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REGULARITY IN PYTHAGOREAN FUZZY NANO TOPOLOGICAL SPACES VIA DELTA BETA -OPEN SETS

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Abstract

In this paper, we introduce Pythagorean fuzzy nano regular spaces and their variants—δ-, δ-pre-, δ-semi-, δα-, and δβ-regular spaces—defined via Pythagorean fuzzy nano δ-type open sets in Pythagorean fuzzy nano topological spaces. We also define the corresponding strongly Pythagorean fuzzy nano regular spaces. Furthermore, we examine interrelationships among these spaces, their connections to existing topological structures, and fundamental properties with characterizations.

1. Introduction

The concept of fuzzy sets, first introduced by Zadeh in 1965 [23], has found wide-ranging applications in fields such as decision theory, artificial intelligence, operations research, expert systems, computer science, data analytics, pattern recognition, management science, and robotics. In 1968, Chang and Warren [8, 20] extended this concept by introducing fuzzy topological spaces (FTS), incorporating fundamental topological notions such as open and closed sets, neighborhoods, interiors, closures, continuity, and compactness. Subsequent studies further explored the applications of fuzzy sets across various domains. Over time, numerous specialized fuzzy topological structures have been developed to address specific theoretical and practical needs.

In 1997, Dogan Coker [6] introduced the concept of intuitionistic fuzzy topological spaces and explored their properties related to compactness and continuity. Building upon intuitionistic fuzzy setsâ€"which account for both membership and non-membership degreesâ€"Pythagorean fuzzy sets (PFS) have gained attention due to their broader applicability. While both set types incorporate membership (μ) and non-membership (λ) degrees, their constraints differ: intuitionistic fuzzy sets satisfy $\mu+\lambda\leq 1$, whereas Pythagorean fuzzy sets satisfy μ^2+λ^2≤1.

To offer greater flexibility in uncertainty modeling, Yager [22] introduced non-standard fuzzy sets in 2013, comparing them with intuitionistic fuzzy sets and proposing the Pythagorean fuzzy set (PFS) as an effective model in decision-making scenarios. PFS has since been applied in areas such as job placement based on academic performance [12] and mask selection during the COVID-19 pandemic using the Pythagorean TOPSIS technique [14].

Subsequently, Murat et al. [11] extended the fuzzy topological framework by developing Pythagorean fuzzy topological spaces (PFTS), inspired by classical fuzzy topological spaces (FTS), and defined Pythagorean fuzzy continuous functions between such spaces.

In 2020, Ajay and Joseline Charisma [2, 3] introduced the concept of Pythagorean fuzzy nano topology respective interior, closure and continuous maps in Pythagorean fuzzy nano topological spaces. Recently, Balasubramaniyan et. al. [7] introduced the concept of Pythagorean fuzzy nano δ open and closed sets, Deivanaiyagai et. al. [10] introduced the concept of some stronger and weaker forms of open maps in Pythagorean fuzzy nano topological spaces.

In parallel developments, Saha [15] introduced the concept of δ-open sets in fuzzy topological spaces. This concept was further extended in 2019 by Acikgoz and Esenbel [1], who introduced neutrosophic soft δ-topologies. Further contributions were made by Aranganayagi et al., Surendra et al., and Vadivel et al. [4, 5, 16, 17, 18, 19], who investigated δ-open sets in neutrosophic, neutrosophic soft, neutrosophic hypersoft, and neutrosophic nano topological spaces, particularly focusing on their mappings and separation axioms.

Similarity measures have emerged as essential tools for quantifying vagueness and evaluating the closeness between fuzzy sets. In this context, Zhang [9] proposed similarity-based techniques for Pythagorean fuzzy multi-attribute decision-making. Peng et al. [13] introduced several new distance and similarity measures aimed at solving problems in pattern recognition, medical diagnosis, and clustering analysis, also examining their transformation properties. Wei and Wei [21] further developed cosine-based similarity functions for decision-making applications.

However, several existing PFS similarity and distance measures suffer from limitations such as division-by-zero issues, inability to distinguish between positive and negative differences, and non-compliance with core axioms (e.g., the third and fourth similarity axioms). These counter-intuitive behaviors [21, 9, 13] hinder the decision-maker's (DMâ€™s) ability to identify optimal or convincing alternatives.

The objective of this paper is to address these challenges by proposing a novel similarity measure for Pythagorean fuzzy sets that overcomes these counter-intuitive limitations and provides a more robust decision-making tool.

Research Gap: To date, no studies have been reported in the literature on Pythagorean fuzzy nano topological spaces that investigate newly defined spaces such as Pythagorean fuzzy nano δβ-regular spaces and strongly Pythagorean fuzzy nano δβ-regular spaces.

In this paper, we introduce Pythagorean fuzzy nano regular spaces, including their variantsâ€"δ, δ pre, δ semi, δα and δβ-regular spacesâ€"as well as their corresponding strongly Pythagorean fuzzy nano δ-type regular spaces. We explore and analyze their fundamental properties within the framework of Pythagorean fuzzy nano topological spaces (PFNts's).

2. Pythagorean Fuzzy nano (resp. δ, δ pre, δ semi, δα and δβ) regular spaces

In this section, we extend the study of Pythagorean fuzzy nano topological spaces by introducing Pythagorean fuzzy nano regular spaces, along with their variantsâ€"namely δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$-regular spaces. These spaces are defined using the corresponding types of Pythagorean fuzzy nano open and closed sets.

Definition 2.1 *Let $(U, \tau_{\mathcal{P}}(F))$ be a $\mathcal{PFN}ts$ is said to be Pythagorean fuzzy nano (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) regular (briefly, $\mathcal{PFN}Reg$ (resp. $\mathcal{PFN}\delta Reg$, $\mathcal{PFN}\delta\mathcal{P}Reg$, $\mathcal{PFN}\delta\mathcal{S}Reg$, $\mathcal{PFN}\delta\alpha Reg$ and $\mathcal{PFN}\delta\beta Reg$)) if for each $\mathcal{PFN}c$ (resp. $\mathcal{PFN}\delta c$, $\mathcal{PFN}\delta\mathcal{P}c$, $\mathcal{PFN}\delta\mathcal{S}c$, $\mathcal{PFN}\delta\alpha c$ and $\mathcal{PFN}\delta\beta c$) set A and a point $x_\alpha \notin A$, there exist disjoint $\mathcal{PFN}o$ (resp. $\mathcal{PFN}\delta o$, $\mathcal{PFN}\delta\mathcal{P}o$, $\mathcal{PFN}\delta\mathcal{S}o$, $\mathcal{PFN}\delta\alpha o$ and $\mathcal{PFN}\delta\beta o$) sets L and M such that $A \subseteq L$, $x_\alpha \in M$.*

Theorem 2.1 *In a $\mathcal{PFN}ts$ $(U, \tau_{\mathcal{P}}(F))$, the following are equivalent:*

(i). U is $\mathcal{PFN}\delta\beta Reg$.

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(ii). For every $x_\alpha \in U$ and every $\mathcal{PFN}\delta\beta o$ set G containing x_α, there exists a $\mathcal{PFN}\delta\beta o$ set L such that $x_\alpha \in L \subseteq \mathcal{PFN}\delta\beta cl(L) \subseteq G$.

(iii). For every $\mathcal{PFN}\delta\beta c$ set F, the intersection of all $\mathcal{PFN}\delta\beta c$ set, $\mathcal{PFN}\delta\beta q$-$nbhd$ of F is exactly F.

(iv). For any Pythagorean fuzzy set A and a $\mathcal{PFN}\delta\beta o$ set B such that $A \cap B \neq 0_p$, there exists a $\mathcal{PFN}\delta\beta o$ set L such that $A \cap L \neq 0_p$ and $\mathcal{PFN}\delta\beta cl(L) \subseteq B$.

(v). For every non-empty Pythagorean fuzzy set A and $\mathcal{PFN}\delta\beta c$ set B such that $A \cap B = 0_p$, there exist disjoint $\mathcal{PFN}\delta\beta o$ sets L and M such that $A \cap L \neq 0_p$ and $B \subseteq M$.

Proof. (i) ⇒ (ii): Suppose U is $\mathcal{PFN}\delta\beta Reg$. Let $x_\alpha \in U$ and let G be a $\mathcal{PFN}\delta\beta o$ set containing x_α. Then $x_\alpha \notin G^c$ and G^c is $\mathcal{PFN}\delta\beta c$ set. Since U is $\mathcal{PFN}\delta\beta Reg$, there exist $\mathcal{PFN}\delta\beta o$ sets L and M such that $L \cap M = 0_p$ and $x_\alpha \in L$, $G^c \subseteq M$. It follows that $L \subseteq M^c \subseteq G$ and hence $\mathcal{PFN}\delta\beta cl(L) \subseteq \mathcal{PFN}\delta\beta cl(M^c) = M^c \subseteq G$. That is $x_\alpha \in L \subseteq \mathcal{PFN}\delta\beta cl(L) \subseteq G$.

(ii) ⇒ (iii): Let F be any $\mathcal{PFN}\delta\beta c$ set and $x_\alpha \notin F$. Then F^c is $\mathcal{PFN}\delta\beta o$ set and $x_\alpha \in F^c$. By assumption, there exists a $\mathcal{PFN}\delta\beta o$ set L such that $x_\alpha \in L \subseteq \mathcal{PFN}\delta\beta cl(L) \subseteq F^c$. Thus $F \subseteq (\mathcal{PFN}\delta\beta cl(L))^c \subseteq L^c$. Now L^c is $\mathcal{PFN}\delta\beta c$ set, $\mathcal{PFN}\delta\beta q$-$nbhd$ of F which does not contain x_α. So, we get the intersection of all $\mathcal{PFN}\delta\beta c$ sets, $\mathcal{PFN}\delta\beta q$-$nbhd$ of F is exactly F.

(iii) ⇒ (iv): Suppose $A \cap B \neq 0_p$ and B is $\mathcal{PFN}\delta\beta o$ set. Let $x_\alpha \in A \cap B$. Since B is $\mathcal{PFN}\delta\beta o$ set, B^c is $\mathcal{PFN}\delta\beta c$ set and $x_\alpha \notin B^c$. By using (iii), there exists a $\mathcal{PFN}\delta\beta c$ set, $\mathcal{PFN}\delta\beta q$-$nbhd$ M of B^c such that $x_\alpha \notin M$. Now for the $\mathcal{PFN}\delta\beta q$-$nbhd$ M of B^c there exists a $\mathcal{PFN}\delta\beta o$ set G such that $B^c \subseteq G \subseteq M$. Take $L = M^c$. Thus L is a $\mathcal{PFN}\delta\beta o$ set containing x_α. Also $A \cap L \neq 0_p$ and $\mathcal{PFN}\delta\beta cl(L) \subseteq G^c \subseteq B$.

(iv) ⇒ (v): Suppose A is a non-empty set and B is $\mathcal{PFN}\delta\beta c$ set such that $A \cap B = 0_p$. Then B^c is $\mathcal{PFN}\delta\beta o$ set and $A \cap (B^c) \neq 0_p$. By our assumption, there exists a $\mathcal{PFN}\delta\beta o$ set L such that $A \cap L \neq 0_p$, and $\mathcal{PFN}\delta\beta cl(L) \subseteq B^c$. Take $M = (\mathcal{PFN}\delta\beta cl(L))^c$ Since $\mathcal{PFN}\delta\beta cl(L)$ is $\mathcal{PFN}\delta\beta c$ set, M is $\mathcal{PFN}\delta\beta o$ set. Also $B \subseteq M$ and $L \cap M \subseteq \mathcal{PFN}\delta\beta cl(L) \cap (\mathcal{PFN}\delta\beta cl(L))^c = 0_p$.

(v) ⇒ (i): Let S be $\mathcal{PFN}\delta\beta c$ set and $x_\alpha \notin S$. Then $S \cap \{x_\alpha\} = 0_p$. By (v), there exist disjoint $\mathcal{PFN}\delta\beta o$ sets L and M such that $L \cap \{x_\alpha\} \neq 0_p$ and $S \subseteq M$. That is L and M are disjoint $\mathcal{PFN}\delta\beta o$ sets containing x_α and S respectively. This proves that $(U, \tau_p(F))$ is $\mathcal{PFN}\delta\beta Reg$.

Corollary 2.1 *In a $\mathcal{PFN}ts$ $(U, \tau_p(F))$, the following are equivalent:*

1. U is $\mathcal{PFN}Reg$.
2. For every $x_\alpha \in U$ and every $\mathcal{PFN}o$ set G containing x_α, there exists a $\mathcal{PFN}\delta\beta o$ set L such that $x_\alpha \in L \subseteq \mathcal{PFN}\delta\beta cl(L) \subseteq G$.
3. For every $\mathcal{PFN}c$ set F, the intersection of all $\mathcal{PFN}\delta\beta c$ set, $\mathcal{PFN}\delta\beta q$-

$nbhd$ of F is exactly F.

4. For any Pythagorean fuzzy set A and a $\mathcal{PF}\mathfrak{N}o$ set B such that $A \cap B \neq 0_p$, there exists a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set L such that $A \cap L \neq 0_p$ and $\mathcal{PF}\mathfrak{N}\delta\beta cl(L) \subseteq B$.

5. For every non-empty Pythagorean fuzzy set A and $\mathcal{PF}\mathfrak{N}c$ set B such that $A \cap B = 0_p$, there exist disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets L and M such that $A \cap L \neq 0_p$ and $B \subseteq M$.

Theorem 2.2 *A $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_p(F))$ is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$ if and only if every $x_\alpha \in U$ and every $\mathcal{PF}\mathfrak{N}\delta\beta q$-$nbhd$ N containing x_α, there exists a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set M such that $x_\alpha \in M \subseteq \mathcal{PF}\mathfrak{N}\delta\beta cl(M) \subseteq N$.*

Proof. Let U be a $\mathcal{PF}\mathfrak{N}\delta\beta Reg$ space. Let N be any $\mathcal{PF}\mathfrak{N}\delta\beta q$-$nbhd$ of x_α. Then there exists a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set G such that $x_\alpha \in G \subseteq N$. Since G^c is $\mathcal{PF}\mathfrak{N}\delta\beta c$ set and $x_\alpha \notin G^c$, by definition there exist $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets L and M such that $G^c \subseteq L$ and $x_\alpha \in M$ and $L \cap M = 0_p$ so that $M \subseteq L^c$. It follows that $\mathcal{PF}\mathfrak{N}\delta\beta cl(M) \subseteq \mathcal{PF}\mathfrak{N}\delta\beta cl(L^c) = L^c$. Also $G^c \subseteq L$ implies $L^c \subseteq G \subseteq N$. Hence $x_\alpha \in M \subseteq \mathcal{PF}\mathfrak{N}\delta\beta cl(M) \subseteq N$. Conversely, suppose for every $x_\alpha \in U$ and every $\mathcal{PF}\mathfrak{N}\delta\beta q$-$nbhd$ N containing x_α, there exists a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set M such that $x_\alpha \in M \subseteq \mathcal{PF}\mathfrak{N}\delta\beta cl(M) \subseteq N$. Let F be any $\mathcal{PF}\mathfrak{N}\delta\beta c$ set and $x_\alpha \notin F$. Then $x_\alpha \in F^c$. Since F^c is $\mathcal{PF}\mathfrak{N}\delta\beta o$ set, F^c is $\mathcal{PF}\mathfrak{N}\delta\beta q$-$nbhd$ containing x_α. By hypothesis there exists a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set M such that $x_\alpha \in M$ and $\mathcal{PF}\mathfrak{N}\delta\beta cl(M) \subseteq F^c$. This implies that, $F \subseteq (\mathcal{PF}\mathfrak{N}\delta\beta cl(M))^c$. Then $(\mathcal{PF}\mathfrak{N}\delta\beta cl(M))^c$ is a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set containing F. Also $M \cap (\mathcal{PF}\mathfrak{N}\delta\beta cl\,(M))^c = 0_p$. Hence the space is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$.

Theorem 2.3 *A* $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{P}}(F))$ *is* $\mathcal{PF}\mathfrak{N}\delta\beta Reg$ *if and only if for each* $\mathcal{PF}\mathfrak{N}\delta\beta c$ *set* F *of* U *and each* $x_\alpha \in F^c$*, there exist* $\mathcal{PF}\mathfrak{N}\delta\beta o$ *sets* L *and* M *of* U *such that* $x_\alpha \in L$ *and* $F \subseteq M$ *and* $\mathcal{PF}\mathfrak{N}\delta\beta cl(L) \cap \mathcal{PF}\mathfrak{N}\delta\beta cl(M) = 0_p$.

Proof. Suppose U is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$ space. Let F be a $\mathcal{PF}\mathfrak{N}\delta\beta c$ set in U and $x_\alpha \notin F$. Then there exist $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets $L(x_\alpha)$ and M such that $x_\alpha \in L(x_\alpha)$, $F \subseteq M$ and $L(x_\alpha) \cap M = 0_p$. This implies that $L(x_\alpha) \cap \mathcal{PF}\mathfrak{N}\delta\beta cl\,(M) = 0_p$. Also $\mathcal{PF}\mathfrak{N}\delta\beta cl(M)$ is a $\mathcal{PF}\mathfrak{N}\delta\beta c$ set and $x_\alpha \notin \mathcal{PF}\mathfrak{N}\delta\beta cl\,(M)$. Since U is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$, there exist $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets G and H of U such that $x_\alpha \in G$, $\mathcal{PF}\mathfrak{N}\delta\beta cl(M) \subseteq H$ and $G \cap H = 0_p$.

This implies $\mathcal{PF}\mathfrak{N}\delta\beta cl(G) \cap H \subseteq \mathcal{PF}\mathfrak{N}\delta\beta cl\,(H^c) \cap H = (H^c) \cap H = 0_p$. Take $L = G$. Now L and M are $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets in U such that $x_\alpha \in L$ and $F \subseteq M$. Also $\mathcal{PF}\mathfrak{N}\delta\beta cl(L) \cap \mathcal{PF}\mathfrak{N}\delta\beta cl\,(M) \subseteq \mathcal{PF}\mathfrak{N}\delta\beta cl(G) \cap H = 0_p$.

Conversely, suppose for each $\mathcal{PF}\mathfrak{N}\delta\beta c$ set F of U and each $x_\alpha \in F^c$, there exist $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets L and M of U such that $x_\alpha \in L$ and $F \subseteq M$ and $\mathcal{PF}\mathfrak{N}\delta\beta cl(L) \cap \mathcal{PF}\mathfrak{N}\delta\beta cl\,(M) = 0_p$.

Now $L \cap M \subseteq \mathcal{PF}\mathfrak{N}\delta\beta cl\,(L) \cap \mathcal{PF}\mathfrak{N}\delta\beta cl(M) = 0_p$. Therefore $L \cap M = 0_p$. This proves that U is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$.

Theorem 2.4 *Let* $(U_1, \tau_{\mathcal{P}}(F_1))$ *and* $(U_2, \tau_{\mathcal{P}}(F_2))$ *be two* $\mathcal{PF}\mathfrak{N}ts$*'s. And let* $h_p: U_1 \to U_2$ *be a bijective function. If* h_p *is* $\mathcal{PF}\mathfrak{N}\delta\beta Irr$*,* $\mathcal{PF}\mathfrak{N}\delta\beta O$ *and* U_1 *is*

$\mathcal{PFN}\delta\beta Reg$, then U_2 is $\mathcal{PFN}\delta\beta Reg$.

Proof. Suppose U_1 is $\mathcal{PFN}\delta\beta Reg$. Let S be any $\mathcal{PFN}\delta\beta c$ set in U_2 such that $y_\beta \notin S$. Since h_p is $\mathcal{PFN}ZIrr$, $h_p^{-1}(S)$ is $\mathcal{PFN}\delta\beta c$ set in U_1. Since h_p is onto, there exists $x_\alpha \in U_1$ such that $y_\beta = h_p(x_\alpha)$. Now $h_p(x_\alpha) = y_\beta \notin S \Rightarrow x_\alpha \notin h_p^{-1}(S)$. Since U_1 is $\mathcal{PFN}\delta\beta Reg$, there exist $\mathcal{PFN}\delta\beta o$ sets L and M in U_1 such that $x_\alpha \in L$, $h_p^{-1}(S) \subseteq M$ and $L \cap M = 0_p$. Now $x_\alpha \in L \Rightarrow h_p(x_\alpha) \in h_p(L)$ and $h_p^{-1}(S) \subseteq M \Rightarrow h_p(M)$. Also $L \cup M = 0_p \Rightarrow h_p(L \cup M) = 0_p \Rightarrow h_p(L) \cup h_p(M) = 0_p$. Since h_p is a $\mathcal{PFN}\delta\beta O$ map, $h_p(L)$ and $h_p(M)$ are disjoint $\mathcal{PFN}\delta\beta o$ sets in U_2 containing y_β and S respectively. Thus U_2 is $\mathcal{PFN}\delta\beta Reg$.

Theorem 2.5 *Let U be a $\mathcal{PFN}\delta\beta Reg$ space.*

(i). Every $\mathcal{PFN}\delta\beta o$ set in U is a union of $\mathcal{PFN}\delta\beta c$ sets.

(ii). Every $\mathcal{PFN}\delta\beta c$ set in U is an intersection of $\mathcal{PFN}\delta\beta o$ sets.

Proof. (i) Suppose U is $\mathcal{PFN}\delta\beta Reg$. Let G be a $\mathcal{PFN}\delta\beta o$ set and $x_\alpha \in G$. Then $F = G^c$ is $\mathcal{PFN}\delta\beta o$ sets $L(x_\alpha)$ and M in U such that $x_\alpha \in L(x_\alpha)$ and $F \subseteq M$. Since $L(x_\alpha) \cap F \subseteq L(x_\alpha) \cap M = 0_p$, we have $L(x_\alpha) \subseteq F^c = G$. Take $M(x_\alpha) = \mathcal{PFN}\delta\beta cl(L(x_\alpha))$. Then $M(x_\alpha)$ is $\mathcal{PFN}\delta\beta c$ set and $M(x_\alpha) \cap M = x_\alpha \in M(x_\alpha) \subseteq F^c = G$. This proves that $G = \cup \{M(x_\alpha): x_\alpha \in G\}$. Thus G is a union of $\mathcal{PFN}\delta\beta c$ sets.

(ii) Follows from (i) and set theoretic properties.

Theorem 2.6

Let $(U_1, \tau_{\mathcal{P}}(F_1))$ and $(U_2, \tau_{\mathcal{P}}(F_2))$ be two $\mathcal{PF}\mathfrak{N}ts$'s. If $h_p : U_1 \to U_2$ is a $\mathcal{PF}\mathfrak{N}\delta\beta Cts$ and $\mathcal{PF}\mathfrak{N}C$ injection of a $\mathcal{PF}\mathfrak{N}ts$ U_1 into a $\mathcal{PF}\mathfrak{N}Reg$ space U_2 and if every $\mathcal{PF}\mathfrak{N}\delta\beta c$ set in U_1 is $\mathcal{PF}\mathfrak{N}c$, then U_1 is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$.

Proof. Let $x_\alpha \in U$ and A be a $\mathcal{PF}\mathfrak{N}\delta\beta c$ set in U not containing x_α. Then by assumption, A is $\mathcal{PF}\mathfrak{N}c$ set in U_1. Since h_p is $\mathcal{PF}\mathfrak{N}C$, $h_p(A)$ is a $\mathcal{PF}\mathfrak{N}c$ set in U_2 not containing $h_p(x_\alpha)$. Since U_2 is $\mathcal{PF}\mathfrak{N}Reg$, there exist disjoint $\mathcal{PF}\mathfrak{N}o$ sets M_1 and M_2 in U_2 such that $h_p(x_\alpha) \in M_1$ and $h_p(A) \subseteq M_2$. Since h_p is $\mathcal{PF}\mathfrak{N}\delta\beta Cts$, $h_p^{-1}(M_1)$ and $h_p^{-1}(M_2)$ are disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets in U_1 containing x_α and A respectively. Hence U_1 is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$.

Theorem 2.7 *Let $(U_1, \tau_{\mathcal{P}}(F_1))$ and $(U_2, \tau_{\mathcal{P}}(F_2))$ be two $\mathcal{PF}\mathfrak{N}ts$'s. If $h_p : U_1 \to U_2$ is a $\mathcal{PF}\mathfrak{N}Cts$, $\mathcal{PF}\mathfrak{N}\delta\beta O$ bijection of a $\mathcal{PF}\mathfrak{N}Reg$ space U_1 into a $\mathcal{PF}\mathfrak{N}ts$ U_2 and if every $\mathcal{PF}\mathfrak{N}\delta\beta c$ set in U_2 is $\mathcal{PF}\mathfrak{N}c$, then U_2 is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$.*

Proof. Let $x_\alpha \in U_2$ and B be a $\mathcal{PF}\mathfrak{N}\delta\beta c$ set in U_2 not containing x_α. Then by assumption, B is $\mathcal{PF}\mathfrak{N}c$ set in U_2. Since h_p is a $\mathcal{PF}\mathfrak{N}Cts$ bijection, $h_p^{-1}(B)$ is a $\mathcal{PF}\mathfrak{N}c$ set in U_1 not containing the point $h_p^{-1}(x_{(\alpha,\beta,\gamma)})$. Since U_1 is $\mathcal{PF}\mathfrak{N}Reg$, there exist disjoint $\mathcal{PF}\mathfrak{N}o$ sets L_1 and L_2 in U_1 such that $h_p^{-1}(x_{(\alpha,\beta,\gamma)}) \in L_1$ and $h_p^{-1}(B) \subseteq L_2$. Since h_p is $\mathcal{PF}\mathfrak{N}\delta\beta O$, $h_p(L_1)$ and $h_p(L_2)$ are disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets in U_2 containing x_α and B respectively. Hence U_2 is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$.

Remark 2.1 *Theorems 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and Corollary 3.1 holds for $\mathcal{PF}\mathfrak{N}o$, $\mathcal{PF}\mathfrak{N}\delta o$, $\mathcal{PF}\mathfrak{N}\delta Po$, $\mathcal{PF}\mathfrak{N}\delta So$ and $\mathcal{PF}\mathfrak{N}\delta\alpha o$ sets.*

3 STRONGLY PYTHAGOREAN FUZZY NANO δ (RESP. δ PRE, δ SEMI, $\delta\alpha$ AND $\delta\beta$) REGULAR SPACES

In this section, we introduce strongly Pythagorean fuzzy nano δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) regular spaces and study their properties.

Definition 3.1 *Let $(U, \tau_{\mathcal{P}}(F))$ be a $\mathcal{PFN}ts$ is said to be strongly Pythagorean fuzzy nano δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) regular (briefly, $\mathcal{PFN}\delta Reg$ (resp. $\mathcal{PFN}\delta\mathcal{P}Reg$, $\mathcal{PFN}\delta\mathcal{S}Reg$, $\mathcal{PFN}\delta\alpha Reg$ and $\mathcal{PFN}\delta\beta Reg$)) if for each $\mathcal{PFN}\delta c$ (resp. $\mathcal{PFN}\delta\mathcal{P}c$, $\mathcal{PFN}\delta\mathcal{S}c$, $\mathcal{PFN}\delta\alpha c$ and $\mathcal{PFN}\delta\beta c$) set A and a point $x_\alpha \notin A$, there exist disjoint $\mathcal{PFN}o$ sets L and M such that $A \subseteq L$ and $x_\alpha \in M$.*

Proposition 3.1 *Let $(U, \tau_{\mathcal{P}}(F))$ be a $\mathcal{PFN}ts$.*

(i). Every $St\mathcal{PFN}\delta\beta Reg$ space is $\mathcal{PFN}\delta\beta Reg$.

(ii). Every $St\mathcal{PFN}\delta\beta Reg$ is $\mathcal{PFN}Reg$.

Proof. (i) Suppose U is $St\mathcal{PFN}\delta\beta Reg$. Let F be a $\mathcal{PFN}\delta\beta c$ set and $x_\alpha \notin F$. Since U is $St\mathcal{PFN}\delta\beta Reg$, there exist disjoint $\mathcal{PFN}\delta\beta o$ sets L and M such that $x_\alpha \in L$ and $F \subseteq M$. Since, every $\mathcal{PFN}o$ set is $\mathcal{PFN}\delta\beta o$ set, L and M are $\mathcal{PFN}\delta\beta o$ sets. This implies that U is $\mathcal{PFN}\delta\beta Reg$. Similarly we can prove the result (ii).

Definition 3.2 *Let $(U, \tau_{\mathcal{P}}(F))$ be a $\mathcal{PFN}ts$ is said to be strongly * Pythagorean fuzzy nano (resp. δ, δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) regular (briefly, $St^*\mathcal{PFN}Reg$ (resp. $St^*\mathcal{PFN}\delta Reg$, $St^*\mathcal{PFN}\delta\mathcal{P}Reg$, $St^*\mathcal{PFN}\delta\mathcal{S}Reg$, $St^*\mathcal{PFN}\delta\alpha Reg$ and $St^*\mathcal{PFN}\delta\beta Reg$)) if for each $\mathcal{PFN}c$ set A and a point $x_\alpha \notin A$, there exist disjoint*

$\mathcal{PF}\mathfrak{N}o$ (resp. $\mathcal{PF}\mathfrak{N}\delta o$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}o$, $\mathcal{PF}\mathfrak{N}\delta\alpha o$ and $\mathcal{PF}\mathfrak{N}\delta\beta o$) sets L and M such that $A \subseteq L$, $x_\alpha \in M$.

Proposition 3.2 *Let $(U, \tau_{\mathcal{P}}(F))$ be a $\mathcal{PF}\mathfrak{N}ts$. Every $\mathcal{PF}\mathfrak{N}\delta\beta Reg$ space is $St^*\mathcal{PF}\mathfrak{N}\delta\beta Reg$.*

Proof. Suppose U is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$. Let F be a $\mathcal{PF}\mathfrak{N}c$ set and $x_\alpha \not\subseteq F$. Then F is $\mathcal{PF}\mathfrak{N}\delta\beta c$ set. Since U is $\mathcal{PF}\mathfrak{N}\delta\beta Reg$, there exist disjoint $\mathcal{PF}\mathfrak{N}\delta\beta o$ sets L and M such that $x_\alpha \in L$ and $F \subseteq M$. This implies that U is $St^*\mathcal{PF}\mathfrak{N}\delta\beta Reg$.

Theorem 3.1 *In a $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{P}}(F))$, the following are equivalent:* (i). U is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$. (ii). For every $x_\alpha \in U$ and every $\mathcal{PF}\mathfrak{N}\delta\beta o$ set G containing x_α, there exists a $\mathcal{PF}\mathfrak{N}o$ set L such that $x_\alpha \in L \subseteq \mathcal{PF}\mathfrak{N}cl(L) \subseteq G$.

(iii). For every $\mathcal{PF}\mathfrak{N}\delta\beta c$ set F, the intersection of all $\mathcal{PF}\mathfrak{N}c$ sets, $\mathcal{PF}\mathfrak{N}q$-$nbhd$ of F is exactly F.

(iv). For any Pythagorean fuzzy set A and a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set B such that $A \cap B \neq 0_p$, there exists a $\mathcal{PF}\mathfrak{N}o$ set L such that $A \cap L \neq 0_p$ and $\mathcal{PF}\mathfrak{N}cl(L) \subseteq B$.

(v). For every non-empty Pythagorean fuzzy set A and $\mathcal{PF}\mathfrak{N}\delta\beta c$ set B such that $A \cap B = 0_p$, there exist disjoint $\mathcal{PF}\mathfrak{N}o$ sets L and M such that $A \cap L \neq 0_p$ and $B \subseteq M$.

Proof. (i) $\Rightarrow$ (ii): Suppose U is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$. Let $x_\alpha \in U$ and let G be a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set containing x_α. Then $x_\alpha \notin G^c$ and G^c is $\mathcal{PF}\mathfrak{N}\delta\beta c$. Since U is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$, there exist $\mathcal{PF}\mathfrak{N}o$ sets L and M such that $L \cap M = 0_p$ and $x_\alpha \in L, G^c \subseteq M$. It

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follows that $L \subseteq M^c \subseteq G$ and hence $\mathcal{PF}\mathfrak{N}cl\,(L) \subseteq \mathcal{PF}\mathfrak{N}cl\,(M^c) = M^c \subseteq G$. That is $x_\alpha \in L \subseteq \mathcal{PF}\mathfrak{N}cl(L) \subseteq G$.

(ii) ⇒ (iii): Let F be any $\mathcal{PF}\mathfrak{N}\delta\beta c$ set and $x_\alpha \not\in F$. Then F^c is $\mathcal{PF}\mathfrak{N}\delta\beta o$ set and $x_\alpha \in F^c$. By assumption, there exists a $\mathcal{PF}\mathfrak{N}o$ set L such that $x_\alpha \in L \subseteq \mathcal{PF}\mathfrak{N}cl(L) \subseteq F^c$. Thus $F \subseteq (\mathcal{PF}\mathfrak{N}cl\,(L))^c \subseteq L^c$. Now L^c is $\mathcal{PF}\mathfrak{N}c$, $\mathcal{PF}\mathfrak{N}q$-$nbhd$ of F which does not contains x_α. So, we get the intersection of all $\mathcal{PF}\mathfrak{N}c$ set, $\mathcal{PF}\mathfrak{N}q$-$nbhd$ of F is exactly F.

(iii) ⇒ (iv): Suppose $A \cap B \neq 0_p$ and B is $\mathcal{PF}\mathfrak{N}\delta\beta o$. Let $x_\alpha \in A \cap B$. Since B is $\mathcal{PF}\mathfrak{N}\delta\beta o$, B^c is $\mathcal{PF}\mathfrak{N}\delta\beta c$ and $x_\alpha \not\in B^c$. By using (iii), there exists a $\mathcal{PF}\mathfrak{N}c$ set, $\mathcal{PF}\mathfrak{N}q$-$nbhd$ M of B^c such that $x_\alpha \not\in M$. Now for the $\mathcal{PF}\mathfrak{N}q$-$nbhd$ M of B^c there exists a $\mathcal{PF}\mathfrak{N}o$ set G such that $B^c \subseteq G \subseteq M$. Take $L = M^c$. Thus L is a $\mathcal{PF}\mathfrak{N}o$ set containing x_α. Also $A \cap L \neq 0_p$ and $\mathcal{PF}\mathfrak{N}cl\,(L) \subseteq G^c \subseteq B$.

(iv) ⇒ (v): Suppose A is a non-empty Pythagorean fuzzy set and B is $\mathcal{PF}\mathfrak{N}\delta\beta c$ set such that $A \cap B = 0_p$. Then B^c is $\mathcal{PF}\mathfrak{N}\delta\beta o$ set and $A \cap B^c \neq 0_p$. By our assumption, there exists a $\mathcal{PF}\mathfrak{N}o$ set L such that $A \cap L \neq 0_p$, and $\mathcal{PF}\mathfrak{N}cl\,(L) \subseteq B^c$. Take $M = (\mathcal{PF}\mathfrak{N}cl\,(L))^c$. Since $\mathcal{PF}\mathfrak{N}cl\,(L)$ is $\mathcal{PF}\mathfrak{N}c$ set, M is $\mathcal{PF}\mathfrak{N}o$ set. Also $B \subseteq M$ and $L \cap M \subseteq \mathcal{PF}\mathfrak{N}cl(L) \cap (\mathcal{PF}\mathfrak{N}cl(L))^c = 0_p$.

(v) ⇒ (i): Let S be $\mathcal{PF}\mathfrak{N}\delta\beta c$ set and $x_\alpha \not\in S$. Then $S \cap \{x_\alpha\} = 0_p$. By (v), there exist disjoint $\mathcal{PF}\mathfrak{N}o$ sets L and M such that $L \cap \{x_\alpha\} \neq 0_p$ and $S \subseteq M$. That is L and M are disjoint $\mathcal{PF}\mathfrak{N}o$ sets containing x_α and S respectively. This proves that $(U, \tau_{\mathcal{P}}(F))$ is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$.

Theorem 3.2 *A $\mathcal{PF}\mathfrak{N}ts$ $(U,\tau_{\mathcal{P}}(F))$ is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$ if and only if for each $\mathcal{PF}\mathfrak{N}\delta\beta c$ set F of U and each $x_\alpha \in F^c$, there exist $\mathcal{PF}\mathfrak{N}o$ sets L and M of U such that $x_\alpha \in L$ and $F \subseteq M$ and $\mathcal{PF}\mathfrak{N}cl(L) \cap \mathcal{PF}\mathfrak{N}cl(M) = 0_{\mathcal{P}}$.*

Proof. Suppose U is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$. Let F be a $\mathcal{PF}\mathfrak{N}\delta\beta c$ set in U and $x_\alpha \notin F$. Then there exist $\mathcal{PF}\mathfrak{N}o$ sets $L(x_\alpha)$ and M such that $x_\alpha \in L(x_\alpha)$, $F \subseteq M$ and $L(x_\alpha) \cap M = 0_{\mathcal{P}}$. This implies that $L(x_\alpha) \cap \mathcal{PF}\mathfrak{N}cl(M) = 0_{\mathcal{P}}$. Also $\mathcal{PF}\mathfrak{N}cl(M)$ is a $\mathcal{PF}\mathfrak{N}c$ set and hence $\mathcal{PF}\mathfrak{N}\delta\beta c$ set and $x_\alpha \notin \mathcal{PF}\mathfrak{N}cl(M)$. Since U is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$, there exist $\mathcal{PF}\mathfrak{N}o$ sets G and H of U such that $x_\alpha \in G$, $\mathcal{PF}\mathfrak{N}cl(M) \subseteq H$ and $G \cap H = 0_{\mathcal{P}}$. This implies $\mathcal{PF}\mathfrak{N}cl(G) \cap H \subseteq \mathcal{PF}\mathfrak{N}cl(H^c) \cap H = (H^c) \cap H = 0_{\mathcal{P}}$. Take $L = G$. Now L and M are $\mathcal{PF}\mathfrak{N}o$ sets in U such that $x_\alpha \in L$ and $F \subseteq M$. Also $\mathcal{PF}\mathfrak{N}cl(L) \cap \mathcal{PF}\mathfrak{N}cl(M) \subseteq \mathcal{PF}\mathfrak{N}cl(G) \cap H = 0_{\mathcal{P}}$.

Conversely, suppose for each $\mathcal{PF}\mathfrak{N}\delta\beta c$ set F of U and each $x_\alpha \in F^c$, there exist $\mathcal{PF}\mathfrak{N}o$ sets L and M of U such that $x_\alpha \in L$ and $F \subseteq M$ and $\mathcal{PF}\mathfrak{N}cl(L) \cap \mathcal{PF}\mathfrak{N}cl(M) = 0_{\mathcal{P}}$. Now $L \cap M \subseteq \mathcal{PF}\mathfrak{N}cl(L) \cap \mathcal{PF}\mathfrak{N}cl(M) = 0_{\mathcal{P}}$. Therefore $L \cap M = 0_{\mathcal{P}}$. This proves that U is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$.

Theorem 3.3 *A $\mathcal{PF}\mathfrak{N}ts$ $(U,\tau_{\mathcal{P}}(F))$ is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$ if and only if every pair consisting of a $\mathcal{PF}\mathfrak{N}\delta\beta\, Comp$ set and a disjoint $\mathcal{PF}\mathfrak{N}\delta\beta c$ set can be separated by $\mathcal{PF}\mathfrak{N}o$ sets.*

Proof. Let U be $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$ and let A be a $\mathcal{PF}\mathfrak{N}\delta\beta\, Comp$ set, B $\mathcal{PF}\mathfrak{N}\delta\beta c$ set with $A \cap B = 0_{\mathcal{P}}$. Since U is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$, for each $x_\alpha \in A$, there exist disjoint $\mathcal{PF}\mathfrak{N}o$ sets $L(x_\alpha)$ and $M(x_\alpha)$ such that $x_\alpha \in L(x_\alpha)$, $B \subseteq M(x_\alpha)$. Obviously,

$\{L(x_\alpha): x_\alpha \in A\}$ is a $\mathcal{PF}\mathfrak{N}o$ covering of A. Since A is $\mathcal{PF}\mathfrak{N}\delta\beta Comp$, there exists a finite subfamily $\{L_{x_i}: i = 1,2,\cdots,n\}$ which covers A. It follows that $A \subseteq \cup \{L_{x_i}: i = 1,2,\cdots,n\}$ and $B \subseteq \cap \{M_{x_i}: i = 1,2,\cdots,n\}$. Put $L = \cup \{L_{x_i}: i = 1,2,\cdots,n\}$ and $M = \cap \{M_{x_i}: i = 1,2,\cdots,n\}$. Then L and M are $\mathcal{PF}\mathfrak{N}o$ in U. Also $L \cap M = 0_P$. Otherwise, if $x_\alpha \in L \cap M \Rightarrow x_\alpha \in L_{x_j}$ for some j and $x_\alpha \in M_{x_i}$ for every i. This implies that $x_\alpha \in L_{x_j} \cap M_{x_j}$, which is a contradiction to $L_{x_j} \cap M_{x_j} = 0_P$. Thus L and M are disjoint $\mathcal{PF}\mathfrak{N}o$ sets containing A and B respectively.

Conversely, suppose every pair consisting of a $\mathcal{PF}\mathfrak{N}\delta\beta Comp$ set and a disjoint $\mathcal{PF}\mathfrak{N}\delta\beta c$ set can be separated by $\mathcal{PF}\mathfrak{N}o$ sets. Let F be a $\mathcal{PF}\mathfrak{N}\delta\beta c$ set and $x_\alpha \notin F$. Then $\{x_\alpha\}$ is $\mathcal{PF}\mathfrak{N}\delta\beta Comp$ set of U and $\{x_\alpha\} \cap F = 0_P$. By our assumption, there exist disjoint $\mathcal{PF}\mathfrak{N}o$ sets L and M such that $x_\alpha \in L$ and $F \subseteq M$. This proves that U is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$.

Corollary 3.1 *If U is a $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$ space, A is a $\mathcal{PF}\mathfrak{N}\delta\beta Comp$ subset of U and B is a $\mathcal{PF}\mathfrak{N}\delta\beta o$ set containing A, then there exists a $\mathcal{PF}\mathfrak{N}ro$ set M such that $A \subseteq M \subseteq \mathcal{PF}\mathfrak{N}cl(M) \subseteq B$.*

Proof. Let U be $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$ and let A be a $\mathcal{PF}\mathfrak{N}\delta\beta Comp$ set, B $\mathcal{PF}\mathfrak{N}\delta\beta o$ set with $A \subseteq B$. Then B^c is $\mathcal{PF}\mathfrak{N}\delta\beta c$ such that $B^c \cap A = 0_P$. Since U is a $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$ space, then there exist disjoint $\mathcal{PF}\mathfrak{N}o$ sets L_1 and L_2 such that $A \subseteq L_1$ and $B^c \subseteq L_2$. Take $M = \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(L_1))$. Then $\mathcal{PF}\mathfrak{N}cl(M) = \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}int(cl(L_1))) \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}cl(L_1)) = \mathcal{PF}\mathfrak{N}cl(L_1)$.

Since L_1 is a $\mathcal{PF}\mathfrak{N}o$ set and $L_1 \subseteq \mathcal{PF}\mathfrak{N}cl(L_1)$, we have $L_1 = \mathcal{PF}\mathfrak{N}int(L_1) \subseteq \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(L_1)) = M$. This implies that $\mathcal{PF}\mathfrak{N}cl(L_1) \subseteq \mathcal{PF}\mathfrak{N}cl(M)$. It follows that $\mathcal{PF}\mathfrak{N}cl(M) = \mathcal{PF}\mathfrak{N}cl(L_1)$ and $\mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(M)) = \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(L_1)) = M$. Thus M is $\mathcal{PF}\mathfrak{N}ro$ set. Now $A \subseteq L_1 = \mathcal{PF}\mathfrak{N}int(L_1) \subseteq \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(L_1)) = M$. This implies that $A \subseteq M$ and $\mathcal{PF}\mathfrak{N}cl(M) = \mathcal{PF}\mathfrak{N}cl(L_1) \subseteq L_2^{\xi} \subseteq B$ imply that $A \subseteq M \subseteq \mathcal{PF}\mathfrak{N}cl(M) \subseteq B$.

Theorem 3.4 *Let $h_p : (U_1, \tau_{\mathcal{P}}(F_1)) \to (U_2, \tau_{\mathcal{P}}(F_2))$ be a bijective function. If h_p is $\mathcal{PF}\mathfrak{N}\delta\beta Irr$, $\mathcal{PF}\mathfrak{N}o$ and U_1 is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$, then U_2 is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$.*

Proof. Suppose U_1 is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$. Let S be any $\mathcal{PF}\mathfrak{N}\delta\beta c$ set in U_2 such that $y_\beta \not\in S$. Since h_p is $\mathcal{PF}\mathfrak{N}\delta\beta Irr$, $h_p^{-1}(S)$ is $\mathcal{PF}\mathfrak{N}\delta\beta c$ set in U_1. Since h_p is onto, there exists $x \in U_1$ such that $y_\beta = h_p(x_\alpha)$. Now $h_p(x_\alpha) = y_\beta \not\in S \Rightarrow x_\alpha \not\in h_p^{-1}(S)$. Since U_1 is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$, there exist $\mathcal{PF}\mathfrak{N}o$ sets L and M in U_1 such that $x_\alpha \in L$, $h_p^{-1}(S) \subseteq M$ and $L \cap M = 0_{\mathcal{P}}$. Now $x_\alpha \in L \Rightarrow h_p(x_\alpha) \in h_p(L)$ and $h_p^{-1}(S) \subseteq M \Rightarrow S \subseteq h_p(M)$. Also $L \cap M = 0_p \Rightarrow h_p(L \cap M) = 0_p \Rightarrow h_p(L) \cap h_p(M) = 0_p$. Since h_p is a $\mathcal{PF}\mathfrak{N}o$ map, $h_p(L)$ and $h_p(M)$ are disjoint $\mathcal{PF}\mathfrak{N}o$ sets in U_2 containing y_β and S respectively. Thus U_2 is $St\mathcal{PF}\mathfrak{N}\delta\beta Reg$.

Remark 3.1 *Theorems 3.1, 3.2, 3.3, 3.4, Propositions 3.1, 3.2 and Corollary 3.1 are holds $\mathcal{PF}\mathfrak{N}o$, $\mathcal{PF}\mathfrak{N}\delta o$, $\mathcal{PF}\mathfrak{N}\delta Po$, $\mathcal{PF}\mathfrak{N}\delta So$ and $\mathcal{PF}\mathfrak{N}\delta\alpha o$ sets.*

Conclusion: In this paper, we have investigated $\mathcal{PF}\mathfrak{N}\delta\beta$-regular spaces and strongly $\mathcal{PF}\mathfrak{N}\delta\beta$-regular spaces by employing $\mathcal{PF}\mathfrak{N}\delta\beta$-open and $\mathcal{PF}\mathfrak{N}\delta\beta$-closed sets. We examined the interrelationships between these newly introduced spaces as well as their connections with existing classes of spaces. Additionally, we explored fundamental properties and provided characterizations of the defined spaces within the framework of Pythagorean fuzzy nano topological structures.

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