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Fekete-Szegö Functional and Second Hankel Determinant for a Subclass of Te-Univalent Functions Connected to Horadam Polynomial.

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***Abstract*:** This paper presents a subclass, referred to as $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1; \nu_1, \Phi(x)]$ of Te-univalent function that employs Horadam polynomial. The research explores this subclass and establishes preliminary coefficient bounds for $|b_2|, |b_3|$, $|b_4|$, Fekete-Szegö inequality and second hankel determinant for this category.Futhermore several corollaries are included to clarify the significance of the results.

KeyWords: *Analytic functions, Te-Univalent functions,Coefficient bounds, Horadam Polynomial and Fekete-Szegö Inequality.*

1 Introduction

Let $\psi(\mathfrak{z})$ be normalized analytic function expressed as follows

$$\psi(\mathfrak{z}) = \mathfrak{z} + b_2\mathfrak{z}^2 + b_3\mathfrak{z}^3 + \ldots \qquad (1.1)$$

where $\mathfrak{z} \in \mathfrak{D}$ and $\mathfrak{D} = \{\mathfrak{z}: \mathfrak{z} \in \mathbb{C} \text{ and } |\mathfrak{z}| < 1\}$.

We denote the class of all such functions as $\mathcal{A}$ and we focus on the subclass $\mathcal{S}$ of $\mathcal{A}$, which is characterized by $\mathcal{S} = \{\psi \in \mathcal{A}: \psi \text{ is univalent in } \mathfrak{D}\}$.

It is well-known result that any function ψ that belongs to $\mathcal{S}$ possesses an inverse function ψ^{-1} which can be represented in the following manner:

$$\psi^{-1}(w) = w + \sum_{n=2}^{\infty} d_n\, w^n. \qquad (1.2)$$

such that $\psi^{-1}(\psi(\mathfrak{z})) = \mathfrak{z}$ for $\mathfrak{z} \in \mathfrak{D}$ and $\psi(\psi^{-1}(w)) = w$ for $w \in \mathfrak{D}$, with $|w| < \rho_0(\psi)$ and $\rho_0(\psi) \geq \frac{1}{4}$. Furthermore, the inverse function ψ^{-1}, as given in (1.2) can be rewritten as

$$\mathfrak{F}(w) = \psi^{-1}(w) = w - b_2w^2 + (2b_2^2 - b_3)w^3 - (5b_2^3 - 5b_2b_3 + b_4)w^4 + \ldots \quad (w \in \mathfrak{D}) \qquad (1.3)$$

The bi-univalent functions introduced by Lewin [13], which are defined as analytic function ψ within the unit disc $\mathfrak{D}$, where both ψ and its inverse ψ^{-1} are univalent in $\mathfrak{D}$. This category of functions is symbolized by σ. Examples of functions that are classified in σ include

$\psi_1(\mathfrak{z}) = \frac{\mathfrak{z}}{1-\mathfrak{z}}$, $\psi_2(\mathfrak{z}) = -log(1-\mathfrak{z})$, $\psi_3(\mathfrak{z}) = \frac{1}{2}log\left(\frac{1+\mathfrak{z}}{1-\mathfrak{z}}\right)$. However the function $\frac{\mathfrak{z}}{(1-\mathfrak{z})^2}$ is included in the class $\mathcal{S}$ but not belong to σ.

For integers $n \geq 1$ and $q \geq 1$, the q^{th} Hankel determinant given by

$$H_q(n) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+q-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n+q-1} & b_{n+q-2} & \cdots & b_{n+2q-2} \end{vmatrix} \qquad (b_1 = 1)$$

The properties of Hankel determinant can be studied in [18].

Let the operator $\mathfrak{J}: \mathcal{A} \longrightarrow \mathcal{A}$ defined by the equation

$$\mathfrak{J}(\psi(\mathfrak{z})) = \mathfrak{z} + \sum_{n=2}^{\infty} t_n b_n \mathfrak{z}^n \qquad (1.4)$$

where $\mathfrak{z} \in \mathfrak{D}$ and $t_n \in \mathbb{C}$.

The concept of Te-univalency associated with an operator was established by Abd-Eltawab[1] which is a extension and generalization of the idea of bi-univalency of the function class defined in (1.4) on $\mathfrak{D}$. Let $\mathfrak{J}_S$ be the class of all functions given by (1.4) that are univalent in $\mathfrak{D}$. Every $\mathfrak{J}\psi \in \mathfrak{J}$ has an inverse defined as $(\mathfrak{J}\psi)^{-1}$.

$$(\mathfrak{J}\psi)^{-1}((\mathfrak{J}\psi)(\mathfrak{z})) = \mathfrak{z}$$

and

$$\mathfrak{J}\psi((\mathfrak{J}\psi)^{-1}(w)) = w \quad \left(|w| < \rho_0(\mathfrak{J}\psi); \rho_0(\mathfrak{J}\psi) \geq \frac{1}{4}\right)$$

where

$$(\mathfrak{J}\psi)^{-1}(w) = w - t_2 b_2 w^2 + (2t_2^2 b_2^2 - t_3 b_3) w^3 - (5t_2^3 b_2^3 - 5t_2 t_3 b_2 b_3 + t_4 b_4) w^4 + \ldots.$$

If both $\mathfrak{J}\psi$ and $(\mathfrak{J}\psi)^{-1}$ are univalent in $\mathfrak{D}$, the function ψ given by (1.1) is Te-univalent in $\mathfrak{D}$ associated with $\mathfrak{J}$. Let $\mathfrak{J}_\sigma$ be the class of all function given by (1.1) that are Te-univalent in $\mathfrak{D}$ and associated with $\mathfrak{J}$

Note that for $\mathfrak{J}\psi = \psi$ we have $\mathfrak{J}_\sigma = \sigma$ and if $\mathrm{t}_n \neq 1$ for some n, then

$$\mathfrak{J}\psi(\mathfrak{J}\mathfrak{F}(w)) = w + 2(\mathrm{t}_3 - \mathrm{t}_2^2) b_2^2\, w^3 + \ldots \neq w$$

For function ψ given by (1.1) and χ is given by

$$\chi(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} c_n \mathfrak{z}^n.$$

The Hadamard product of ψ and χ is defined by

$$(\psi * \chi)\mathfrak{z} = \mathfrak{z} + \sum_{n=2}^{\infty} b_n c_n \mathfrak{z}^n = (\chi * \psi)\mathfrak{z}.$$

For complex parameters $\mu_1, \mu_2, \mu_3, \ldots. \mu_q$ and $\nu_1, \nu_2, \nu_3, \ldots. \nu_\varsigma$ $(\nu_j \neq 0, -1, -2, \ldots., j = 1,2, \ldots.., \varsigma)$ the generalized hypergeometric function ${}_q\mathfrak{T}_\varsigma$ is defined as follows

$${}_q\mathfrak{T}_\varsigma(\mu_1, \mu_2, \mu_3, \ldots. \mu_q, \nu_1, \nu_2, \nu_3, \ldots. \nu_\varsigma; \mathfrak{z}) = \sum_{n=0}^{\infty} \frac{(\mu_1)_n \cdots (\mu_q)_n}{(\nu_1)_n \cdots (\nu_\varsigma)_n} \frac{\mathfrak{z}^n}{n!}, \quad (\mathfrak{z} \in \mathfrak{D})$$

where $q, \varsigma \in N_0 := N \cup \{0\}$ with $\varsigma + 1 \geq q$ and $(\gamma)_n$ is the shift factorial(or Pochhamber symbol) defined in terms of the gamma function Γ, by

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \begin{cases} 1, & if \ n = 0 \\ \gamma(\gamma + 1) \ldots. (\gamma + n - 1), & for \ n \in N \end{cases}$$

corresponding a function

$$\mathfrak{F}(\mu_1, \mu_2, \ldots., \mu_q; \nu_1, \nu_2, \ldots..., \nu_\varsigma; \mathfrak{z}) = \mathfrak{z}\, {}_q\mathfrak{T}_\varsigma(\mu_1, \mu_2, \ldots., \mu_q; \nu_1, \nu_2, \ldots..., \nu_\varsigma; \mathfrak{z}) \quad (\mathfrak{z} \in \mathfrak{D}) \qquad (1.5)$$

Dziok and Srivastava [6] investigated that a linear operator is defined as

$$\mathfrak{G}(\mu_1, \mu_2, \ldots., \mu_q; \nu_1, \nu_2, \ldots..., \nu_\varsigma): \mathcal{A} \longrightarrow \mathcal{A}$$

defined as the following: Hadamard Product

$$\mathfrak{G}(\mu_1, \mu_2, \ldots., \mu_q; \nu_1, \nu_2, \ldots..., \nu_\varsigma)\psi(\mathfrak{z}) = \mathfrak{F}(\mu_1, \mu_2, \ldots., \mu_q; \nu_1, \nu_2, \ldots..., \nu_\varsigma; \mathfrak{z}) * \psi(\mathfrak{z})$$

where $\mathfrak{F}$ is defined as (1.5), and $q, \varsigma \in N_0$ with $\varsigma + 1 \geq q$.

If $\psi \in \mathcal{A}$ is given by (1.1) then

$$\mathfrak{G}(\mu_1, \mu_2, \ldots., \mu_q; \nu_1, \nu_2, \ldots..., \nu_\varsigma)\psi(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} \Gamma_n[\mu_1; \nu_1] b_n \mathfrak{z}^n \quad (\mathfrak{z} \in \mathfrak{D}) \quad (1.6)$$

where

$$\Gamma_n[\mu_1;\nu_1] = \frac{(\mu_1)_{n-1}\cdots(\mu_q)_{n-1}}{(\nu_1)_{n-1}\cdots(\nu_\varsigma)_{n-1}}\frac{1}{(n-1)!} \quad (1.7)$$

To simplify the notation, we write:

$$\mathfrak{G}(\mu_1,\mu_2,\ldots,\mu_q;\nu_1,\nu_2,\ldots,\nu_\varsigma)\psi = \mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi$$

Let $\mathfrak{T}_\varsigma^{q,\varsigma}$ represents the class of functions given by (1.6) that are univalent in $\mathfrak{D}$. Every function $\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi \in \mathfrak{T}_\varsigma^{q,\varsigma}[\mu_1;\nu_1]$ has an inverse, $\mathfrak{g} = \left(\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi\right)^{-1}$ defined as [14]

$$\mathfrak{g}\left(\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi(\mathfrak{z})\right) = \mathfrak{z} \qquad (\mathfrak{z} \in \mathfrak{D}),$$

and

$$\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi(\mathfrak{g}(w)) = w, \qquad \left(|w| < \rho_0(\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi)\text{and}\rho_0(\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi) \geq \frac{1}{4}\right)$$

where

$$\begin{aligned}\mathfrak{g}(w) &= \left(\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi\right)^{-1}(w)\\ &= w - \Gamma_2[\mu_1;\nu_1]b_2w^2 + [2(\Gamma_2[\mu_1;\nu_1])^2b_2^2 - \Gamma_3[\mu_1;\nu_1]b_3]w^3\\ &\quad -[5(\Gamma_2[\mu_1;\nu_1])^3b_2^3 - 5\Gamma_2[\mu_1;\nu_1]\Gamma_3[\mu_1;\nu_1]b_2b_3 + \Gamma_4[\mu_1;\nu_1]b_4]w^4 + \ldots \qquad (1.8)\end{aligned}$$

and $\Gamma_n[\mu_1;\nu_1]$ is given by (1.7). A function ψ given by (1.1) is said to be Te-univalent [1] in $\mathfrak{D}$ associated with operator $\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]$. If both $\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi$ and $\left(\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi\right)^{-1}$ are univalent in $\mathfrak{D}$. Let $\mathfrak{T}_\sigma^{q,\varsigma}[\mu_1;\nu_1]$ be the class of all functions given by (1.1) that are Te-univalent in $\mathfrak{D}$ associated with $\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]$.

Note that for $q = 2, \varsigma = 1$, $\mu_1 = \nu_1 = c$ and $\mu_2 = 1$ we have $\mathfrak{T}_\sigma^{q,\varsigma}[c,1;c] = \sigma$ and if $\Gamma_n[\mu_1;\nu_1] \neq 1$ for some n, we have

$$\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi\left(\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\mathfrak{F}(w)\right) = w + 2[\Gamma_3[\mu_1;\nu_1] - (\Gamma_2[\mu_1;\nu_1])^2]b_2^2w^3 + \ldots \neq w.$$

where $\mathfrak{F}$ is given by (1.3).

In 1985, Horadam and Mohan [9] defined the Horadam polynomials $\varphi_n(x) = \varphi_n(\gamma,\delta;l,m)$ by the following recurrence relation:

$$\varphi_n(x) = lx\varphi_{n-1}(x) + m\varphi_{n-2}(x), \text{for} n \geq 3. \qquad (1.9)$$

with intial values

$$\varphi_1(x) = \gamma,\ \varphi_2(x) = \delta\, x \text{ and } \varphi_3(x) = l\delta\, x^2 + m\gamma. (1.10)$$

Further, the generating function of Horadam polynomial is

$$\Phi(x) = \sum_{n=1}^{\infty} \varphi_n(x)\mathfrak{z}^{n-1} = \frac{\gamma + (\delta - \gamma\, l)x\mathfrak{z}}{1 - l\, x\mathfrak{z} - m\mathfrak{z}^2}.$$

In this paper, the argument of $x \in \mathbb{R}$ is independent of the argument $\mathfrak{z} \in \mathfrak{D}$; that is $x \neq \mathcal{R}(\mathfrak{z})$. By giving values for γ, δ, l and m the Horadam polynomials leads to several known polynomials as follows:

- If $\gamma = \delta = l = m = 1$, we get Fibonacci polynomials $\mathbb{F}_n(x)$
- If $\gamma = 2$ and $\delta = l = m = 1$, we get Lucas polynomials $L_n(x)$.
- If $\gamma = m = 1$ and $\delta = l = 2$, we get Pell polynomials $P_n(x)$.
- If $\gamma = \delta = l = 2$ and $m = 1$, we get Pell-Lucas polynomials $Q_n(x)$.
- If $\gamma = \delta = -m = 1$ and $l = 2$, we get the Chebyshev Polynomials $T_n(x)$ of the first kind.
- If $\gamma = -m = 1$ and $\delta = l = 2$ we get the Chebyshev Polynomials $\mathcal{U}_n(x)$ of the second kind.

For more information see [2, 3, 4, 5, 9, 10, 12, 15, 16, 17, 19].

2 Basic Preliminaries and Lemmas

Definition: A function $\psi \in \sigma$ is said to be in the class $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,\Phi(x)]$ if it satisfies :

$$(1-\rho)\frac{\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi(\mathfrak{z})}{\mathfrak{z}} + \rho(\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi(\mathfrak{z}))' \prec \Phi(x,\mathfrak{z}) + 1 - \gamma \qquad (\mathfrak{z} \in \mathfrak{D}) \qquad (2.1)$$

and

$$(1-\rho)\frac{\mathfrak{g}(w)}{w} + \rho\mathfrak{g}'(w) \prec \Phi(x,w) + 1 - \gamma \qquad (w \in \mathfrak{D}) \qquad (2.2)$$

where $\mathfrak{g}(w) = (\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi)^{-1}(w)$, $\prec$ means subordination and $0 \le \rho \le 1$.

Lemma 2.1 [11] Let $k, j \in \mathbb{R}$ and $x, y \in \mathbb{C}$.If $|x| < r \; and \; |y| < r,$

$$|(k+j)x + (k-j)y| \le \begin{cases} 2|k|r, & if \quad |k| \ge j \\ 2|j|r, & if \quad |k| \le j \end{cases}$$

Lemma 2.2 [7] Suppose that $\mathcal{P}$ is the set of all analytic functions of the form

$$\mathfrak{p}(\mathfrak{z}) = 1 + \sum_{n=1}^{\infty} \alpha_n \mathfrak{z}^n$$

satisfying $R(\mathfrak{p}(\mathfrak{z})) > 0, \mathfrak{z} \in \mathbb{C}$ and $\mathfrak{p}(0) = 1$. Then $|\alpha_n| \le 2 \quad n = 1,2,3,\ldots.$ For any value of $n = 1,2,3\ldots$, this inequality is sharp. For example the function $\alpha(\mathfrak{z}) = \frac{1+\mathfrak{z}}{1-\mathfrak{z}}$ is equal for all n.

Lemma 2.3 [8] Suppose that $\mathcal{P}$ is the set of all analytic functions of the form

$\mathfrak{p}(\mathfrak{z}) = 1 + \sum_{n=1}^{\infty} \alpha_n \mathfrak{z}^n$ satisfying $R(\mathfrak{p}(\mathfrak{z})) > 0, \mathfrak{z} \in \mathbb{C}$ and $\mathfrak{p}(0) = 1$. Then

$$2\alpha_2 = \alpha_1^2 + (4-\alpha_1^2)y$$

$$4\alpha_3 = \alpha_1^3 + 2(4-\alpha_1^2)\alpha_1 y - (4-\alpha_1^2)\alpha_1 y^2 + 2(4-\alpha_1^2)(1-|y|^2)\mathfrak{z}$$

for some $y, \mathfrak{z}$ with $|y| \le 1, |\mathfrak{z}| \le 1$.

3 Coefficient Bounds

Theorem 3.1 If ψ as defined in (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,\Phi(x)]$. Then

$$|b_2| \le \frac{|\delta\, x|\sqrt{|\delta\, x|}}{\Gamma_2[\mu_1;\nu_1]\sqrt{|[(2\rho+1)(\delta\, x+1)+\rho^2]\delta\, x-[l\delta\, x^2+m\gamma](\rho+1)^2|}}. (3.1)$$

$$|b_3| \le \frac{|\delta\, x|}{\Gamma_3[\mu_1;\nu_1]}\left|\frac{1}{(2\rho+1)} + \frac{\delta\, x}{(\rho+1)^2}\right|. \qquad (3.2)$$

and

$$|b_4| \le \frac{|l^2\delta\, x^3 + lmx\gamma + mx\delta|}{|3\rho+1|\Gamma_4[\mu_1;\nu_1]} + \frac{5x^2\delta^2}{2|(\rho+1)(2\rho+1)|\Gamma_4[\mu_1;\nu_1]} \qquad (3.3)$$

Proof. Let $\psi \in \mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,\Phi(x)]$. Then

$$(1-\rho)\frac{\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi(\mathfrak{z})}{\mathfrak{z}} + \rho(\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi(\mathfrak{z}))' = \Phi(x,c(\mathfrak{z})) + 1 - \gamma \qquad (\mathfrak{z} \in \mathfrak{D}) \; (3.4)$$

and

$$(1-\rho)\frac{\mathfrak{g}(w)}{w} + \rho\mathfrak{g}'(w) = \Phi(x,d(w)) + 1 - \gamma \qquad (w \in \mathfrak{D}) \qquad (3.5)$$

where $\mathfrak{g}(w) = (\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi)^{-1}(w)$ is given by (1.8).

Where $\alpha, \beta \in \mathcal{P}$ as follows:

$\alpha(\mathfrak{z}) = \frac{1+c(\mathfrak{z})}{1-c(\mathfrak{z})} = 1 + \alpha_1\mathfrak{z} + \alpha_2\mathfrak{z}^2 + \alpha_3\mathfrak{z}^3 + \ldots$

$\Longrightarrow c(\mathfrak{z}) = \frac{\alpha(\mathfrak{z})-1}{\alpha(\mathfrak{z})+1} \qquad (\mathfrak{z} \in \mathfrak{D}) \qquad (3.6)$

and

$\beta(\mathfrak{z}) = \frac{1+d(w)}{1-d(w)} = 1 + \beta_1 w + \beta_2 w^2 + \beta_3 w^3 + \ldots$

$\Longrightarrow d(w) = \frac{\beta(w)-1}{\beta(w)+1} \qquad (w \in \mathfrak{D}) \qquad (3.7)$

From (3.6)and (3.7)

$$c(\mathfrak{z}) = \frac{\alpha_1}{2}\mathfrak{z} + \left(\frac{\alpha_2}{2} - \frac{\alpha_1^2}{4}\right)\mathfrak{z}^2 + \left(\frac{\alpha_3}{2} - \frac{\alpha_1\alpha_2}{2} + \frac{\alpha_1^3}{8}\right)\mathfrak{z}^3 + \ldots \quad (3.8)$$

and

$$d(w) = \frac{\beta_1}{2}w + \left(\frac{\beta_2}{2} - \frac{\beta_1^2}{4}\right)w^2 + \left(\frac{\beta_3}{2} - \frac{\beta_1\beta_2}{2} + \frac{\beta_1^3}{8}\right)w^3 + \ldots \quad (3.9)$$

From (3.8) and (3.9)

$$\Phi(x, c(\mathfrak{z})) + 1 - \gamma = 1 + \frac{\varphi_2(x)}{2}\alpha_1\mathfrak{z} + \left[\frac{\varphi_2(x)}{2}\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\varphi_3(x)}{4}\alpha_1^2\right]\mathfrak{z}^2 + \ldots \quad (3.10)$$

and

$$\Phi(x, d(w)) + 1 - \gamma = 1 + \frac{\varphi_2(x)}{2}\beta_1 w + \left[\frac{\varphi_2(x)}{2}\left(\beta_2 - \frac{\beta_1^2}{2}\right) + \frac{\varphi_3(x)}{4}\beta_1^2\right]w^2 + \ldots \quad (3.11)$$

Also

$$(1-\rho)\frac{\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi(\mathfrak{z})}{\mathfrak{z}} + \rho(\mathfrak{G}_{q,\varsigma}[\mu_1;\nu_1]\psi(\mathfrak{z}))' = 1 + (\rho+1)\Gamma_2[\mu_1;\nu_1]b_2\mathfrak{z}$$
$$+(2\rho+1)\Gamma_3[\mu_1;\nu_1]b_3\mathfrak{z}^2 + (3\rho+1)\Gamma_4[\mu_1;\nu_1]b_4\mathfrak{z}^3 + \ldots \quad (3.12)$$

and

$$(1-\rho)\frac{\mathrm{g}(w)}{w} + \rho\mathrm{g}'(w) = 1 - (\rho+1)\Gamma_2[\mu_1;\nu_1]b_2 w + (2\rho+1)(2(\Gamma_2[\mu_1;\nu_1])^2 b_2^2 - \Gamma_3[\mu_1;\nu_1]b_3)w^2$$
$$-(3\rho+1)(5(\Gamma_2[\mu_1;\nu_1]^3 b_2^3 - 5\Gamma_2[\mu_1;\nu_1]\Gamma_3[\mu_1;\nu_1]b_2b_3 + \Gamma_4[\mu_1;\nu_1]b_4)w^3 + \ldots \quad (3.13)$$

From (3.10), (3.11), (3.12) and (3.13)

$$(\rho+1)\Gamma_2[\mu_1;\nu_1]b_2 = \frac{\varphi_2(x)}{2}\alpha_1. \quad (3.14)$$

$$(2\rho+1)\Gamma_3[\mu_1;\nu_1]b_3 = \frac{\varphi_2(x)}{2}\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\varphi_3(x)}{4}\alpha_1^2. \quad (3.15)$$

$$(3\rho+1)\Gamma_4[\mu_1;\nu_1]b_4 = \frac{\varphi_2(x)}{2}\left(\alpha_3 - \alpha_1\alpha_2 + \frac{\alpha_1^3}{4}\right) + \frac{\varphi_3(x)}{2}\alpha_1\left(\alpha_2 - \frac{\alpha_1^2}{2}\right) + \frac{\varphi_4(x)}{8}\alpha_1^3. \quad (3.16)$$

and

$$-(\rho+1)\Gamma_2[\mu_1;\nu_1]b_2 = \frac{\varphi_2(x)}{2}\beta_1. \quad (3.17)$$

$$(2\rho+1)(2(\Gamma_2[\mu_1;\nu_1])^2 b_2^2 - \Gamma_3[\mu_1;\nu_1]b_3) = \frac{\varphi_2(x)}{2}\left(\beta_2 - \frac{\beta_1^2}{2}\right) + \frac{\varphi_3(x)}{4}\beta_1^2. \quad (3.18)$$

$$-(3\rho+1)(5(\Gamma_2[\mu_1;\nu_1])^3 b_2^3 - 5\Gamma_2[\mu_1;\nu_1]\Gamma_3[\mu_1;\nu_1]b_2b_3 + \Gamma_4[\mu_1;\nu_1]b_4) =$$
$$\frac{\varphi_2(x)}{2}\left(\beta_3 - \beta_1\beta_2 + \frac{\beta_1^3}{4}\right) + \frac{\varphi_3(x)}{2}\beta_1\left(\beta_2 - \frac{\beta_1^2}{2}\right) + \frac{\varphi_4(x)}{8}\beta_1^3. \quad (3.19)$$

From (3.14)and (3.17)

$$\alpha_1 = -\beta_1, \ \alpha_1^2 = \beta_1^2 \text{ and } \alpha_1^3 = -\beta_1^3. \qquad (3.20)$$

Squaring and adding (3.14)and (3.17)

$$2(\rho+1)^2(\Gamma_2[\mu_1;\nu_1])^2 b_2^2 = \frac{\varphi_2^2(x)}{4}(\alpha_1^2+\beta_1^2). \qquad (3.21)$$

Implies

$$b_2^2 = \frac{\varphi_2^2(x)(\alpha_1^2+\beta_1^2)}{8(\rho+1)^2(\Gamma_2[\mu_1;\nu_1])^2}, \qquad (3.22)$$

Adding (3.15)and (3.18)

$$4(2\rho+1)(\Gamma_2[\mu_1;\nu_1])^2 b_2^2 = \varphi_2(x)(\alpha_2+\beta_2) + (\varphi_3(x)-\varphi_2(x))\alpha_1^2. \quad (3.23)$$

Applying (3.20) in (3.22)

$$\alpha_1^2 = \frac{4(\rho+1)^2(\Gamma_2[\mu_1;\nu_1])^2 b_2^2}{\varphi_2^2(x)} \quad (3.24)$$

In (3.23), replacing α_1^2

$$4(2\rho+1)(\Gamma_2[\mu_1;\nu_1])^2 b_2^2 = \varphi_2(x)(\alpha_2+\beta_2) + \frac{4(\varphi_3(x)-\varphi_2(x))(\rho+1)^2(\Gamma_2[\mu_1;\nu_1])^2 b_2^2}{\varphi_2^2(x)}.$$

Thus

$$b_2^2 = \frac{\varphi_2^3(x)(\alpha_2+\beta_2)}{4(\Gamma_2[\mu_1;\nu_1])^2[(2\rho+1)\varphi_2^2(x)-(\rho+1)^2(\varphi_3(x)-\varphi_2(x))]} \qquad (3.25)$$

Applying Lemma 2.2 and using (1.10)

$$|b_2| \leq \frac{|\delta\, x|\sqrt{|\delta\, x|}}{\Gamma_2[\mu_1;\nu_1]\sqrt{|[(2\rho+1)(\delta\, x+1)+\rho^2]\delta\, x-[l\delta\, x^2+m\gamma](\rho+1)^2|}}.$$

Subtracting (3.18) from (3.15)

$$b_3 = \frac{\varphi_2(x)(\alpha_2-\beta_2)}{4(2\rho+1)\Gamma_3[\mu_1;\nu_1]} + \frac{(\Gamma_2[\mu_1;\nu_1])^2 b_2^2}{\Gamma_3[\mu_1;\nu_1]}. \qquad (3.26)$$

$$b_3 = \frac{\varphi_2(x)(\alpha_2-\beta_2)}{4(2\rho+1)\Gamma_3[\mu_1;\nu_1]} + \frac{\varphi_2^2(x)\alpha_1^2}{4(\rho+1)^2\Gamma_3[\mu_1;\nu_1]}. \qquad (3.27)$$

Applying Lemma 2.2 and using (1.10)

$$|b_3| \leq \frac{|\delta\, x|}{\Gamma_3[\mu_1;\nu_1]}\left|\frac{1}{(2\rho+1)} + \frac{\delta\, x}{(\rho+1)^2}\right|.$$

By removing (3.19) from (3.16)

$$b_4 = \frac{\varphi_2(x)(\alpha_3-\beta_3)}{4(3\rho+1)\Gamma_4[\mu_1;\nu_1]} + \frac{[\varphi_3(x)-\varphi_2(x)]\alpha_1(\alpha_2+\beta_2)}{4(3\rho+1)\Gamma_4[\mu_1;\nu_1]}$$

$$+\frac{5\varphi_2^2(x)\alpha_1(\alpha_2-\beta_2)}{16(\rho+1)(2\rho+1)\Gamma_4[\mu_1;\nu_1]} + \frac{[\varphi_2(x)-2\varphi_3(x)+\varphi_4(x)]\alpha_1^3}{8(3\rho+1)\Gamma_4[\mu_1;\nu_1]}. \qquad (3.28)$$

Applying Lemma 2.2 and using (1.10)

$$|b_4| \leq \frac{|l^2\delta\, x^3 + lmx\gamma + m\delta\, x|}{|3\rho+1|\Gamma_4[\mu_1;\nu_1]} + \frac{5\delta^2\, x^2}{2|(\rho+1)(2\rho+1)|\Gamma_4[\mu_1;\nu_1]}$$

The proof of theorem 3.1 is completed.

The subsequent statements are merely corollaries associated with the specific instances of Horadam polynomials.

Corollary 3.2 If ψ given by (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,\mathbb{F}_n(x)]$. Then

$$|b_2|\le \frac{|x|\sqrt{|x|}}{\Gamma_2[\mu_1;\nu_1]\sqrt{|(2-\rho x)\rho x+(1+\rho^2)x-(\rho+1)^2|}}.$$

$$|b_3|\le \frac{|x|}{\Gamma_3[\mu_1;\nu_1]}\left|\frac{1}{(2\rho+1)}+\frac{x}{(\rho+1)^2}\right|.$$

$$|b_4|\le \frac{|x^3+2x|}{|3\rho+1|\Gamma_4[\mu_1;\nu_1]}+\frac{5x^2}{2|\rho+1||2\rho+1|\Gamma_4[\mu_1;\nu_1]}$$

Corollary 3.3 If ψ given by (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,L_n(x)]$. Then

$$|b_2|\le \frac{|x|\sqrt{|x|}}{\Gamma_2[\mu_1;\nu_1]\sqrt{|(2-\rho x)\rho x+(1+\rho^2)x-2(\rho+1)^2|}}.$$

$$|b_3|\le \frac{|x|}{\Gamma_3[\mu_1;\nu_1]}\left|\frac{1}{(2\rho+1)}+\frac{x}{(\rho+1)^2}\right|.$$

$$|b_4|\le \frac{|x^3+3x|}{|3\rho+1|\Gamma_4[\mu_1;\nu_1]}+\frac{5x^2}{2|\rho+1||2\rho+1|\Gamma_4[\mu_1;\nu_1]}.$$

Corollary 3.4 If ψ given by (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,P_n(x)]$. Then

$$|b_2|\le \frac{2|x|\sqrt{|2x|}}{\Gamma_2[\mu_1;\nu_1]\sqrt{|(1-\rho x)4\rho x+(1+\rho^2)2x-(\rho+1)^2|}}.$$

$$|b_3|\le \frac{2|x|}{\Gamma_3[\mu_1;\nu_1]}\left|\frac{1}{(2\rho+1)}+\frac{2x}{(\rho+1)^2}\right|.$$

$$|b_4|\le \frac{4x|2x^2+1|}{|3\rho+1|\Gamma_4[\mu_1;\nu_1]}+\frac{10x^2}{|\rho+1||2\rho+1|\Gamma_4[\mu_1;\nu_1]}.$$

Corollary 3.5 If ψ given by (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,Q_n(x)]$. Then

$$|b_2|\le \frac{2|x|\sqrt{|2x|}}{\Gamma_2[\mu_1;\nu_1]\sqrt{|(1-\rho x)4\rho x+(1+\rho^2)2x-2(\rho+1)^2|}}.$$

$$|b_3|\le \frac{2|x|}{\Gamma_3[\mu_1;\nu_1]}\left|\frac{1}{(2\rho+1)}+\frac{2x}{(\rho+1)^2}\right|.$$

$$|b_4|\le \frac{2x|4x^2+3|}{|3\rho+1|\Gamma_4[\mu_1;\nu_1]}+\frac{10x^2}{|\rho+1||2\rho+1|\Gamma_4[\mu_1;\nu_1]}.$$

Corollary 3.6 If ψ given by (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,T_n(x)]$. Then

$$|b_2|\le \frac{|x|\sqrt{|x|}}{\Gamma_2[\mu_1;\nu_1]\sqrt{|(1-\rho x)2\rho x+(1+\rho^2)x-(1+2\rho)x^2+(\rho+1)^2|}}.$$

$$|b_3|\le \frac{|x|}{\Gamma_3[\mu_1;\nu_1]}\left|\frac{1}{(2\rho+1)}+\frac{x}{(\rho+1)^2}\right|.$$

$$|b_4|\le \frac{x|4x^2-3|}{|3\rho+1|\Gamma_4[\mu_1;\nu_1]}+\frac{5x^2}{2|\rho+1||2\rho+1|\Gamma_4[\mu_1;\nu_1]}.$$

Corollary 3.7 If ψ given by (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,U_n(x)]$. Then

$$|b_2|\le \frac{2|x|\sqrt{|2x|}}{\Gamma_2[\mu_1;\nu_1]\sqrt{|(1-\rho x)4\rho x+(1+\rho^2)2x+(\rho+1)^2|}}.$$

$$|b_3|\le \frac{2|x|}{\Gamma_3[\mu_1;\nu_1]}\left|\frac{1}{(2\rho+1)}+\frac{2x}{(\rho+1)^2}\right|.$$

$$|b_4|\le \frac{4x|2x^2-1|}{|3\rho+1|\Gamma_4[\mu_1;\nu_1]}+\frac{10x^2}{|\rho+1||2\rho+1|\Gamma_4[\mu_1;\nu_1]}.$$

4 Fekete-Szegö Inequality for the class $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,\Phi(x)]$

Theorem 4.1 If ψ given by (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,\Phi(x)]$. Then for some $\eta \in \mathbb{R}$

$$|b_3-\eta\, b_2^2| \le \begin{cases} 4|\delta\, x||\Delta| & if \quad |\Delta| \ge \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \\ \frac{|\delta\, x|}{\Gamma_3[\mu_1;\nu_1]|2\rho+1|} & if \quad |\Delta| \le \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \end{cases}$$

where

$$|\Delta| = \frac{\delta^2\, x^2}{4|[(2\rho+1)(\delta\, x+1)+\rho^2]\delta\, x-(\rho+1)^2(l\delta\, x^2+m\gamma)|}\left|\frac{1}{\Gamma_3[\mu_1;\nu_1]}-\frac{\eta}{(\Gamma_2[\mu_1;\nu_1])^2}\right|. \quad (4.1)$$

Proof. From (3.26)

$$b_3-\eta\, b_2^2 = \frac{\varphi_2(x)(\alpha_2-\beta_2)}{4(2\rho+1)\Gamma_3[\mu_1;\nu_1]}+\left[\frac{(\Gamma_2[\mu_1;\nu_1])^2}{\Gamma_3[\mu_1;\nu_1]}-\eta\right]b_2^2.$$

Using (3.25)

$$b_3-\eta\, b_2^2 = \varphi_2(x)\left\{\left(\Delta(\eta,\rho)+\frac{1}{4(2\rho+1)\Gamma_3[\mu_1;\nu_1]}\right)\alpha_2+\left(\Delta(\eta,\rho)-\frac{1}{4(2\rho+1)\Gamma_3[\mu_1;\nu_1]}\right)\beta_2\right\}$$

Where

$$\Delta(\eta,\rho) = \frac{\varphi_2^2(x)}{4[(2\rho+1)\varphi_2^2(x)-(\rho+1)^2(\varphi_3(x)-\varphi_2(x))]}\left[\frac{1}{\Gamma_3[\mu_1;\nu_1]}-\frac{\eta}{(\Gamma_2[\mu_1;\nu_1])^2}\right]$$

Using Lemma 2.1

$$|b_3-\eta\, b_2^2| \le \begin{cases} 4|\delta\, x||\Delta| & if \quad |\Delta| \ge \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \\ \frac{|\delta\, x|}{\Gamma_3[\mu_1;\nu_1]|2\rho+1|} & if \quad |\Delta| \le \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \end{cases}$$

Corollary 4.2 If ψ given by (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,\mathbb{F}_n(x)]$. Then

$$|b_3-\eta\, b_2^2| \le \begin{cases} 4|x||\Delta| & if \quad |\Delta| \ge \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \\ \frac{|x|}{\Gamma_3[\mu_1;\nu_1]|2\rho+1|} & if \quad |\Delta| \le \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \end{cases}$$

Where

$$|\Delta| = \frac{x^2}{4|[\rho\, x(2-\rho\, x)+x(1+\rho^2)-(\rho+1)^2|}\left|\frac{1}{\Gamma_3[\mu_1;\nu_1]}-\frac{\eta}{(\Gamma_2[\mu_1;\nu_1])^2}\right|.$$

Corollary 4.3 If ψ given by (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,L_n(x)]$. Then

$$|b_3-\eta\, b_2^2| \le \begin{cases} 4|x||\Delta| & if \quad |\Delta| \ge \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \\ \frac{|x|}{\Gamma_3[\mu_1;\nu_1]|2\rho+1|} & if \quad |\Delta| \le \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \end{cases}$$

Where

$$|\Delta| = \frac{x^2}{4|[\rho\, x(2-\rho\, x)+x(1+\rho^2)-2(\rho+1)^2|}\left|\frac{1}{\Gamma_3[\mu_1;\nu_1]}-\frac{\eta}{(\Gamma_2[\mu_1;\nu_1])^2}\right|.$$

Corollary 4.4 If ψ given by (1.1) is in $\mathfrak{T}_{\sigma}^{q,\varsigma}[\mu_1;\nu_1,P_n(x)]$. Then

$$|b_3 - \eta\, b_2^2| \leq \begin{cases} 8|x||\Delta| & if \quad |\Delta| \geq \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \\ \frac{2|x|}{\Gamma_3[\mu_1;\nu_1]|2\rho+1|} & if \quad |\Delta| \leq \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \end{cases}$$

Where

$$|\Delta| = \frac{x^2}{|[4\rho\, x(1-\rho\, x)+2x(1+\rho^2)-(\rho+1)^2|}\left|\frac{1}{\Gamma_3[\mu_1;\nu_1]} - \frac{\eta}{(\Gamma_2[\mu_1;\nu_1])^2}\right|.$$

Corollary 4.5 If ψ given by (1.1) is in $\mathfrak{T}_\sigma^{q,\varsigma}[\mu_1;\nu_1, Q_n(x)]$. Then

$$|b_3 - \eta\, b_2^2| \leq \begin{cases} 8|x||\Delta| & if \quad |\Delta| \geq \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \\ \frac{2|x|}{\Gamma_3[\mu_1;\nu_1]|2\rho+1|} & if \quad |\Delta| \leq \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \end{cases}$$

Where

$$|\Delta| = \frac{x^2}{|[4\rho\, x(1-\rho\, x)+2x(1+\rho^2)-2(\rho+1)^2|}\left|\frac{1}{\Gamma_3[\mu_1;\nu_1]} - \frac{\eta}{(\Gamma_2[\mu_1;\nu_1])^2}\right|.$$

Corollary 4.6 If ψ given by (1.1) is in $\mathfrak{T}_\sigma^{q,\varsigma}[\mu_1;\nu_1, T_n(x)]$. Then

$$|b_3 - \eta\, b_2^2| \leq \begin{cases} 4|x||\Delta| & if \quad |\Delta| \geq \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \\ \frac{|x|}{\Gamma_3[\mu_1;\nu_1]|2\rho+1|} & if \quad |\Delta| \leq \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \end{cases}$$

Where

$$|\Delta| = \frac{x^2}{4|[(2\rho+1)(1-x)+\rho^2(1-2x)]x+(\rho+1)^2|}\left|\frac{1}{\Gamma_3[\mu_1;\nu_1]} - \frac{\eta}{(\Gamma_2[\mu_1;\nu_1])^2}\right|.$$

Corollary 4.7 If ψ given by (1.1) is in $\mathfrak{T}_\sigma^{q,\varsigma}[\mu_1;\nu_1, U_n(x)]$. Then

$$|b_3 - \eta\, b_2^2| \leq \begin{cases} 8|x||\Delta| & if \quad |\Delta| \geq \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \\ \frac{2|x|}{\Gamma_3[\mu_1;\nu_1]|2\rho+1|} & if \quad |\Delta| \leq \frac{1}{4\Gamma_3[\mu_1;\nu_1]|2\rho+1|} \end{cases}$$

Where

$$|\Delta| = \frac{x^2}{|[4\rho\, x(1-\rho\, x)+2x(1+\rho^2)+(\rho+1)^2|}\left|\frac{1}{\Gamma_3[\mu_1;\nu_1]} - \frac{\eta}{(\Gamma_2[\mu_1;\nu_1])^2}\right|.$$

5 Second Hankel Determinant for $\mathfrak{T}_\sigma^{q,\varsigma}[\mu_1;\nu_1, \Phi(x)]$

Theorem 5.1 If ψ given by (1.1) is in $\mathfrak{T}_\sigma^{q,\varsigma}[\mu_1;\nu_1, \Phi(x)]$. Then

$$|b_2 b_4 - b_3^2| \leq \begin{cases} \mathcal{J}(x,2); & B_1 \geq 0 \; and \; B_2 \geq 0 \\ max\left\{\frac{\delta^2 x^2}{(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2}, \mathcal{J}(x,2)\right\}; & B_1 > 0 \; and \; B_2 < 0 \\ \frac{\delta^2 x^2}{(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2}; & B_1 \leq 0 \; and \; B_2 \leq 0 \\ max\{\mathcal{J}(x,\alpha_0), \mathcal{J}(x,2)\}; & B_1 < 0 \; and \; B_2 > 0 \end{cases}$$

Where

$$\mathcal{J}(x,2) = \frac{\delta^2 x^2}{(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2} + \frac{B_1+B_2}{2(\rho+1)^4(2\rho+1)^2(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]},$$

$$\mathcal{J}(x,\alpha_0) = \frac{\delta^2 x^2}{(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2} - \frac{B_2^2}{8B_1(\rho+1)^4(2\rho+1)^2(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]},$$

$$B_1 = 2(\rho+1)^3(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2\delta\, x[(l\delta\, x^2 + m\gamma)(l\, x - 2) + \delta\, x(m-1)]$$
$$+2(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]\delta^2 x^2[(\rho+1)^4 - (2\rho+1)^2\delta^2 x^2]$$
$$+(5(\Gamma_3[\mu_1;\nu_1])^2 - 4\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1])(\rho+1)^2(2\rho+1)(3\rho+1)\delta^3\, x^3,$$

and

$$B_2 = 2(\rho+1)^3(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2\delta\, x[\delta\, x(1+2lx) + 2m\gamma]$$
$$-4(\rho+1)^4(3\rho+1)\delta^2\, x^2\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]$$
$$+(5(\Gamma_3[\mu_1;\nu_1])^2 - 4\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1])(\rho+1)^2(2\rho+1)(3\rho+1)\delta^3\, x^3.$$

Proof. From (3.14), (3.27) and (3.28)

$$b_2b_4 - b_3^2 = \frac{\varphi_2(x)[\varphi_2(x) - 2\varphi_3(x) + \varphi_4(x)](\rho+1)^3(\Gamma_3[\mu_1;\nu_1])^2 - \varphi_2^4(x)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}{16(\rho+1)^4(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}\alpha_1^4$$

$$+\frac{\varphi_2(x)[\varphi_3(x) - \varphi_2(x)]\alpha_1^2(\alpha_2+\beta_2)}{8(\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]} + \frac{\varphi_2^2(x)\alpha_1(\alpha_3-\beta_3)}{8(\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}$$

$$+\frac{[5(\Gamma_3[\mu_1;\nu_1])^2 - 4\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]]\varphi_2^3(x)\alpha_1^2(\alpha_2-\beta_2)}{32(\rho+1)^2(2\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]} - \frac{\varphi_2^2(x)(\alpha_2-\beta_2)^2}{16(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2}.$$

using Lemma[2.3]

$$\alpha_2 - \beta_2 = \frac{(4-\alpha_1^2)(x-y)}{2}, \quad (5.1)$$

$$\alpha_2 + \beta_2 = \frac{(4-\alpha_1^2)(x+y)}{2} + \alpha_1^2, \quad (5.2)$$

and

$$\alpha_3 - \beta_3 = \frac{\alpha_1^3}{2} + \frac{4-\alpha_1^2}{2}\alpha_1(x+y) - \frac{4-\alpha_1^2}{4}\alpha_1(x^2+y^2) + \frac{4-\alpha_1^2}{2}[(1-|x|^2)z - (1-|y|^2)w]. (5.3)$$

for some x, y, z, w with $|x| \le 1$, $|y| \le 1$, $|z| \le 1$, $|w| \le 1$, $|\alpha_1| \in (0,2)$ and substituting $\alpha_2 - \beta_2$, $\alpha_2 + \beta_2$ and $\alpha_3 - \beta_3$ and after some simplifications

$$b_2b_4 - b_3^2 = \frac{\varphi_2(x)\varphi_4(x)(\rho+1)^3(\Gamma_3[\mu_1;\nu_1])^2 - \varphi_2^4(x)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}{16(\rho+1)^4(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}\alpha_1^4$$

$$+\frac{\varphi_2(x)\varphi_3(x)\alpha_1^2(4-\alpha_1^2)(x+y)}{16(\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]} - \frac{\varphi_2^2(x)(4-\alpha_1^2)\alpha_1^2(x^2+y^2)}{32(\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}$$

$$+\frac{[5(\Gamma_3[\mu_1;\nu_1])^2 - 4\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]]\varphi_2^3(x)\alpha_1^2(4-\alpha_1^2)(x-y)}{64(\rho+1)^2(2\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]} - \frac{\varphi_2^2(x)(4-\alpha_1^2)^2(x-y)^2}{64(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2}$$

$$+\frac{\varphi_2^2(x)(4-\alpha_1^2)\alpha_1[(1-|x|^2)\mathfrak{z} - (1-|y|^2)w]}{16(\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}.$$

Let $\alpha = \alpha_1$, without any restriction it can be assumed that $\alpha \in [0,2]$, $\xi_1 = |x| \le 1$, $\xi_2 = |y| \le 1$ and applying triangular inequality,

$$|b_2b_4 - b_3^2| \le \frac{\varphi_2(x)\varphi_4(x)(\rho+1)^3(\Gamma_3[\mu_1;\nu_1])^2 - \varphi_2^4(x)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}{16(\rho+1)^4(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}\alpha^4$$

$$+\frac{\varphi_2^2(x)\alpha(4-\alpha^2)}{8(\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]} + \frac{\varphi_2^2(x)(4-\alpha^2)\alpha(\alpha-2)(\xi_1^2+\xi_2^2)}{32(\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}$$

$$+\frac{4(\rho+1)(2\rho+1)(\Gamma_3[\mu_1;\nu_1])^2\varphi_2(x)\varphi_3(x)\alpha^2(4-\alpha^2)(\xi_1+\xi_2)}{64(\rho+1)^2(2\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}$$

$$+\frac{(5(\Gamma_3[\mu_1;\nu_1])^2-4\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1])\varphi_2^3(x)(3\rho+1)\alpha^2(4-\alpha^2)(\xi_1+\xi_2)}{64(\rho+1)^2(2\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}$$

$$+\frac{\varphi_2^2(x)(4-\alpha^2)^2(\xi_1+\xi_2)^2}{64(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2}.$$

That is

$$|b_2b_4-b_3^2|\leq T_1(x,\alpha)+T_2(x,\alpha)(\xi_1+\xi_2)+T_3(x,\alpha)(\xi_1^2+\xi_2^2)+T_4(x,\alpha)(\xi_1+\xi_2)^2$$
$$=F(\xi_1,\xi_2). \qquad (5.4)$$

Where,

$$T_1(x,\alpha)=\frac{\varphi_2(x)\varphi_4(x)(\rho+1)^3(\Gamma_3[\mu_1;\nu_1])^2-\varphi_2^4(x)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}{16(\rho+1)^4(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}\alpha^4$$
$$+\frac{\varphi_2^2(x)\alpha(4-\alpha^2)}{8(\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}\geq 0,$$

$$T_2(x,\alpha)=\frac{4(\rho+1)(2\rho+1)(\Gamma_3[\mu_1;\nu_1])^2\varphi_2(x)\varphi_3(x)\alpha^2(4-\alpha^2)}{64(\rho+1)^2(2\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}$$
$$+\frac{(5(\Gamma_3[\mu_1;\nu_1])^2-4\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1])\varphi_2^3(x)(3\rho+1)\alpha^2(4-\alpha^2)}{64(\rho+1)^2(2\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}\geq 0,$$

$$T_3(x,\alpha)=\frac{\varphi_2^2(x)(4-\alpha^2)\alpha(\alpha-2)}{32(\rho+1)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}\leq 0,$$

$$T_4(x,\alpha)=\frac{\varphi_2^2(x)(4-\alpha^2)^2}{64(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2}\geq 0, \quad \text{and} \quad 0\leq\alpha\leq 2.$$

As we have to maximize $F(\xi_1,\xi_2)$ in the closed square

$C_s=\{(\xi_1,\xi_2),0\leq\xi_1\leq 1,0\leq\xi_2\leq 1\}$.

Let $\alpha=0$, $\alpha=2$ and $\alpha\in(0,2)$

When $\alpha=0$,

$$F(\xi_1,\xi_2)=T_4(x,0)=\frac{\varphi_2^2(x)(\xi_1+\xi_2)^2}{4(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2}$$

and in this case $F(\xi_1,\xi_2)$ reaches it maximum at (ξ_1,ξ_2) and

$$max\{F(\xi_1,\xi_2):\xi_1,\xi_2\in[0,1]\}=F(1,1)=\frac{\varphi_2^2(x)}{(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2}.$$

When $\alpha=2$, $F(\xi_1,\xi_2)$ is a constant function.

$$F(\xi_1,\xi_2)=T_1(x,2)=\frac{\varphi_2(x)\varphi_4(x)(\rho+1)^3(\Gamma_3[\mu_1;\nu_1])^2-\varphi_2^4(x)(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]}{(\rho+1)^4(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}.$$

When $\alpha\in(0,2)$, let $\xi_1+\xi_2=e$ and $\xi_1.\xi_2=g$, then

$$F(\xi_1,\xi_2)=T_1(x,\alpha)+T_2(x,\alpha)e+(T_3(x,\alpha)+T_4(x,\alpha))e^2-2T_3(x,\alpha)g=H(e,g).$$

where $e\in[0,2]$ and $g\in[0,1]$. Now we have to investigate the maximum of

$$H(e,g)\in\Theta=\{(e,g):e\in[0,2],g\in[0,1]\}\ (5.5)$$

Differentiating 5.5 partially,

$$\frac{\partial H}{\partial e}=T_2(x,\alpha)+2\big(T_3(x,\alpha)+T_4(x,\alpha)\big)e=0 \text{ and } \frac{\partial H}{\partial g}=-2T_3(x,\alpha)=0$$

reveals $H(e,g)$ does not have a critical point in closed square C_s and so $F(\xi_1,\xi_2)$ does not have a critical point in the square C_s.

So $F(\xi_1,\xi_2)$ cannot have maximum value in the interior of closed square. The maximum of $F(\xi_1,\xi_2)$ on the boundary of C_s has to be investigated.

For $\xi_1 = 0, \xi_2 \in [0,1]$ (similarly for $\xi_2 = 0, \xi_1 \in [0,1]$) and

$$F(0,\xi_2) = T_1(x,\alpha) + T_2(x,\alpha)\xi_2 + [T_3(x,\alpha) + T_4(x,\alpha)]\xi_2^2 = M(\xi_2).$$

Since $T_3(x,\alpha) + T_4(x,\alpha) \geq 0$, we have

$$M'(\xi_2) = T_2(x,\alpha) + 2[T_3(x,\alpha) + T_4(x,\alpha)]\xi_2 > 0,$$

which implies that $M(\xi_2)$ is an increasing function. Therefore for a fixed $\alpha \in [0,2)$ the maximum occurs at $\xi_2 = 1$ and

$$Max\, M(\xi_2) = M(1) = T_1(x,\alpha) + T_2(x,\alpha) + T_3(x,\alpha) + T_4(x,\alpha) = F(0,1).$$

For $\xi_1 = 1$, $\xi_2 \in [0,1]$ (similarly for $\xi_2 = 1$, $\xi_1 \in [0,1]$) and

$$F(1,\xi_2) = T_1(x,\alpha) + T_2(x,\alpha) + T_3(x,\alpha) + T_4(x,\alpha)$$
$$+[T_2(x,\alpha) + 2T_4(x,\alpha)]\xi_2 + [T_3(x,\alpha) + T_4(x,\alpha)]\xi_2^2 = G(\xi_2).$$

As $T_3(x,\alpha) + T_4(x,\alpha) \geq 0$ then

$$G'(\xi_2) = T_2(x,\alpha) + 2T_4(x,\alpha) + 2[T_3(x,\alpha) + T_4(x,\alpha)]\xi_2 > 0.$$

Therefore $G(\xi_2)$ is an increasing function and the maximum is attained at $\xi_2 = 1$.

$$Max\{F(1,\xi_2);\quad \xi_2 \in [0,1]\} = F(1,1) = T_1(x,\alpha) + 2[T_2(x,\alpha) + T_3(x,\alpha)] + 4T_4(x,\alpha). \quad (5.6)$$

Now for each $\alpha \in (0,2)$, we have

$$T_1(x,\alpha) + 2[T_2(x,\alpha) + T_3(x,\alpha)] + 4T_4(x,\alpha) > T_1(x,\alpha) + T_2(x,\alpha) + T_3(x,\alpha) + T_4(x,\alpha).$$

Thus

$max\{F(\xi_1,\xi_2);\quad \xi_1 \in [0,1],\quad \xi_2 \in [0,1]\} = T_1(x,\alpha) + 2[T_2(x,\alpha) + T_3(x,\alpha)] + 4T_4(x,\alpha)$. Since $M(1) \leq G(1)$ for $\alpha \in [0,2]$. we have, $max\{F(\xi_1,\xi_2)\} = F(1,1)$ occus on the boundary of closed square.

Let $\mathcal{J}: (0,2) \longrightarrow \mathbb{R}$ be the function defined by

$$\mathcal{J}(x,\alpha) = max\{F(\xi_1,\xi_2)\} = F(1,1) = T_1(x,\alpha) + 2[T_2(x,\alpha) + T_3(x,\alpha)] + 4T_4(x,\alpha). \quad (5.7)$$

By putting the values of $T_1(x,\alpha)$, $T_2(x,\alpha)$, $T_3(x,\alpha)$, $T_4(x,\alpha)$ in (5.7)

$$\mathcal{J}(x,\alpha) = \frac{\varphi_2^2(x)}{(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2} + \frac{B_1\alpha^4 + 4B_2\alpha^2}{32(\rho+1)^4(2\rho+1)^2(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]} \quad (5.8)$$

where

$$B_1 = 2(\rho+1)^3(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2\varphi_2(x)[\varphi_4(x) - 2\varphi_3(x) - \varphi_2(x)]$$
$$+2(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]\varphi_2^2(x)[(\rho+1)^4 - (2\rho+1)^2\varphi_2^2(x)]$$
$$+\big[5(\Gamma_3[\mu_1;\nu_1])^2 - 4\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]\big](\rho+1)^2(2\rho+1)(3\rho+1)\varphi_2^3(x).$$

and

$$B_2 = 2(\rho+1)^3(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2\varphi_2(x)[2\varphi_3(x) + \varphi_2(x)]$$
$$-4(\rho+1)^4(3\rho+1)\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]\varphi_2^2(x)$$
$$+\big[5(\Gamma_3[\mu_1;\nu_1])^2 - 4\Gamma_2[\mu_1;\nu_1]\Gamma_4[\mu_1;\nu_1]\big](\rho+1)^2(2\rho+1)(3\rho+1)\varphi_2^3(x).$$

If $\mathcal{J}(x,\alpha)$ has a maximum value in the interior of $\alpha \in [0,2]$ we have

$$\mathcal{J}'(x,\alpha) = \frac{B_1\alpha^3 + 2B_2\alpha}{8(\rho+1)^4(2\rho+1)^2(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}$$

As the sign of $\mathcal{J}'(x,\alpha)$ depends on the sign of B_1 and B_2, we have the following cases

Result 1: When $B_1 \geq 0$ and $B_2 \geq 0$ we have $\mathcal{J}'(x,\alpha) \geq 0$. So $\mathcal{J}(x,\alpha)$ is an increasing function. Therefore $max\{\mathcal{J}(x,\alpha); \quad \alpha \in (0,2)\} = \mathcal{J}(x,2)$

$$= \frac{\varphi_2^2(x)}{(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2} + \frac{B_1+B_2}{2(\rho+1)^4(2\rho+1)^2(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}.$$

Thus $Max\{ Max\{ F(\xi_1,\xi_2); 0 \leq \xi_1,\xi_2 \leq 1\}: 0 < \alpha < 2\} = \mathcal{J}(x,2)$

Result 2: When $B_1 > 0$ and $B_2 < 0$. we have,

$$\mathcal{J}'(x,\alpha) = \frac{(B_1\alpha^2+2B_2)\alpha}{8(\rho+1)^4(2\rho+1)^2(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}.$$

So $\mathcal{J}'(x,\alpha) = 0$ at point $\alpha_0 = \sqrt{\frac{-2B_2}{B_1}}$ which is the critical point of $\mathcal{J}(x,\alpha)$.

Now

$$\mathcal{J}''(x,\alpha) = \frac{-4B_2}{8(\rho+1)^4(2\rho+1)^2(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]} > 0 \text{ at } \alpha_0.$$

Therefore α_0 is the minimum point of the function $\mathcal{J}(x,\alpha)$. Hence $\mathcal{J}(x,\alpha)$ cannot have a maximum.

Result 3: If $B_1 \leq 0$ and $B_2 \leq 0$ then $\mathcal{J}'(x,\alpha) \leq 0$, therefore $\mathcal{J}(x,\alpha)$ is a decreasing function on the interval $(0,2)$. Hence

$$max\{\mathcal{J}(x,\alpha); \alpha \in (0,2)\} = \mathcal{J}(x,0) = \frac{\varphi_2^2(x)}{(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2}. \qquad (5.9)$$

Result 4: If $B_1 < 0$ and $B_2 > 0$ then $\mathcal{J}''(x,\alpha) < 0$ at α_0. Hence α_0 is the maximum point of $\mathcal{J}(x,\alpha)$ and the maximum value occurs at $\alpha = \alpha_0$. Thus

$$max\{\mathcal{J}(x,\alpha); \alpha \in (0,2)\} = \mathcal{J}(x,\alpha_0)$$

$$\mathcal{J}(x,\alpha_0) = \frac{\varphi_2^2(x)}{(2\rho+1)^2(\Gamma_3[\mu_1;\nu_1])^2} - \frac{B_2^2}{8B_1(\rho+1)^4(2\rho+1)^2(3\rho+1)\Gamma_2[\mu_1;\nu_1](\Gamma_3[\mu_1;\nu_1])^2\Gamma_4[\mu_1;\nu_1]}$$

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