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Total Cordial Labeling of Special Classes of Snake Graphs

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Abstract:- A graph G is said to be *cordial* if there exist a vertex labeling $f : V(G) \to \{0,1\}$ which induces an edge labeling $f^* : E(G) \to \{0,1\}$ defined by $f^*(uv) = |f(u) - f(v)|$ for each edge $uv \in E$, such that $|V_f(0) - V_f(1)| \leq 1$ and $|E_{f^*}(0) - E_{f^*}(1)| \leq 1$ where $V_f(0)$ is the number of vertices labeled with 0, $V_f(1)$ is the number of vertices labeled with 1, $E_{f^*}(0)$ is the number of edges labeled with 0 and $E_{f^*}(1)$ is the number of edges labeled with 1. A cordial graph in which the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1(i.e) $\left|\left(V_f(0) + E_{f^*}(0)\right) - \left(V_f(1) + E_{f^*}(1)\right)\right| \leq 1$ is called as a *total cordial*graph. In this paper, we have proved the existence of total cordial labeling of special classes of snake graphs.

Keywords: *Graph Labeling, CordialLabeling, Product CordialLabeling.*

1. Introduction

By a *graph G* we mean a finite, undirected, connected graph without any loops or multiple edges. Let *V(G)* and *E(G)* be the set of vertices and edges of a graph *G* respectively. The order and size of a graph G are denoted as $p = |V(G)|$ and $q = |E(G)|$ respectively. For general graph theoretic notions we refer to Harary [3].

Assigning numbers to the graph elements (vertices or edges or both) subject to certain condition(s) is known as *graph labeling*. There are several types of labeling available in the literature and a detailed survey of many of them can be found in the dynamic survey of graph labeling by J.A. Gallian [2].

Cahit [1] has introduced a variation of both graceful and harmonious labeling and named it as cordial labeling, which is defined as follows:

A graph Gis said to be *cordial* if there exist a vertex labeling$f : V(G) \to \{0,1\}$ which induces an edge labeling $f^* : E(G) \to \{0,1\}$ defined by$f^*(uv) = |f(u) - f(v)|$ for each edge $uv \in E(G)$, such that $|V_f(0) - V_f(1)| \leq 1$ and $|E_{f^*}(0) - E_{f^*}(1)| \leq 1$ where $V_f(0)$ is the number of vertices labeled with 0, $V_f(1)$ is the number of vertices labeled with 1, $E_{f^*}(0)$ is the number of edges labeled with 0 and $E_{f^*}(1)$ is the number of edges labeled with 1.

In [6], Sundaram and Ponraj introduced the notion of product cordial labeling. They defined a *product cordial labeling* of a graph G as a function *f* from V(G) to {0, 1} such that if each edge *uv*in E(G) is assigned the label *f(u)f(v)*, then the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.

Also they have introduced the notion of total product cordial labeling. They defined a *total product cordial labeling* of a graph G as a function *f* from V(G) to {0, 1} such that if each edge *uv*in E(G) is assigned the label *f(u)f(v)*then the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. A graph with a total product cordial labeling is called a *total product cordial graph*.

As an extension of the above, we define the*total cordial labeling* of a graph G as a cordial labeling such that number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. (i.e) A graph Gis said to be *total cordial* if there exist a vertex labeling$f : V(G) \rightarrow \{0, 1\}$ which induces an edge labelling $f^* : E(G) \rightarrow \{0, 1\}$ defined by$f^*(uv) = |f(u) - f(v)|$ for each edge $uv \in E$, such that

(i) $\left|V_f(0) - V_f(1)\right| \leq 1$

(ii) $\left|E_{f^*}(0) - E_{f^*}(1)\right| \leq 1$ and

(iii) $\left|\left(V_f(0) + E_{f^*}(0)\right) - \left(V_f(1) + E_{f^*}(1)\right)\right| \leq 1$

where $V_f(0)$ is the number of vertices labeled with 0, $V_f(1)$ is the number of vertices labeled with 1, $E_{f^*}(0)$ is the number of edges labeled with 0 and $E_{f^*}(1)$ is the number of edges labeled with 1.

A triangular snake graph T_n is obtained from a path P_n by replacing each edge of the path by a cycle C_3. An alternate triangular snake graph $A(T_n)$ is obtained from a path P_n by replacing every alternate edge of path by a cycle C_3. A double alternate triangular snake graph $DA(T_n)$ consists of two alternate triangular snakes that have a common path. In [4], Jeyanthi et al. have investigated the existence of 3-product cordial labeling of triangular snake, alternate triangular snake and double alternate triangular snake graphs.

In [5], Muhammad Imran et al. have discussed about the cordial labeling of a new class of triangular snake graph which is described as follows: Let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of a path P_n. They called the graph obtained by joining the vertices u_i and u_{i+2} to a new vertex v_i for $1 \leq i \leq (n-2)$ as alternate triangular snake graph. To distinguish this class of graph from the alternate triangular snake graph $A(T_n)$, we call this graph as alternative triangular snake graph and denote it as $A^*(T_n)$.

In this paper, we have investigated the existence of total cordial labeling of some classes of triangular snake graphs.

2. Total Cordial Labeling of Some Classes of Snake Graphs

In this section we have proved the total cordial labeling of triangular snake graph, alternative triangular snake graph and double alternative triangular snake graph.

Theorem 2.1: For $n \not\equiv 3 \ (mod\ 4)$, the triangular snake graph $T_n$has a total cordial labeling.

Proof : Let the vertex set and edge set of the triangular snake graphG = $T_n$be described as follows:

$V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-1}\}$ and

$E(G) = \{u_i u_{i+1} : 1 \leq i \leq (n-1)\} \cup \{u_i v_i, u_{i+1} v_i : 1 \leq i \leq (n-1)\}$.

Define a function$f : V(G) \rightarrow \{0, 1\}$ such that,

$$f(u_i) = \begin{cases} 0 ; & if\ i \equiv 1, 2\ (mod\ 4) \\ 1 ; & if\ i \equiv 0, 3\ (mod\ 4) \end{cases}$$

$$f(v_i) = \begin{cases} 1 ; & if\ i\ is\ odd \\ 0 ; & if\ i\ is\ even \end{cases}$$

Let $f^* : E(G) \rightarrow \{0,\ 1\}$ be an induced functionsuch that,for every edge $uv \in E(G)$,

$$f^*(uv) = |f(u) - f(v)|.$$

From the above, we observe the following:

Case (i):$n \equiv 0\ (mod\ 4)$

$$V_f(0) = n - 1$$

$$V_f(1) = n$$

$$E_{f^*}(0) = \frac{3n-2}{2}$$

$$E_{f^*}(1) = \frac{3n-4}{2}$$

Thus, we have

$$\left|\left(V_f(0) + E_{f^*}(0)\right) - \left(V_f(1) + E_{f^*}(1)\right)\right| = 0.$$

Case (ii):$n \equiv 1\ (mod\ 4)$

$$V_f(0) = n$$

$$V_f(1) = n - 1$$

$$E_{f^*}(0) = \frac{3n-3}{2}$$

$$E_{f^*}(1) = \frac{3n-3}{2}$$

Thus, we have

$$\left|\left(V_f(0) + E_{f^*}(0)\right) - \left(V_f(1) + E_{f^*}(1)\right)\right| = 1.$$

Case (iii):$n \equiv 2\ (mod\ 4)$

$$V_f(0) = n$$

$$V_f(1) = n - 1$$

$$E_{f^*}(0) = \frac{3n-4}{2}$$

$$E_{f^*}(1) = \frac{3n-2}{2}$$

Thus, we have

$$\left|\left(V_f(0) + E_{f^*}(0)\right) - \left(V_f(1) + E_{f^*}(1)\right)\right| = 0.$$

Hence f is a total cordial labeling of the triangular snake graphT_n.

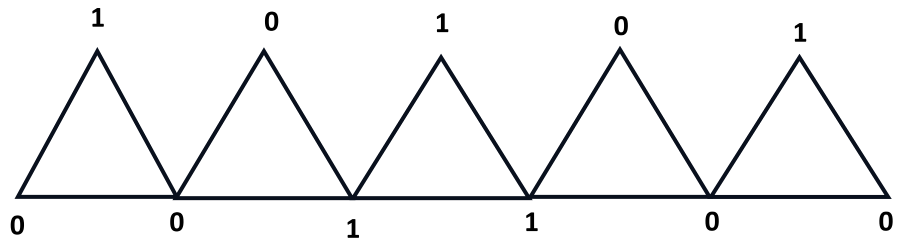

Fig. 2.1 Total cordial labeling of T_6

Theorem 2.2: For $n \not\equiv 1\ (mod\ 4)$, the alternative triangular snake graph A*(T_n) has a total cordial labeling.

Proof: Let the vertex set and edge set of the triangular snake graph G = A*(T_n) be described as follows:

$V(G) = \{u_1,\ u_2, \dots, u_n, v_1,\ v_2, \dots, v_{n-2}\}$ and

$E(G) = \{u_i u_{i+1} : 1 \leq i \leq (n-1)\} \cup \{u_i v_i, u_{i+2} v_i : 1 \leq i \leq (n-2)\}$.

Define a function $f : V(G) \to \{0,\ 1\}$ such that,

$$f(u_i) = \begin{cases} 0\,; & if\ i \equiv 0, 1\ (mod\ 4) \\ 1; & if\ i \equiv 2, 3\ (mod\ 4) \end{cases}$$

$$f(v_i) = \begin{cases} 1\,; & if\ i\ is\ odd \\ 0\,; & if\ i\ is\ even \end{cases}$$

Let $f^* : E(G) \to \{0,\ 1\}$ be an induced functionsuch that, for every edge $uv \in E(G)$,

$f^*(uv) = |f(u) - f(v)|$.

From the above, we observe the following:

Case (i):$n \equiv 0, 2\ (mod\ 4)$

$V_f(0) = n - 1$

$V_f(1) = n - 1$

$$E_{f^*}(0) = \frac{3n-6}{2}$$

$$E_{f^*}(1) = \frac{3n-4}{2}$$

Thus, we have

$$\left|\left(V_f(0) + E_{f^*}(0)\right) - \left(V_f(1) + E_{f^*}(1)\right)\right| = 1.$$

Case (ii):$n \equiv 3\ (mod\ 4)$

$V_f(0) = n - 1$

$V_f(1) = n - 1$

$$E_{f^*}(0) = \frac{3n-5}{2}$$

$$E_{f^*}(1) = \frac{3n-5}{2}$$

Thus, we have

$$\left|\left(V_f(0) + E_{f^*}(0)\right) - \left(V_f(1) + E_{f^*}(1)\right)\right| = 0.$$

Hence *f* is a total cordial labeling of the alternative triangular snake graph A*(T_n).

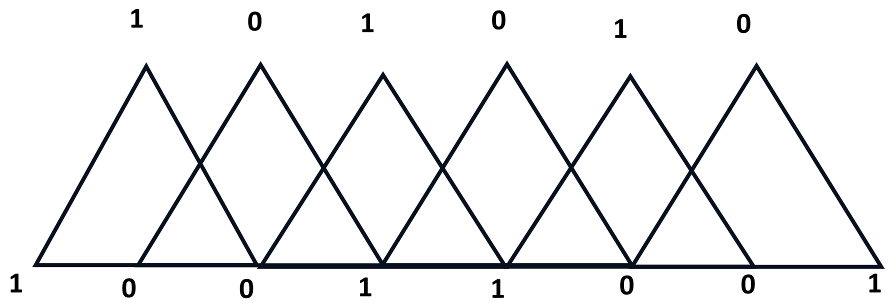

Fig. 2.2 Total cordial labeling of A*(T_8)

A double alternative triangular snake graph DA*(T_n), is a graph obtained from an alternative triangular snake graph A*(T_n), by joining the vertices u_i and u_{i+2} to a new vertex w_i for $1 \leq i \leq (n-2)$.

Theorem 2.3: For$n \not\equiv 1 \ (mod\ 4)$, the double alternative triangular snake graph DA*(T_n) has a total cordial labeling.

Proof :

Let the vertex set and edge set of the triangular snake graph G = DA*(T_n) be described as follows:

$V(G) = \{u_1,\ u_2, \dots, u_n, v_1,\ v_2, \dots, v_{n-2}, w_1,\ w_2, \dots, w_{n-2}\}$ and

$E(G) = \{u_i u_{i+1} : 1 \leq i \leq (n-1)\} \cup \{u_i v_i, u_{i+2} v_i : 1 \leq i \leq (n-2)\} \cup \{u_i w_i, u_{i+2} w_i : 1 \leq i \leq (n-2)\}.$

Define a function $f : V(G) \rightarrow \{0,\ 1\}$ such that,

$$f(u_i) = \begin{cases} 0\ ; & if\ i \equiv 0, 1\ (mod\ 4) \\ 1; & if\ i \equiv 2, 3\ (mod\ 4) \end{cases}$$

$$f(v_i) = \begin{cases} 1\ ; & if\ i\ is\ odd \\ 0\ ; & if\ i\ is\ even \end{cases}$$

$$f(w_i) = \begin{cases} 1\ ; & if\ i\ is\ odd \\ 0\ ; & if\ i\ is\ even \end{cases}$$

Let $f^* : E(G) \rightarrow \{0,\ 1\}$ be an induced functionsuch that, for every edge $uv \in E(G)$,

$$f^*(uv) = |f(u) - f(v)|.$$

From the above, we observe the following:

Case (i):$n \equiv 3\ (mod\ 4)$

$$V_f(0) = \frac{3n-5}{2}$$

$$V_f(1) = \frac{3n-3}{2}$$

$$E_{f^*}(0) = \frac{5n-9}{2}$$

$$E_{f^*}(1) = \frac{5n-9}{2}$$

Thus, we have

$$\left|\left(V_f(0) + E_{f^*}(0)\right) - \left(V_f(1) + E_{f^*}(1)\right)\right| = 1.$$

Case (ii): $n \equiv 0, 2\ (mod\ 4)$

$$V_f(0) = \frac{3n-4}{2}$$

$$V_f(1) = \frac{3n-4}{2}$$

$$E_{f^*}(0) = \frac{5n-10}{2}$$

$$E_{f^*}(1) = \frac{5n-8}{2}$$

Thus, we have

$$\left|\left(V_f(0) + E_{f^*}(0)\right) - \left(V_f(1) + E_{f^*}(1)\right)\right| = 1.$$

Hence *f* is a total cordial labeling of the double alternative triangular snake graph DA*(T_n).

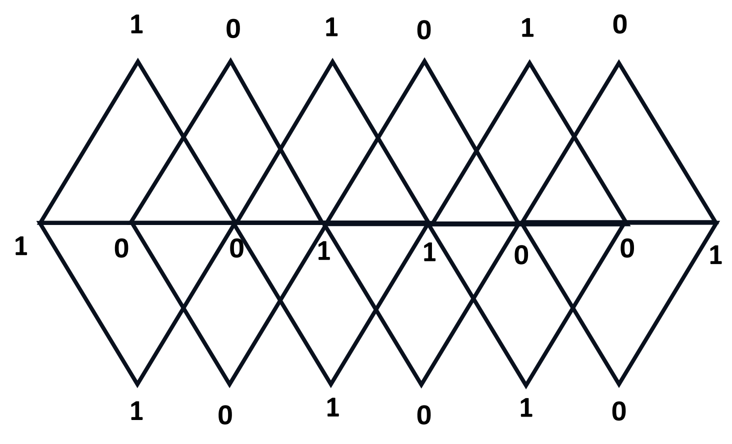

Fig. 2.3 Total cordial labeling of DA*(T_8)

3. Conclusion

In this paper, we have proved the existence of total cordial labeling for triangular snake graph, alternative triangular snake graph and double alternative triangular snake graph. The existence of total cordial labeling for other classes of snake graphs can also be investigated.

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