Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 47 No. 1 (2026)

Multi-Criteria Decision-Making Methods based on Aggregation Operators under Pythagorean Fuzzy Sets

P.Revathi[1] and M.Gowdhami[2]
Assistant professor,[1] Department of Mathematics,
Annamalai University, Annamalai nagar-608 002.
revathimathsau@gmail.com
Research Scholar[2],Department of Mathematics,
Annamalai University, Annamalai nagar-608 002.
gowdhami1984@gmail.com

Abstract

In this article, we present a new notion of Pythagorean fuzzy sets and define several fundamental operators, namely union, intersection, complement, score function, and accuracy function. In addition, we introduce various aggregation operators, including the Pythagorean Fuzzy Weighted Average (PFWA) operator, Pythagorean Fuzzy Weighted Geometric (PFWG) operator, Pythagorean Fuzzy Weighted Power Average (PFWPA) operator, and Pythagorean Fuzzy Weighted Power Geometric (PFWPG) operator. Furthermore, we formulate a multi-criteria decision-making (MCDM) problem and propose an algorithm based on these aggregation operators under the Pythagorean fuzzy set framework.

Keywords: Pythagorean Fuzzy Set, Score Function, Accuracy Function, Aggregation Operators, PFWA, PFWG, PFWPA, PFWPG, MCDM.

1 Introduction

The concept of a fuzzy set (FS) was introduced by Zadeh. Subsequently, Atanassov generalized the fuzzy set theory by proposing the intuitionistic fuzzy set (IFS) [1].

Later, Yager further extended this framework by introducing the Pythagorean fuzzy set (PFS). The PFS [2] represents vagueness using a membership degree (MD) function and a non-membership degree (NMD) function , where the sum of their squares lies between 0 and 1, that is,(i.e) $0 \leq \mu^2 + \vartheta^2 \leq 1$. In this article, we investigate the concept of Pythagorean fuzzy sets, along with entropy measures, score functions, accuracy functions, and various aggregation operators such as PFWA, PFWG, PFWPA, and PFWPG. Furthermore, an algorithm based on these aggregation operators under the Pythagorean fuzzy set framework is proposed for application in multi-criteria decision-making (MCDM) problems, supported by a suitable illustrative example [3].

2 preliminary

Definition 2.1. Fuzzy Set[2]
Let $\mathcal{U}$ be a non empty set collection of objects denote by y. Then an FS $\mathcal{F}$ is a set having the form $\mathcal{F} = [y, \mu(y), /y \in \mathcal{U}]$, where $\mu_{\mathcal{F}}$, represents the MD of $\mathcal{F}$ respectively. The mapping $\mu_{\mathcal{F}}, : \mathcal{U} \rightarrow [0,1]$.

Definition 2.2. Intuitionistic Fuzzy Set [10]
Let $\mathcal{U}$ be a non empty set collection of objects denote by y. Then an IFS $\mathcal{F}$ is a set having the form $\mathcal{F} = [y, \mu(y), \vartheta(y)/y \in \mathcal{U}]$, where $\mu_{\mathcal{F}}, \vartheta_{\mathcal{F}}$ represents the Membership degree and Non-Membership Degree of $\mathcal{F}$ respectively. The mapping $\mu_{\mathcal{F}}, \vartheta_{\mathcal{F}} : \mathcal{U} \rightarrow [0,1]$ and $0 \leq \mu_{\mathcal{F}} + \vartheta_{\mathcal{F}} \leq 1$. Then, there degree of indeterminacy of $y \in \mathcal{U}$ to $\mathcal{F}$ defined by $\pi_{\mathcal{F}} = 1 - [\mu_{\mathcal{F}} + \vartheta_{\mathcal{F}}]$, clearly that $\mu_{\mathcal{F}} + \vartheta_{\mathcal{F}} + \pi_{\mathcal{F}} = 1$, otherwise $\pi_{\mathcal{F}} = 0$ then $\mu_{\mathcal{F}} + \vartheta_{\mathcal{F}} = 1$.

Definition 2.3. Pythagorean Fuzzy Set[2]
Let $\mathcal{U}$ be a non empty set collection of objects denote by y. Then an IFS $\mathcal{F}$ is a set having the form $\mathcal{F} = [y, \mu(y), \vartheta(y)/y \in \mathcal{U}]$, where $\mu_{\mathcal{F}}, \vartheta_{\mathcal{F}}$ represents the MD and NMD of $\mathcal{F}$ respectively. The mapping $\mu_{\mathcal{F}}, \vartheta_{\mathcal{F}} : \mathcal{U} \rightarrow [0,1]$ and $0 \leq \mu_{\mathcal{F}}^2 + \vartheta_{\mathcal{F}}^2 \leq 1$.

Definition 2.4 (6)**.** Let $\mathcal{B} = [\mu(y), \vartheta(y)]$,$\mathcal{B}_1 = [\mu_1, \vartheta_1]$ and $\mathcal{B}_2 = [\mu_2, \vartheta_2]$ be the Pythagorean fuzzy sets and $\Phi \geq 0$, then the following operators hold
(i) $\mathcal{B}_1 \oplus \mathcal{B}_2 = (\sqrt[2]{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, (\vartheta_1\vartheta_2))$
(ii) $\mathcal{B}_1 \otimes \mathcal{B}_2 = ((\mu_1\mu_2), \sqrt[2]{\vartheta_1^{1/2} + \vartheta_2^{1/2} - \vartheta_1^{1/2}\vartheta_2^{1/2}})$
(iii)$\Phi\mathcal{B} = (\sqrt[2]{1 - (1 - \mu^2)^{\Phi}}, \vartheta^{\Phi})$
(iv) $\mathcal{B}^{\Phi} = (\mu^{\Phi}, \sqrt[2]{1 - (1 - \sqrt{\vartheta})^{\Phi}}$

3 Operation of Pythagorean Fuzzy Set

Definition 3.1. Let $\mathcal{B} = [\mu, \vartheta], \mathcal{B}_1 = [\mu_1, \vartheta_1], \mathcal{B}_2 = [\mu_2, \vartheta_2]$ be the Pythagorean fuzzy sets, then[2]
(i) Intersection:
$\mathcal{B}_1 \wedge \mathcal{B}_2 = min(\mu_1, \mu_2), max(\vartheta_1, \vartheta_2)$
(ii)Union:
$\mathcal{B}_1 \vee \mathcal{B}_2 = max(\mu_1, \mu_2), min(\vartheta_1, \vartheta_2)$
(iii) Complement
$(\mathcal{B}^C = [(\mu)^4, \vartheta]$
Note that $[\vartheta^2 + \mu^2] = (\vartheta)^2 + \mu^2$
$= (0.6)^2 + (0.5)2 = 0.61 < 1$
so, $(\mathcal{B})^C$ is a Pythagorean fuzzy set.
We know that ,
$(\mathcal{B}^C)^C = (\vartheta^2, \mu(y))^C = (\mu, \vartheta) = \mathcal{B}$.

Example 3.2. Let Assume that $\mathcal{B}_1 = (\mu_1 = 0.57, \vartheta_1 = 0.43) and \mathcal{B}_2 = (\mu_2 = 0.47, \vartheta_2 = 0.56)$ are the Pythagorean fuzzy sets , then

(i) $\mathcal{B}_1 \wedge \mathcal{B}_2 = min(\mu_1, \mu_2), max(\vartheta_1, \vartheta_2$
= (min(.57, .47), max(.43, .56))

= (.47,.56)

(ii) $\mathcal{B}_1 \vee \mathcal{B}_2 = min(\mu_1, \mu_2), max(\vartheta_1, \vartheta_2)$

= (max(.57, .47), min(.43, .56))

=(.57, .43)

(iii)$(\mathcal{B})^C = [(\mu)^2, \vartheta^2]^2$

$=((.32)^4), (.54)^4)$

=(.010,.085)

Definition 3.3. Score Function Let $H\mathcal{B} = (\mu_{\mathcal{B}}, \vartheta_{\mathcal{B}})$ be the PFS then the score function is written as $H(\mathcal{B}) = \mu_{\mathcal{B}}^2 - (\vartheta_{\mathcal{B}}^2)$ ......(I) and noted that $S(\mathcal{B}) \in$ [-1,1]

Example: Let $\mathcal{B} = [0.7, 0.6]$be the PFS then $H(\mathcal{B}) = 0.7 - 0.6 = 0.13$

Definition 3.4. Accuracy Function Let $\mathcal{B} = (\mu_{\mathcal{B}}, \vartheta_{\mathcal{B}})$ be the PFS then the score function is written as $A(\mathcal{B}) = \mu_{\mathcal{B}}^2 + \vartheta_{\mathcal{B}}^2$ and noted that $A(\mathcal{B})[0, 1]$

Example: Let $\mathcal{B} = [0.4, 0.8]$be the PFS then $A(\mathcal{B}) = 0.7 + 0.6 = 0.85$

Theorem 3.5. For three pythagorean fuzzy number $\mathcal{B}$, $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_2$ and Φ_1, Φ_2 ¿0 following are valid

(i) $\mathcal{B}_1 \wedge \mathcal{B}_2 = \mathcal{B}_2 \wedge \mathcal{B}_1$ (ii)$\mathcal{B}_1 \vee \mathcal{B}_2 = \mathcal{B}_2 \vee \mathcal{B}_1$

(iii)$\mathcal{B}_1 \wedge (\mathcal{B}_2 \wedge \mathcal{B}_2) = (\mathcal{B}_1 \wedge \mathcal{B}_2) \wedge \mathcal{B}_2$ (iv)$\mathcal{B}_1 \vee (\mathcal{B}_2 \vee \mathcal{B}_2) = (\mathcal{B}_1 \vee \mathcal{B}_2) \vee \mathcal{B}_2$

(v) $\Phi(\mathcal{B}_1 \vee \mathcal{B}_2) = \Phi\mathcal{B}_1 \vee \Phi\mathcal{B}_2$ (iv)$(\mathcal{B}_1 \vee \mathcal{B}_2)^{\Phi} = (\mathcal{B}_1)^{\Phi} \vee (\mathcal{B}_2)^{\Phi}$

Proof:

(i) $\mathcal{B}_1 \wedge \mathcal{B}_2 = min(\mu_1, \mu_2), max(\vartheta_1, \vartheta_2)$

$=min(\mu_2, \mu_1), max(\vartheta_2, \vartheta_1)$

$=\mathcal{B}_2 \wedge \mathcal{B}_1$

(ii) $\mathcal{B}_1 \vee \mathcal{B}_2 = max(\mu_1, \mu_2), min(\vartheta_1, \vartheta_2)$

$=max(\mu_2, \mu_1), min(\vartheta_2, \vartheta_1)$

$=\mathcal{B}_2 \vee \mathcal{B}_1$

(iii)$\mathcal{B}_1 \wedge (\mathcal{B}_2 \wedge \mathcal{B}_2) = \mathcal{B}_1 \wedge (min(\mu_2, \mu_2), max(\vartheta_2, \vartheta_2)$

$=min(\mu_1, (min(\mu_2, \mu_3))), max(\vartheta_1, (max(\vartheta_2, \vartheta_3)))$

$=min((\mu_1, \mu_2), min\mu_3), (max(\vartheta_1, \vartheta_2), max\vartheta_3))$

$=(min(\mu_1, \mu_2), max(\vartheta_1, \vartheta_2) \wedge \mathcal{B}_3$

$$=(\mathcal{B}_1 \wedge \mathcal{B}_2) \wedge \mathcal{B}_3$$

$$\text{(iv)}\mathcal{B}_1 \vee (\mathcal{B}_2 \vee \mathcal{B}_3) = \mathcal{B}_1 \vee (max(\mu_2(y), \mu_3(y)), min(\vartheta_2(y), \vartheta_3(y))$$

$$=max(\mu_1, (max(\mu_2(y), \mu_3(y)))), min(\vartheta_1, (min(\vartheta_2(y), \vartheta_3(y))))$$

$$=max((\mu_1, \mu_2(y)), max\mu_3(y)), (min(\vartheta_1, \vartheta_2(y)), min\vartheta_3(y)))$$

$$=(max(\mu_1(y), \mu_2(y)), min(\vartheta_1(y), \vartheta_2(y)) \vee \mathcal{B}_3$$

$$= (\mathcal{B}_1 \vee \mathcal{B}_2) \vee \mathcal{B}_3$$

$$\text{(v) } \Phi(\mathcal{B}_1 \vee \mathcal{B}_2) = \Phi(max(\mu_1(y), \mu_2(y)), min(\vartheta_1(y), \vartheta_2(y)))$$

$$= (\sqrt[2]{1 - (1 - max(\mu_1^2, \mu_2^2)^\Phi}), min(\vartheta_1^\Phi, \vartheta_2^\Phi))$$

$$\Phi\mathcal{B}_1 \vee \Phi\mathcal{B}_2 = (\sqrt[2]{1 - (1 - \mu_1^2)^\Phi}, \vartheta_1^\Phi) \vee (\sqrt[2]{1 - (1 - \mu_2^2)^\Phi}, \vartheta_2^\Phi)$$

$$=max(\sqrt[2]{1 - (1 - \mu_1^2)^\Phi}, \sqrt[2]{1 - (1 - \mu_2^2)^\Phi}, min(\vartheta_1^\Phi, \vartheta_2^\Phi))$$

$$=(\sqrt[2]{1 - (1 - max(\mu_1^2), \mu_2^2)^\Phi}, min(\vartheta_1^\Phi, \vartheta_2^\Phi))$$

$$=\Phi(\mathcal{B}_1 \vee \mathcal{B}_2)$$

Theorem 3.6. For two pythagorean fuzzy number $\mathcal{B}_1 = (\mu_1, \vartheta_1), \mathcal{B}_2 = (\mu_2, \vartheta_2)$ following are valid
(i) $(\mathcal{B}_1 \wedge \mathcal{B}_2) \vee \mathcal{B}_2 = \mathcal{B}_2$
(ii) $(\mathcal{B}_1 \vee \mathcal{B}_2) \wedge \mathcal{B}_2 = \mathcal{B}_2$
proof:
(i) $(\mathcal{B}_1 \wedge \mathcal{B}_2) \vee \mathcal{B}_2 = (min(\mu_1, \mu_2), max(\vartheta_1, \vartheta_2) \vee (\mu_2, \vartheta_2)$
$= (max(min(\mu_1, \mu_2), \mu_2), min(max(\vartheta_1, \vartheta_2), \vartheta_2))$
$= (\mu_2, \vartheta_2)$
$= \mathcal{B}_2$
(ii) $(\mathcal{B}_1 \vee \mathcal{B}_2) \wedge \mathcal{B}_2 = (max(\mu_1, \mu_2), min(\vartheta_1, \vartheta_2)) \wedge (\mu_2, \vartheta_2)$
$= (min(max(\mu_1, \mu_2), \mu_2), max(min(\vartheta_1, \vartheta_2), \vartheta_2))$
$= (\mu_2, \vartheta_2)$
$= \mathcal{B}_2$

Aggregation of Pythagorean Fuzzy Set[6]

Definition 3.7. Let $\mathcal{B}_z = (\mu_{\mathcal{B}z}, \vartheta_{\mathcal{B}z})$ (z =1 to n) be the Pythagorean Fuzzy Set and $\tau = (\tau_1, \tau_2, ...\tau_n)^T$ be the weight vector of $\mathcal{B}_z$ with $\sum_{z=1}^{n} \tau = 1$. Then a (PFWA) is a function PFWA: $\mathcal{B}^n \rightarrow \mathcal{B}$, where
PFWA $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) = [\sum_{z=1}^{n} \tau_z\mu_z, \sum_{z=1}^{n} \tau_z\vartheta_z]$......(1)

Definition 3.8. Let $\mathcal{B}_z = (\mu_{\mathcal{B}z}, \vartheta_{\mathcal{B}z})$ (z =1 to n) be the Pythagorean Fuzzy Set and $\tau = (\tau_1, \tau_2, ...\tau_n)^T$ be the weight vector of $\mathcal{B}_z$ with $\sum_{z=1}^{n} \tau = 1$. Then a (PFWG) is a function PFWG: $\mathcal{B}^n \rightarrow \mathcal{B}$, where
PFWG $(\mathcal{B}_1, \mathcal{B}2, \mathcal{B}_3, ..., \mathcal{B}_n) = [\prod_{z=1}^{n} \mu_z^{\tau_z}, \prod_{z=1}^{n} \vartheta_z^{\tau_z}]$......(2)

Definition 3.9. Let $\mathcal{B}_z = (\mu_{\mathcal{B}z}, \vartheta_{\mathcal{B}_z})$ (z =1 to n) be the Pythagorean Fuzzy Set and $\tau = (\tau_1, \tau_2, ...\tau_n)^T$ be the weight vector of $\mathcal{B}_z$ with $\sum_{z=1}^{n} \tau = 1$. Then a (PFWPA) is a function PFWPA: $\mathcal{B}^n \rightarrow \mathcal{B}$, where
PFWPA $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) = [(\sum_{z=1}^{n} \tau_z\mu_z^2), (\sum_{z=1}^{n} \tau_z\vartheta_z^2)]$......(3)

Definition 3.10. Let $\mathcal{B}_z = (\mu_{\mathcal{B}z}, \vartheta_{\mathcal{B}z})$ (z =1 to n) be the Pythagorean Fuzzy Set and $\tau = (\tau_1, \tau_2, ...\tau_n)^T$ be the weight vector of $\mathcal{B}_z$ with $\sum_{z=1}^{n} \tau = 1$. Then a (PFWPG) is a function PFWPG: $\mathcal{B}^n \rightarrow \mathcal{B}$, where
PFWPG $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) = [((1 - \prod_{z=1}^{n} \mu_z^2)^{\tau_z}), ((1 - \prod_{z=1}^{n} \vartheta_z^2)^{\tau_z})]$......(4)

Example 3.11. *Let $\mathcal{B}_1$ =(0.5,0.2),$\mathcal{B}_2$=(0.6,0.7), $\mathcal{B}_3$=(0.5,0.9) be three PFS and assume that τ=(0.2, 0.4, 0.4)^T is weight vector of $\mathcal{B}_z$ (z=1 to n), then*

(1)PFWA $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) = [\sum_{z=1}^{n} \tau_z\mu_z, \sum_{z=1}^{n} \tau_z\vartheta_z]$
$= (0.5 \times 0.2 + 0.6 \times 0.4 + 0.5 \times 0.4), (0.2 \times 0.2 + 0.7 \times 0.4 + 0.5 \times 0.4)$

= (0.64, 0.6)

(2)PFWG $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_2, ..., \mathcal{B}_n) = [\prod_{z=1}^{n} \mu_z^{\tau_z}, \prod_{z=1}^{n} \vartheta_z^{\tau_z}]$
$= (0.5 \times 0.2 \times 0.6 \times 0.4 \times 0.5 \times 0.6), (0.2 \times 0.2 \times 0.7 \times 0.4 \times 0.9 \times 0.6)$
= (0.46,0.58)

(3)PFWPA $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) = [(\sum_{z=1}^{n} \tau_z\mu_z)^2, (\sum_{z=1}^{n} \tau_z\vartheta_z)^2]$
$= (0.2 \times 0.5^2 + 0.4 \times 0.4^2 + 0.4 \times 0.5^2)^2, (0.2 \times 0.2^2 + 0.4 \times 0.7^2 + 0.4 \times 0.9^2)^2$
= (0.0457, 0.36)

(4)PFWPG $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) = [((1 - \prod_{z=1}^{n} \mu_z^2)^{\tau_z}), ((1 - \prod_{z=1}^{n} \vartheta_z^2)^{\tau_z})]$
$= ((1 - (1 - 0.5))^{0.2}(1 - (1 - 0.6))^{0.4}(1 - (1 - 0.5))^{0.4})^2,$

$((1 - (1 - (0.2)^{0.2}((1 - (1 - 0.7)^{0.4}((1 - (1 - (0.9))^{0.4})$

=(0.013, 0.486)

Theorem 3.12. *Let* $\mathcal{B}_z = (\mu_{\mathcal{B}z}, \vartheta_{\mathcal{B}z})$ *(z =1 to n) be the Pythagorean Fuzzy Set and* $\tau = (\tau_1, \tau_2, ...\tau_n)^T$ *be the weight vector of* $\mathcal{B}_z$ *with* $\tau_z > 0$ *and* $\sum_{z=1}^{n} \tau_z = 1$. *Then a (PFWA) is a function PFWPA:* $\mathcal{B}^n \to \mathcal{B}$, *where*
PFWA $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n)$ *=is an PFS.*
Proof:
For any PFS $\mathcal{B}_z = (\mu_{\mathcal{B}z}, \vartheta_{\mathcal{B}z})$, *we have* $0 \le \mu_{\mathcal{B}z}^2 \le 1, 0 \le \sqrt[2]{\vartheta}_{\mathcal{B}z} \le 1$

$$0 \le \mu_{\mathcal{B}z}^2 + \vartheta_{\mathcal{B}z}^2 \le 1$$

Then , we given that

$$0 \le \tau_1 \mu_{\mathcal{B}1}^2 + \tau_1 \vartheta_{\mathcal{B}1}^2 \le \tau_1,$$
$$0 \le \tau_2 \mu_{\mathcal{B}2}^2 + \tau_2 + \vartheta_{\mathcal{B}2}^2 \le \tau_2,$$
$$.................................,$$
$$0 \le \tau_n \mu_{\mathcal{B}n}^2 + \tau_n \vartheta_{\mathcal{B}n}^2 \le \tau_n$$

And also,
$0(\le \tau_1 \mu_{\mathcal{B}1}^2 + \tau_1 \vartheta_{\mathcal{B}1}^2) + (\tau_2 \mu_{\mathcal{B}2}^2 + \tau_2 \vartheta_{\mathcal{B}2}^2) + ... + (\tau_n \mu_{\mathcal{B}n}^2 + \tau_n \vartheta_{\mathcal{B}n}^2) \le (\tau_1 + \tau_2 + ... + \tau_n)$

$0 \le \sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}z}^2 + \sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}z}^2 \le \sum_{z=1}^{n} \tau_z = 1$
Therefore
$0 \le ((\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}_z}^2)^2 + ((\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}_z}^2)^2) \le \sum_{z=1}^{n} \tau_z = 1$
$= \sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}z}^2 + \sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}z}^2 = 1$
it is obvious that,
$0 \le ((\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}z}^2)) \le 1, 0 \le ((\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}z})^2) \le 1$
Then PFWPA $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n)$ *=is an PFS.*

Theorem 3.13. *Let* $\mathcal{B}_z = (\mu_{\mathcal{B}_z}, \vartheta_{\mathcal{B}_z})$ *(z =1 to n) and* $\mathcal{B} = (\mu_{\mathcal{B}}, \vartheta_{\mathcal{B}})$ *be the Pythagorean Fuzzy Sets and* $\tau = (\tau_1, \tau_2, ...\tau_n)^T$ *be the weight vector of* $\mathcal{B}_z$ *with* $\sum_{z=1}^{n} \tau_z = 1$. *Then a PFWPA* $(\mathcal{B}_1 \bigoplus \mathcal{B}_2 \bigoplus \mathcal{B}_3 \bigoplus ... \bigoplus \mathcal{B}_n \bigoplus \mathcal{B}) \ge$*PFWPA* $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) \bigotimes \mathcal{B}$.
Proof:
Let $\mathcal{B}_z = (\mu_{\mathcal{B}_z}, \vartheta_{\mathcal{B}_z})$ *(z =1 to n) and* $\mathcal{B} = (\mu_{\mathcal{B}}, \vartheta_{\mathcal{B}})$ *be the Pythagorean Fuzzy Sets, we have*
$[\sum_{z=1}^{n} \tau_z (\mu_{\mathcal{B}_z}^2 + \mu_{\mathcal{B}}^2 - \mu_{\mathcal{B}_z}^2 \mu_{\mathcal{B}}^2)] \ge [\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}_z}^2 \mu_{\mathcal{B}}^2]^2 = (\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}_z}^2)^2 \mu_{\mathcal{B}}$
$[\sum_{z=1}^{n} \tau_z (\vartheta_{\mathcal{B}_z}^2 + \vartheta_{\mathcal{B}}^2 - \vartheta_{\mathcal{B}_z}^2 \vartheta_{\mathcal{B}}^2)] \ge [\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}_z}^2 \vartheta_{\mathcal{B}}^2]^2 = \sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}_z}^2 \mu_{\mathcal{B}}^2$
We have,
PFPA $(\mathcal{B}_1 \bigoplus \mathcal{B}_2 \bigoplus \mathcal{B}_3 \bigoplus ... \bigoplus \mathcal{B}_n \bigoplus \mathcal{B}) = ([\sum_{z=1}^{n} \tau_z (\mu_{\mathcal{B}_z}^2 + \mu_{\mathcal{B}}^2 - \mu_{\mathcal{B}_z}^2 \mu_{\mathcal{B}}^2)], [\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}_z}^2 \vartheta_{\mathcal{B}}^2])$
PFWPA $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) \bigotimes \mathcal{B} = ((\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}_z}^2)), (\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}_z}^2))) \bigotimes (\mu_{\mathcal{B}}, \vartheta_{\mathcal{B}})$

$=((\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}_z}^2)\mu_{\mathcal{B}}^2, \vartheta_{\mathcal{B}_z}^2 + \vartheta_{\mathcal{B}}^2 - \vartheta_{\mathcal{B}_z}\vartheta_{\mathcal{B}}))$

It follows that

$PFWPA\ (\mathcal{B}_1 \bigoplus \mathcal{B}_2 \bigoplus \mathcal{B}_3 \bigoplus ... \bigoplus \mathcal{B}_n \bigoplus \mathcal{B}) \geq PFWPA\ (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) \bigotimes \mathcal{B}.$

Theorem 3.14. *(Boundedness)*

Suppose that $\mu_{\mathcal{B}}^* = min_{1\leq z\leq n}(\mu_{\mathcal{B}_z}), \mu_{\mathcal{B}}^\epsilon = max_{1\leq z\leq n}(\mu_{\mathcal{B}_z}), \vartheta_{\mathcal{B}}^* = min_{1\leq z\leq n}(\vartheta_{\mathcal{B}_z}), \vartheta_{\mathcal{B}}^\epsilon = max_{1\leq z\leq n}(\vartheta_{\mathcal{B}_z})$ *then* $(\mu_{\mathcal{B}}^*, \vartheta_{\mathcal{B}}^\epsilon) \leq C\mathcal{B}FWPA(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) \leq (\mu_{\mathcal{B}}^\epsilon, \vartheta_{\mathcal{B}}^*)$.

Proof: For any $\mathcal{B}_z = (\mu_{\mathcal{B}_z}, \vartheta_{\mathcal{B}_z})$*(z= 1 to n),*

We have $\mu_{\mathcal{B}}^* \leq \mu_{\mathcal{B}_z} \leq \mu_{\mathcal{B}}^\epsilon$ *and* $\vartheta_{\mathcal{B}}^\epsilon \leq \vartheta_{\mathcal{B}_z} \leq \vartheta_{\mathcal{B}}^*$

The inequatities for MD value are

$\mu_{\mathcal{B}}^* = [\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}_z}^{*2}]^2 \leq [\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}_z}^2]^2 \leq [\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}_z}^{\epsilon 2}]^2 = \mu_{\mathcal{B}}^\epsilon$

Similarly for NMD value are

$\vartheta_{\mathcal{B}}^* = [\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}_z}^*] \leq [\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}_z}^2]^2 \leq [\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}_z}^\epsilon] = \vartheta_{\mathcal{B}}^\epsilon$

Hence $PFWPA\ (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) \leq (\mu_{\mathcal{B}}^\epsilon, \vartheta_{\mathcal{B}}^*)$.

Theorem 3.15. *(Monotonicity)*

Let $\mathcal{B}_Z = (\mu_{\mathcal{B}_Z}, \vartheta_{\mathcal{B}_Z})$ *and* $\mathcal{E}_Z = (\mu_{\mathcal{E}_Z}, \vartheta_{\mathcal{E}_Z})$*(z= 1 to n) be the two PFS. If* $\mu_{\mathcal{B}_Z} \leq \mu_{\mathcal{E}_Z}$ *and* $\vartheta_{\mathcal{B}_Z} \leq \vartheta_{\mathcal{E}_Z}$ *then,*

$PFWPA\ (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) \leq PFWPA\ (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, , ..., \mathcal{E}_n,)$

Proof:

For all z, we have,

$\mu_{\mathcal{B}_Z} \leq \mu_{\mathcal{E}_Z}$ *and* $\vartheta_{\mathcal{B}_Z} \leq \vartheta_{\mathcal{E}_Z}$ *then* $[\sum_{z=1}^{n} \tau_Z \mu_{\mathcal{B}_Z}^2] \leq [\sum_{z=1}^{n} \tau_Z \mu_{\mathcal{E}_Z}^2]$ *and*
$[\sum_{z=1}^{n} \tau_Z \vartheta_{\mathcal{B}_Z}^2] \leq [\sum_{z=1}^{n} \tau_Z \vartheta_{\mathcal{E}_Z,}^2]$

Therefore

$PFWPA\ (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) \leq ([\sum_{z=1}^{n} \tau_Z \mu_{\mathcal{B}_Z}^2], [\sum_{z=1}^{n} \tau_Z \vartheta_{\mathcal{B}_Z}^2])$
$\leq ([\sum_{z=1}^{n} \tau_Z \mu_{\mathcal{E}_Z,}^2], [\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{E}_Z}]^2)$
$=PFWPA\ (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, ..., \mathcal{E}_n)$

Hence $FWPA\ (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) \leq PFWPA\ (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, , ..., \mathcal{E}_n,)$

Theorem 3.16. *(Idempotency)*

Let $\mathcal{B}_Z = (\mu_{\mathcal{B}_Z}, \vartheta_{\mathcal{B}_Z})$ *(z= 1 to n) than Fuzzy Sets such that* $\mathcal{B}_Z = \mathcal{B} = (\mu_{\mathcal{B}}, \vartheta_{\mathcal{B}})$ *and* $\tau = (\tau_1, \tau 2, ..., \tau_n)^T$ *be the weight vectors of* $\mathcal{B}_z$ *with then* $PFWPA\ (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) = \mathcal{B}$

Proof:

Since $\mathcal{B}_Z = \mathcal{B} = (\mu_{\mathcal{B}}, \vartheta_{\mathcal{B}})$*(z= 1 to n) then*

$PFWPA\ (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) = ([\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}z}^2], [\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}_z}^2])$
$= ([\sum_{z=1}^{n} \tau_z \mu_{\mathcal{B}}^2]^2, [\sum_{z=1}^{n} \tau_z \vartheta_{\mathcal{B}}]^2)$
$=(\mu_{\mathcal{B}}, \vartheta_{\mathcal{B}})$

$= \mathcal{B}$

$Hence\ PFWPA\ (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ..., \mathcal{B}_n) = \mathcal{B}$

4 MCDM Problems entropy using Aggregation Operators in Pythagorean Fuzzy Set

Algorithm:

Step:1

Consider $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 ..., \mathcal{E}_n$ be alternatives and $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, , \mathcal{G}_n$ be the criteria. Suppose that the pythagorean fuzzy sets $\mathcal{B}_{\mathcal{L}} = (\mu_{\mathcal{L}}, \vartheta_{\mathcal{L}})$, ($\mathcal{L}$ = 1 to n) where $\mu_{\mathcal{L}}$ represent the Membership Degree of the alternative $\mathcal{E}_{\mathcal{L}}$ ($\mathcal{L}$=1 to n)for the criteria$\mathcal{G}_{\mathcal{L}}$ ($\mathcal{L}$=1 to n) .similarly, ϑ_L represent the Non-Membership Degree of the alternative $\mathcal{E}_{\mathcal{L}}$ ($\mathcal{L}$= 1 to n) for the criteria $\mathcal{G}_{\mathcal{L}}$ ($\mathcal{L}$=1 to n).

Step 2:

Pythagorean fuzzy set are used to assign weight $\mathcal{W}_i (i = 1, 2, 3, .., n)$ to different criteria for a set of group.

Step: 3

Calculate the Pythagorean values using the Aggregation Operators (equation (1),(2), (3), (4)) .

Step: 4

Calculate the Score values using the formula (equation (I)).

Step:5 Determine the alternative is smaller . As a output, the rank the alternative in descending order.

Numerical Example

To illustrate the applicability of the proposed methodology, a faculty recruitment problem is considered. Four candidates are evaluated based on multiple criteria such as communication skills, experience, speaking ability, computer knowledge, and professional competence. The assessments are expressed using Pythagorean fuzzy numbers to capture uncertainty and hesitation in expert judgements. ($\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, ..., $\mathcal{E}_n$) . Consider a set of criteria $\mathcal{G}_1$, $\mathcal{G}_2$,G_3, ..., $\mathcal{G}_n$ the criteria stands for Communication , Experience , Good speaking and Computer knowledge respectively.

Step 1: Construct that the alternatives corresponding to each criteria is given in the form of Pythagorean Fuzzy Set

$$\begin{bmatrix} & \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \mathcal{E}_4 \\ \mathcal{G}_1 & [0.6, 0.3] & [0.4, 0.3] & [0.4, 0.6] & [0.5, 0.3] \\ \mathcal{G}_2 & [0.5, 0.4] & [0.2, 0.6] & [0.5, 0.4] & [0.3, 0.6] \\ \mathcal{G}_3 & [0.4, 0.7] & [0.8, 0.2] & [0.3, 0.7] & [0.4, 0.3] \\ \mathcal{G}_4 & [0.6, 0.4] & [0.4, 0.5] & [0.3, 0.6] & [0.4, 0.3] \end{bmatrix}$$

Step:2

Suppose that we take the weight τ_k(L = 1,2,3,4,5) in the form of Pythagorean Fuzzy set with $\tau_z > 0$ and $\sum_{z=1}^{n} \tau_z = 1$, weight $\tau_1 = 0.14, \tau_2 = 0.22, \tau_3 = 0.37$ $\tau_4 = 0.03$ and $\tau_5 = 0.24$.

Step:3 Calculate the PFS values using the Aggregation Operators (equation (1),(2), (3), (4)), as shows in the table.

$$\begin{bmatrix} & \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \mathcal{E}_4 \\ PFWA & [0.398, 0.423] & [0.398, 0.48] & [0.483, 0.05] & [0.397, 0.31] \\ PFWG & [0.449, 0.398] & [0.372, 0.470] & [0.445, 0.424] & [0.383, 0.471] \\ PFWPA & [0.045, 0.040] & [0.030, 0.057] & [0.074, 0.096] & [0.019, 0.062] \\ PFWPG & [0.480, 0.467] & [0.354, 0.508] & [0.376, 0.505] & [0.447, 0.393] \end{bmatrix}$$

Step:4 Determine the Score values using the formula (equation (I)) , as shows in the table

$$\begin{bmatrix} & \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \mathcal{E}_4 \\ PFWA & [0.0249] & [0.0237] & [0.0007] & [0.00074] \\ PFWG & [0.0445] & [0.0249] & [0.0027] & [0.0005] \\ PFWPA & [0.1] & [0.1131] & [0.012] & [0.0032] \\ PFWPG & [0.1155] & [0.1584] & [0.0024] & [0.0015] \end{bmatrix}$$

Step:5

The alternative can be calculate based on their smallar distance of above table are generated.

PFWA $\mathcal{E}_2 > \mathcal{E}_1 > \mathcal{E}_3 > \mathcal{E}_4$
PFWG $\mathcal{E}_2 > \mathcal{E}_1 > \mathcal{E}_3 > \mathcal{E}_4$
PFWPA $\mathcal{E}_2 > \mathcal{E}_1 > \mathcal{E}_3 > \mathcal{E}_4$
PFWPG $\mathcal{E}_2 > \mathcal{E}_1 > \mathcal{E}_3 > \mathcal{E}_4$

The Pythagorean fuzzy decision matrix is constructed, criteria weights are assigned, and aggregation operators are applied. The score function is then used to rank the candidates. The results obtained from all aggregation operators consistently identify as the most suitable candidate, demonstrating the effectiveness and robustness of the proposed approach.

Along these results we find out the $\mathcal{E}_2$ is the best candidate when compared with other candidates.

5 Conclusions

In this study, a collection of operators, including the union, intersection, Pythagorean Fuzzy Weighted Average (PFWA) operator, Pythagorean Fuzzy Weighted Geometric (PFWG) operator, Pythagorean Fuzzy Weighted Power Average (PFWPA) operator, and Pythagorean Fuzzy Weighted Power Geometric (PFWPG) operator, has been examined and analyzed for the class of B fuzzy sets, together with proofs of their essential properties. A multi-criteria decision-making (MCDM) framework is adopted to solve decision-making problems based on B fuzzy set operators and to effectively assess the significance of the derived results. As a direction for future work, the proposed method will be expanded by applying Pythagorean fuzzy sets to various application areas in order to further demonstrate its validity and effectiveness.

reference

[1] K.T.Atanasssov, "Intuitionistic fuzzy sets", Fuzzy sets and Systems 20(1986), 87-96.

[2] Khan.F, Abdullah.s, Mahmood.T, Shakeel.M and Rahim.M(2019 B). "Pythagorean Cubic fuzzy aggregation information based on confidence levels and its application

[3]Muhammad.S, Muhammad Haris.S, Rimsha.S, Salvatore.S Umar.I and Ferdinando D.M, "A theoretical Development of cubic pythagorean fuzzy soft set with its application in multi-attribute decision making",Symmetry 2022, 14, 2639.

[4] K.Balamurugan. Dr.R.Nagarajan, "Applications of cubic root fuzzy sets in decision making approach". European Journal of Molecular and Clinical Medicine , ISSN 2515-8260 Vol 10, 2023.

[5] Khan.F,Khan M.S.A.,Shahzad M., and Abdullah. S, (2019 a). "Pythagorean cubic fuzzy aggregation operators and their application to multi-criteria decision

making process". Journal of Intelligent and Fuzzy systems, 36(1),595-607.

[6] V.Chinnadurai, S.Thayalan, A.Bobin, "Multi-criteria decision-making in complex Fermatean fuzzy environment", J.Math.Comput. Sci. 11(2021), No.6, 7209-7227.

[7]Paul Augustine Ejegwa, "Pythgorean Fuzzy set and its application in career place- ments based on academic performance using max-min-max composition", Complec Intelligent systems (2019) 5:165-175.

[8]V.Chinnadurai, S.Thayalan, A.Bobin, "Multi-criteria decision-making process using complex cubic interval valued intuitionistic fuzzy set", Joural of Physics conference series 1850(2021) 012094.

[9] Rodrigo barraza, Juanm Sepulveda and Ivan Derpich, "Application of Fermatean fuzzy matric in co-design of urban Projects", Procedia computer science 199 (2022) 463-470.

[10] Anjan Mukherjee, Sadhan Sarkar and Somen Debnath, "Application of Intuitionistic Fuzzy soft set in decision making based on real life problems",Communication in Mathematical Modeling and Application, CMMA 1. No.3,16-21(2016)

[11] M.Palanikumar, K.Arulmozhi, Aiyared Iampan, "Novel Possibility Pythagorean Cubic fuzzy soft sets and their application", InternationalJournal of Innovative Computing Information and Control, Vol 19, N,2 (2023)

[12] L.A. Zadeh, "Fuzzy Sets ", Inf. Control, 8, 338-353.

[13] D.Liu, Y.Liu and L.wang ,"Distance Measure for Fermatean Fuzzy linguistic term sets based on linguistic scale function: An illustration of the TODIM and TOPSIS methods" Intell syst., Vol. wiley, No. DOI:10.1002/int.22162, pp1-28,2019.

[14] Yagar.R.R,, Abbasov. A.M, "Pythagorean membership grades, complex numbers and decising making", Int , Jouranl Intell., Syst., 2013, 28,pp436-452.

[15] Y.A. Salih, H.Z.Inrajo, Cubic fuzzy sets and their applications, J.Math Coomputer Sci., 28 (2023),pp 171-181.

[16] Hariwan Z.Ibrahim, Tareq M. Al-Shami and Abdelwaheb Mhemdi , Applications of nth power Root Fuzzy sets in Multicriteria Decision Making, Hindawi Journal of Mathematics Vol (2023), pp 14.

[17] H. Garg and D. Rani, some generalized complex intuitionistic fuzzy Aggregation Operators and their Application to Multicriteria Decision-Making Process, Arab,J. Sci.Eng.44(2019),2679-2698.

[18] G.Wei and some induced geometric aggregation operators based on einstein operation, Int.J. Intell. Syst 26(11)(2011),1049-1075.

[19]Z.Xu, Intuitionistic fuzzy aggregation operations,IEEE Trans. Fuzzy Syst.15(2007),1179-1187.

[20] Z.Xu, chequet integrals of weighted intuitionistic fuzzy information inform. Sci 180(2010), 726-736.

[21]R.R. Yager and A. M.Abbasov, pythagorean membership grades complex and decision making intell. syst 22(2014) 958-965.

[22] R.R. Yager, OWA aggregation of intuitionistic fuzzy set int.J.Gen. Syst.(2009), 617-641.

[23] V.Chinnadurai,S. Thayalana and A. Bobin,complex cubic intuitionistic fuzzy set and its decision making Adv.math 9(2020), 7933-7946.

[24] H.Garg and D.Rani, some generalized complex intuitionistic fuzzy aggregation operators and their application to multicriteria decision making process,Arab,J. Sci.eng. 44(2019), 2679-2698.

[25] R.R. Yager, pythagorean membership grades in multi-criteria decision making,IEEE Trans. Fuzzy Syst. 22(2014),958-965.

[26] V.Chinnadurai, S. Thayalana and A. Babin complex cubic intuitionistic fuzzy set and its decision making, Adv.Math.9(2020),7933-7946.