4,$
$\lambda_4 = 1.9, \lambda_5 = 1.8, \lambda_6 = 2.2, \alpha = 1.4, \beta = 0.8, \gamma_1 = 1.7, \gamma_2 = 1.3, \delta = 0.5, \delta_1 = 0.06,$
$\delta_2 = 0.74, r = 0.85, q = 0.92, p = 0.39, K = 18, z_0 = 0.89, \mu_{12} = 1.3, \mu_2 = 1.8$

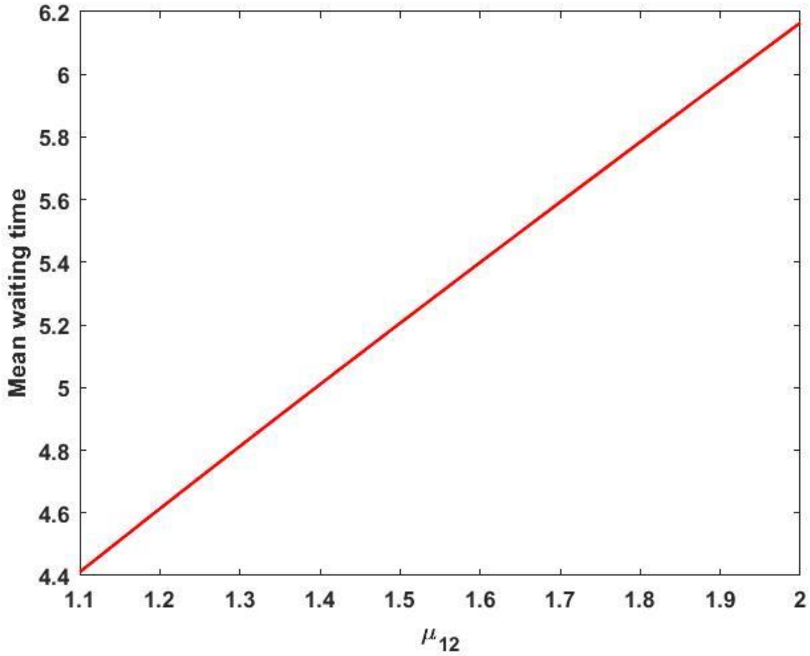

Figure 7.9: Mean waiting time: $\lambda_0 = 1.9, \lambda_1 = 2.9, \lambda_2 = 1.5, \lambda_3 = 1.4,$
$\lambda_4 = 1.9, \lambda_5 = 1.8, \lambda_6 = 2.2, \alpha = 1.4, \beta = 0.8, \gamma_1 = 1.7, \gamma_2 = 1.3, \delta = 0.5, \delta_1 = 0.06,$

$\delta_2 = 0.74, r = 0.85, q = 0.92, p = 0.39, K = 18, z_0 = 0.89, \mu_{11} = 1.3, \mu_2 = 1.8$

In the figures for arrival rates, service rates and repair rates, the mean waiting time is calculated and the graphs are shown in these figures 7.7 to 7.12.

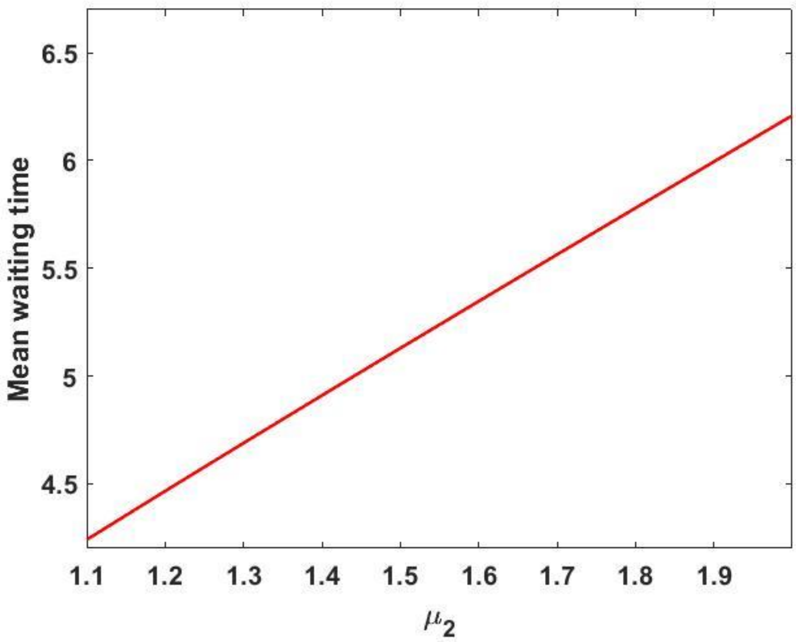

Figure 7.10: Mean waiting time: $\lambda_0 = 1.9, \lambda_1 = 2.9, \lambda_2 = 1.5, \lambda_3 = 1.4,$
$\lambda_4 = 1.9, \lambda_5 = 1.8, \lambda_6 = 2.2, \alpha = 1.4, \beta = 0.8, \gamma_1 = 1.7, \gamma_2 = 1.3, \delta = 0.5, \delta_1 = 0.06,$
$\delta_2 = 0.74, r = 0.85, q = 0.92, p = 0.39, K = 18, z_0 = 0.89, \mu_{11} = 1.3, \mu_{12} = 1.8$

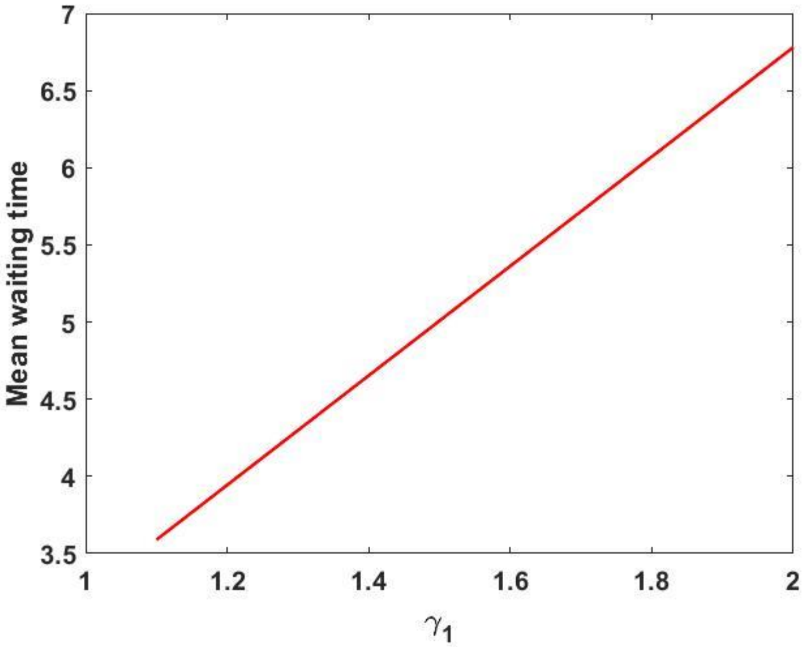

Figure 7.11: Mean waiting time: $\lambda_0 = 1.9, \lambda_1 = 2.9, \lambda_2 = 1.5, \lambda_3 = 1.4, \lambda_4 = 1.9, \lambda_5 = 1.8, \lambda_6 = 2.2, \alpha = 1.4, \beta = 0.8, \gamma_2 = 1.3, \delta = 0.5, \delta_1 = 0.06, \delta_2 = 0.74, r = 0.85, q = 0.92, p = 0.39, K = 18, z_0 = 0.89, \mu_{11} = 1.3, \mu_{12} = 1.8, \mu_2 = 1.7$

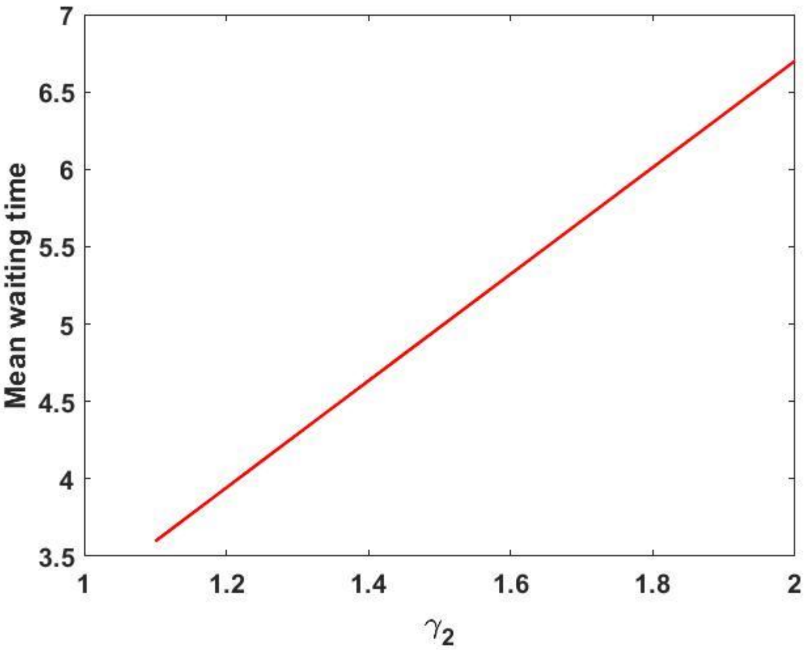

Figure 7.12: Mean waiting time: $\lambda_0 = 1.9, \lambda_1 = 2.9, \lambda_2 = 1.5, \lambda_3 = 1.4, \lambda_4 = 1.9, \lambda_5 = 1.8, \lambda_6 = 2.2, \alpha = 1.4, \beta = 0.8, \gamma_1 = 1.3, \delta = 0.5, \delta_1 = 0.06, \delta_2 = 0.74, r = 0.85, q = 0.92, p = 0.39, K = 18, z_0 = 0.89, \mu_{11} = 1.3, \mu_{12} = 1.8, \mu_2 = 1.7$

8. Conclusion

This article deals with a single server non-Markovian batch arrival, fixed batch service queue. The server provides three type of services and the server takes Bernoulli vacation at each service completion epoch. The server may breakdown, and there are two types of repairs. Also, the arrival rates depend on the server state. This model is completely analyzed using supplementary variable technique. The results are derived in time independent domain. To show the practical applicability of the model some numerical examples are provided. The model can be extended by adding some more queue related characters, such as balking, reneging etc.

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