Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 46 No. 04 (2025)

Left P-Injective Rings on Matrices over Von Neumann Regular Rings

Rajashekhar Upase and Asha Saraswathi B.

Rajashekhar, Upase Faculty Member's, Upase Education Institute, Jayanagar, Dharwad-580001, Karnataka.

upase1580@gmail.com , upase2000@yahoo.com

Asha Saraswathi B. Professor and H.O.D, Department of Mathematics, Srinivas University, Srinivas Nagar, Mukka. Surathkal. Mangalore – 574146. India

ashasaraswathib@gmail.com,

Abstract: Entire in this paper denotes commutative with an identity, Von Neumann regular ring and be the ring of matrices over. Our motivation is to think of as the sort of ring that occurs in functional analysis and we prove how to uniquely write down all idempotents in terms of arbitrary parameters

Keywords: Matrix ring, division ring, simple matrix ring, R- module, lattice of all R-sub-modules, R_n -modules and isomorphism left p-injective rings, von-Neumann regular and Jacobson radical

INTRODUCTION: In all modules are unitary left ideal of R respectively. A left $R-$ module M is said to be left principally injective or simply left $P-$ injective. If for any principal left ideal P of R and any left $R-$ module homomorphism $f: P \to M$ can be extended to R . A ring R is said to be left $P-$ injective, if R is $P-$ injective as a left $R-$ module. A left self-injective ring is clearly $P-$ injective and von Neumann regular ring is also left $P-$ injective. However, in general the converse does not holds in either case. The connection between von Neumann regular rings, self-injective rings and $P-$ injective rings are studied a several papers, for example Utumi [13], than Chan Young Hang, Jin Young Kim and Nam Kyun Kim have proved in right self –injective rings which are either semi prime PI or semi prime rings of all essential left ideals are two-sided are von Neumann regular [6], However Hirano [4] showed that there exists a semi prime PI left injective rings but not von Neumann regular. It is well known that reduces or semi prime left duo left $P-$ injective rings are von Neumann regular [10, 14].

.**2. MOTIVATION:** From these facts, we may ask the following question. In particular question was also raised by Yue Chi Ming [15, 16]. Is a semi prime left $P-$ injective ring, all of whose essential left ideals are two sided von Neumann regular? and is a factor ring module the Jacobson radical of a left $P-$ injective ring over von Neumann regular?

We give a negative solution for these questions and we say conditions for a semi prime left p-injective ring all of whose essential left ideals are two sided to be von-Neumann regular. As a byproduct, we can show that a factor ring modulo the Jacobson radical of a left p-injective ring need not be von Neumann regular and also we characterize here a von Neumann regular ring by every left ideal is either a maximal left annihilator or a projective left annihilator of an element of R.

. LATTICE REGULAR ELEMENT OF A VON-NEUMANN REGULAR RINGS

Definition:3.1 If for any regular element $a \in R$ then there exist an element $x \in R$ such that $a = axa$ is called von Neumann regular and it is denoted by

$$aRa = \{ a \in R / \, for\, some\, x \in R \} .$$

Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 46 No. 04 (2025)

Example:3.2 Let R be von Neumann regular ring then shows that the center $Z(R)$ is also von Neumann regular.

Solution: Let $c \in R$ by definition $c = cxc$.

Now we can always replace x by $y = xcx$ then we have $cyc = cxcxc = (cxc)xc = cxc = c$. Thus we are done if we can show that whenever c is central, so is y. now we assume that $c \in Z(R)$ and let $r \in R$ then $yr = (xcx)r = x(cx)r = x(xc)r = x^2cr = x^2rc = x^2r(cxc) = x^2r[c(xc)] = x^2r[c(cx)] = x^2rc^2x = x^2c^2rx = x(cr)x.$

Similarly; $ry = x(rc)x.$

Example:3.4 Let R is in a decomposibe ring the shows that the center $Z(R)$ is field.

Solution: Suppose R is in-decomposibe then $R \neq 0$ and the only central idempotent's are 0 and 1, for some $0 \neq c \in Z(R)$. Consider the equation $c = cyc$. Since $cy \neq 0$ is a central idempotent, we have $cy = 1$. This shows that $Z(R)$ is field.

Example:3.5 Let $e(= e^2) \in R$ be von Neumann regular ring then so is $s = e\mathrm{Re}$.

Solution: Suppose R is von Neumann regular ring. Let $a \in S$, we write $a = axa$,where $x \in R$. Since $ae = a = ea$ Now let $a = axa = (ae)x(ea) = a(exe)a = aya$, where $y = exe$ .this verifies that S is also von Neumann regular ring.

Proposition:3.6 For e is an idempotent in a von Neumann regular ring R,

(i) If e is right irreducible then $e\mathrm{Re}$ is a division ring.

(ii) converse is true if R is a semi-prime ring.

Proof: (i) it follows form the Schur's Lemma, since by taking $e^{'} = e$, now we have the group isomorphism $\lambda : End_R(R) \to eRe$, it suffices to shows that λ is a ring isomorphism. Let $(\theta \cdot \theta^{'}) \in End_R(R)$ and let $m = \theta(e) \in \mathrm{R}_e$ then we have $\lambda(\theta \cdot \theta^{'}) = \theta^{'}\theta(e) = \theta^{'}m = \theta^{'}(em) = \theta^{'}(e)m = \lambda(\theta^{'}) \cdot \lambda(\theta)$. Hence $End_R(R) \cong e\mathrm{Re}$.

(ii) Suppose R is semi-prime and $e\mathrm{Re}$ is division ring, now we consider any non- zero element $er \in e\mathrm{R}$ where $r \in \mathrm{R}$. Since R is a semi-prime, $er\mathrm{Re}r \neq 0$, hence $ers\mathrm{e} \neq 0$, for some $s \in R$. Let $et\mathrm{e}$ be the inverse of $ers\mathrm{e}$ in $e\mathrm{Re}$ then $(ers\mathrm{e})(et\mathrm{e}) = e$. Therefore $erR = eR$, so that eR is an irreducible R-module.

Proposition:3.7 For e is an idempotent in a von Neumann regular ring R is primitive iff e is a left irreducible iff $e\mathrm{Re}$ is a division ring.

Proof: If $(e\mathrm{R})_R$ is irreducible it is certainly indecomposable. Conversely suppose $(eR)_R$ is not irreducible it would contain a proper sub-module $aR \neq 0$ but aR is a direct summand in R_R and hence also in $(e\mathrm{R})_R$ is a contradiction. This prove the first iff statement and the second iff statement follows form the above proposition.

Preposition:3.8. For $e = e^2$ is idempotent in ring R, if R is semi-local ring (resp von Neumann regular , unit regular, strongly regular) then so is $s = e\mathrm{Re}$

Proof: First we assume R is semi-local Let e be an independent in R and let $J = rad(\mathrm{R})$ then $rad(e\mathrm{Re}) = J \cap (e\mathrm{Re}) = eJe$ moreover $\left[e\mathrm{Re}/\, rad(e\mathrm{Re})\right] \cong \overline{e}\overline{\mathrm{R}}\overline{e}$, where $\overline{e}$ is the image of e in $\overline{\mathrm{R}} = R/J$, then we have $\left[S/rad(S)\right] \cong \overline{e}\overline{\mathrm{R}}\overline{e}$,where $\overline{\mathrm{R}} = R/J$. Therefore, $\overline{\mathrm{R}}$ is a semi-simple ring.

(a) Let $e \neq 0$ be any idempotent ring R, if R is Jacobson semi-simple (resp semi-simple , simple-prime, semi-prime, left noetherian, left artinian then the same holds for $e\mathrm{Re})$ then it follows that $\overline{e}\overline{\mathrm{R}}\overline{e}$ is also semi-simple. Therefore, S is semi-local ring.

(b) Now we assume that R is von Neumann regular. Let $a \in S$ then we have $a = axa$ where for some $x \in R$. Since $ae = a = ea$ we have $a = (ae)x(ea) = a(exe)a = aya$ where $y = exe \in S$. This satisfies that S also von Neumann regular.

(c) Now we assume that R is strangely regular. Let $a \in S$ then we have $a = a^2x$ where for some $x \in R$. Since $ae = a = ea$ we have $a = (a^2x)e = a^2(xe) = a^2(exe) = a^2y$ where $y = exe \in S$. This satisfies that S also von Neumann regular.

Finally, we assume that R is unit regular identifying S with $End(eR)_R$, we will show that the unit regularity of S. Let $eR = N \oplus K = N' \oplus K'$ and $N \cong N'$ implies $K \cong K'$. Let $R = (f_R \oplus N) \oplus K = (f_R \oplus N') \oplus K'$, ${}_Lf = (1-e)$ and $f_R \oplus N \cong f_R \oplus N'$ then we conclude $K \cong K'$. since $R \cong End(eR)_R$ There for this satisfies that R is a unite regular.

Lemma: 3.9. (McCoy's Lemma [16]) An element a in a ring R is regular if and only if there an exist $x \in R$ such that $axa - a$ is regular.

Proof: Only if: Since, by hypothesis a is regular, then there exists $x \in R$ such that $a = axa$. So, $axa - a = o$, which is trivially regular.

If: Let us assume that $axa - a$ is regular for some $x \in R$. So, we can choose $y \in R$ such that $axa - a = (axa - a)y(axa - a)$ implies $axa - a = (axay - ay)(axa - a) = a(xayax - xay - yax + y)a = aza$, where $z = (xayax - xay - yax + y) \in aRa$. Hence, $a = a(x - z)a$, where $x - y \in R$. Hence a is regular.

Proposition: 3.10. For $n \in \mathbb{N}$,R is von Neumann regular if and only if $S = M_n(R)$ is von Neumann regular where $n \geq 1$.

Proof: If: By hypothesis, S is a von Neumann regular and $e = E_{11}$ is an idempotent in S. So, by eSe is von Neumann regular. But, by Proposition (1.10) $eSe \cong R$ and hence, R is von Neumann regular.

Only if: By hypothesis, R is von Neumann regular. We now prove, by induction on k, $M_{2^k}(R)$ is von Neumann regular. We first consider the case $k = 1$ .We show that any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$ is regular. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. Since, by hypothesis, R is von Neumann regular, $c = cxc$ for some $x \in R$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} o & ax \\ o & cx \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$=\begin{pmatrix} axc & axd \\ cxc & cxd \end{pmatrix}-\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$=\begin{pmatrix} axc-a & axd-b \\ cxc-c & cxd-d \end{pmatrix}=\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

By McCoy's lemma (Lemma 1.11), it suffices to show that any $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is regular. Let $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\in M_2(R)$. Since, by hypothesis, a,d are regular $a=aya$, and $d=dzd$, for some y,z in R.

We have $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}-\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}=\begin{pmatrix} ay & bz \\ 0 & dz \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}-\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$$=\begin{pmatrix} aya & ayb+bzd \\ 0 & dzd \end{pmatrix}-\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$=\begin{pmatrix} aya-a, & (ayb+bzd)-b \\ 0 & dzd-d \end{pmatrix}=\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

So, we are now reduced to showing that any $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is regular, this follows by the noting that if $b=bwb$ then we have $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} bw & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & bwb \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. This proves that $M_2(R)$ is von Neumann regular.

Now suppose $k>1$ and our required assertion is true for $k-1$. Since, by Proposition (1.10), $M_{2^k}(R)\cong M_2\left(M_{2^{k-1}}(R)\right)$, our assertion follows. We now consider the general case. Let $n\in\mathbb{N}$. Choose $k\in\mathbb{N}$ such that $2^k\geq n$.

Let $A\in M_n(R)$ then $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\in M_{2^k}(R)$. Since this is regular in $M_{2^k}(R)$ then $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ This implies that $A=APA$ and hence A is regular. This proves that $M_n(R)$ is regular.

Definition: 4.1. for any non-empty subset X of a ring R, the left annihilator of X will denoted by $\ell(X)$. We recall that a ring R is ELT (essential left ideal), every essential left ideal of R is two sided. Obviously a left duo ring is an ELT ring.

Example: 4.2. There exists a semi prime ELT left $P-$ injective ring which is not von Neumann regular.

Let $A=\left\{\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}/a,b\in\mathbb{Z}_2\right\}$ be a sub ring of all 2×2 full matrix ring $Mat_2(\mathbb{Z}_2)$, where $\mathbb{Z}_2$ is the ring of all integers modulo 2, then A is a commutative $P-$ injective rings. Let R be the set of all 2×2 matrices over $\mathbb{Z}_2$ which are eventually in A, that is $R=\{<a>/a_i\in Mat_2(\mathbb{Z}_2)$ then there exist $k\in Z^+$ such that $a_i\in A$, $if\ i>k\}$, then R is a ring under $<a_i>$. Let $<a_i>+<b_i>=<a_i+b_i>+<a_i>\cdot<b_i>=<a_ib_i>$ .thus it is clearly, we say, that R is a semi-prime ring but not von Neumann regular.

Example: 4.3. A ring R is left non-singular left $P-$ injective ring then R satisfies

(i) R is left non-singular.

(ii) R is left non-singular left $P-$ injective.

(iii) $r(I\cap J)=r(I)+r(J)$, for any non-zero left ideals $I,J\ of\ R.$

Solution: Since A ring R is left non-singular left $P-$ injective ring condition (i) and (ii) follows example (4 .4)

case (i) Let $I=\left\{<a_i>/a_i\in\begin{pmatrix}\mathbb{Z}_2 & 0\\ \mathbb{Z}_2 & 0\end{pmatrix},\ a_i\ is\ eventully\ in\ \begin{pmatrix}0 & 0\\ \mathbb{Z}_2 & 0\end{pmatrix}\right\}$ and

$J=\left\{<b_i>/b_i\in\begin{pmatrix}0 & \mathbb{Z}_2\\ 0 & \mathbb{Z}_2\end{pmatrix},\ b_i\ is\ eventully\ in\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}$ then $I\ and\ J$ are left ideals of R.

Since, $I\cap J=<(0)>,\ r(I\cap J)+R$, but $r(I)=\left\{<a_i>/a_i\in\begin{pmatrix}0 & 0\\ \mathbb{Z}_2 & \mathbb{Z}_2\end{pmatrix},\ a_i\ is\ eventally\ in\ \begin{pmatrix}0 & 0\\ \mathbb{Z}_2 & 0\end{pmatrix}\right\}$ and

$r(J)=\left\{<b_i>/b_i\in\begin{pmatrix}\mathbb{Z}_2 & \mathbb{Z}_2\\ 0 & 0\end{pmatrix},\ b_i s\ eventally\ in\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}$

So, $r(I)+r(J)=<(c_i)>/c_i\in Mat_2(\mathbb{Z}_2)\ \ b_i's\ eventually\ in\ \begin{pmatrix}0 & 0\\ \mathbb{Z}_2 & 0\end{pmatrix}\Bigg\}\neq 0$. Hence R does not satisfies the above condition (iii) moreover R is not von Neumann regular,

Case (ii) a ring $\mathbb{Z}_4$ satisfies the above condition in (ii) & (iii) but $\mathbb{Z}_4$ is neither non-singular nor von Neumann regular

Case (iii) Let $R=\mathbb{Z}$ where $\mathbb{Z}$ is the ring of integer then it can be checked that R satisfies the above condition (i) and (ii). But R is neither left $P-$ injective nor von Neumann regular. The following proposition shows that conditions on examples (1.4),

(i) R is left non singular

(ii) R is left $P-$injective

(iii) $r(I\cap J)=r(I)+r(J)$, for any non-zero left ideals $I,J\ of\ R.$

Preposition: 4.5. Let R be a left non-singular left $P-$ injective ring. Suppose $r(I\cap J)=r(I)+r(J)$ for any non-zero left ideal $I,J\ of\ R$ then R is von – Neumann regular.

Proof: suppose a principal left ideal xR for any non-zero $x\in R$, since R is left $P-$injective by (4.4), $xR=r((x)l)$. By hypothesis R is left non-singular. So $r(x)$ is not essential left ideal of R. Hence $r(x)\oplus M$ is an essential left ideal, for same non zero left ideal M of R. Now we have $r((x)\ell)+r(M)=r((x)\ell\cap M)=R$, while $r((x)\ell)\cap r(M)\subseteq r((x)l\cap M)=0$ since $((x)\ell+M)$ is essential. Hence $xR=l((x)r)$ is a direct summand of R. Therefor R is von-Neumann regular.

Lemma: 4.6. Let R be a semi prime essential left ideal left $P-$injective ring and suppose $r(I\cap J)=r(I)+r(J)$, for any non-zero left ideals of $I,J\ of\ R$ then R is von Neumann regular.

Proof: Let x be an essential left ideal of R then x is a two sided ideal and also $(x)r$ is a two sided ideal of R. Now $(x\cap(x)r)^2\subseteq x((x)r)=0$. Since R is a semi prime we have $(x\cap(x)r)=0$, hence $(e)r=0$ implies R is left non-singular. Thus by above preposition (1.4). Hence R is von –Neumann regular.

Definition: 5.1. A left ideal I of R is a maximal left annihilator, if $I=(x)r$ for some non-empty subset of $x\neq\{0\}$ of R and for any left annihilator J with $I\subseteq J$ is either $J=I\, or\, J=R$ In that case, $I=(x)r$ for any $0\neq x\in S$. A ring R is called to be a left SF- ring if every simple left $R-$ modal is flat. Clearly Von-Neumann regular ring is an left (right) SF-rings.

Preposition: 5.2. A ring is left $P-$ injective iff every principal left ideal of R is a left annihilator.

Proof: obvious.

Theorem: 5.3. A ring R is Von-Neumann regular iff R is a left non-singular left $SF-$ring such that every principal left annihilator or a projective left annihilator of an elements of R.

Proof: If R is Von-Neumann regular then every principal left ideal is generated by an Idempotent. Hence every principal left ideal is a projective left annihilator of an elements of R.

Conversely : we assume that R is not Von Neumann regular then there exists a principal left ideal of I which is not direct summand of R, so we have $I=(x)r$ for some non-zero element $x\in R$. Note that Rx ,is not projective. Thus xR must be a maximal left annihilator say $xR=(y)r$ for some $y\in R$. Obviously $y\neq 0$, hence $(y)r$ is not essential left ideal of R is left non-singular. So we can choose a non-zero element z such that $(y)r\cap zR=0$, since $xR=(y)r$ is not projective. $xR\oplus zR\neq R$. Thus, there exists a maximal left ideal M of R such that $xR\oplus yR\subseteq M$. Since R/M is flat, there exist $c\in M$ such that $c=cx$ and $d=xd$ .Hence $x,d\in(1-x)r$ and $(y)r=Rx\subseteq(1-a)r\neq R$. Whenever $(1-x)r=Rx$ is contradiction to our hypothesis $(y)r\cap Rd=0$. Hence R is Von Neumann regular.

Proposition: 5.4. Let R be a ring for which every finitely generated ideal is a maximal left annihilator then

(i) Every proper non zero left ideal is maximal

(ii) Every proper non zero left ideal is simple

(iii) Exactly one of the following their cases oceans

(a) R is a division ring

(b) R Contains exactly one proper nor – zero left ideal which is two sided and nilpotent.

(c) R Contains exactly two proper non zero left ideals I_1, I_2 with $I_1\oplus I_2=R.$.

Proof: Let I be a left ideal of R and $a\in I$ then Ra is a maximal left annihilator and for any $b\notin I$, such that $Ra+Rb$ is contains Ra ; so $Ra+Rb=R$. I is then maximal. (ii) follows from (i) and (iii) can be argued by considering the cases where R is uniform and note in this case where there is only one left ideal I. that $I=J(R)$ and it is then two sided and $Ia=0$ for some $0\neq a\in R$ then $0=Ia=RIa\supseteq I^2$ and I is nil point.

CONCLUSION:

A ring R is called to be a left SF ring if every simple left $R-$ modal is flat, Clearly Von-Neumann regular ring is an left (right) SF rings but converse is still an open question which raised.

REFERENCES

[1] K. R. Goodearl, von Neumann regular Rings,Pitman, New York, 1978.

[2] C.Barneet and V.camillo, Idempotents in matrices over commutative von Neumann regular Rings, comm.. Algebra 18(1990), 3905 - 3911 .

[3] R.S.Pierce,Modules over commutative over commutative von Neumann regular rings Mem. Amer. Math. Soc., vol 70, Amer. Mah. Soc., Providence, RI, 1967.

[4] Chen Yong Hong, Jin Yong Kim and Nam Kyun Kim. Communication in Algebra, 28(2), 791-801 (2000).

[5] J. Chen, on von Neumann regular rings and SF-rings, Math. Japonic 36(6) (1991), 1123-1127

[6] J.W. Fisher and R. L. Snider, On the von Neumann regularity of rings with regular prime factor rings, pacific J. Math. 54(1) (1974), 135-144.

[7] K.R. Goddearl, On von Neumann regular rings, pitman publishing Limited, New York, 1979

[8] Y. Hirano, On nonsingular p-injective rings, Pub. Mathematiques 38 (1994), 455-461.

[9] M. Ikeda and T. Nakayama, On some characteristic properties of quasi- Frobenius and regular rings, Proc. Amer. Math. Soc.5 (1954),15-19.

[10] S.K. Jain, S.H. Mohamed and Surjeet Singh, Rigs in which every right, ideal is quasi- injective, pacific J. Math. [31] (1) (1969),73-79

[11] J. Y. Kim and park, On reduced rings and generalized duo rings, Comm. Korean Math. Soc. 8 (1993), 345-349

[12] J. Y. Kim and park, when is a regular ring a semi simple Artinian rings?, Math Japonica 45(2) 91997), 311-313.

[13] Y. Lee, N.K. Kim and C.Y. Hong, Counter examples on Baer rings, Comm. in Algebra 25(2), (1997),497-507,

[14] W.K. Nicholson and M.F. Yousif, Principally injective rings, J. of Algebra 174 (1995), 77-93.

[15] V. S. Ramamurthi, on the injectivity and flatness of certain cyclic modules, proc. Amer. Math. Soc. 48 (1975), 21-25.

[16] J.J. Rotman, An Introduction to Homological Algebra, Academic press, New York, 1979

[17] Y. Utumi, On continuous rings and self-injective rings, Trans. Amer. Math.Soc.118 (1965), 158-173.

[18] R. Yue Chi Ming, On von Neumann regular rings Proc. Edinburgh Math. Soc 19 (1974), 89-91.

[19] R. Yue Chi Ming, On von Neumann regular rings V, Math. J. Okayama Univ. 22 (1980), 151-160.

[20] R Yue Chi Ming, On V- rings, p-V-rings and injectivity, Kyungpook Math. J. 32 (1992), 219-227.

[21] Chan Youg Hong, et.: “On Von Neumann Regular Rings”. Communication in algebra 28(2), 791-801 (1991).

[22] Francisco Jose Costa-Cano, et: “On Semi Regular Infinite Matrix Rings”. Communication in algebra 27(12), 5737-57401 (1999).

[23] J .L . Garcia: “The finite column matrix ring of a ring”, Proceedings of a first Spanish-Belgion week on Algebra and Geometry, EdeJ .L .Bues, M .I. Segura, A.Verschoren, R.U.C.A.,Antwerpen 1988, pp 64- 74.

[24] W . K. Nicholson: “Semi-regular modules and rings” can. J. Math (1976), 1105 – 1120.

[25] R. F. Shany: “Regular endomorphism rings of free modules”, J. London. Math. Soc.2(4),(1971), 353 – 345.

[26] Bo Stenstrm: "Rings of quotients", Springer – Verlag, Berlian1975.

[27] R. Ware and J . Zelmanowitz: "The Jacobson radical of the endomorphism ring of a projective module", Proc. AMS 26 (1970), 15 -20.

[28] Y. Utumi: " on Continuous rings and self injective rings: Trans, Amer. Math. Soc. 118, (1965), 158 – 173.

[29] S. K Jain, et : " Rings in which every right ideal is quasi injective" , Pacific J. Mat, 31 (1), (19969), 73 -79.

[30] N.E. Sexaur et: "The Radical of the row finite matrices over an arbitrary ring", Proc. AMS 26 (1969), 287 – 295.

[31] V. S. Ramamurthi: " On the injective and fitness of certain cyclic modules", Proc. AMS 48 (1975), 21 -25.

[32] Rimhak Ree: " on the projective geometry over full matrix rings", The University of British Columbia. (1955), p.no.144 -150.

[33] R. Baer: " Linear algebra and Projective Geometry", Academic Press. (1952).

[34] J.Lambek: " Lectures on rings and modules", Blaisdell. (1966).

[35] L.Levy: " Torsion free and divisible modules over non-integral-dimensions ", Can. Jour. Math., Vol.15, No.1. (1963).