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An analysis of (0,1;0) interpolation based on roots of Pál-type interpolatory polynomials

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Abstract: - The objective of this paper is to investigate an interpolation process based on the roots of the polynomial $\pi_n(x)$ and it's subject to an additional conditional Point $x_0 = 0$. Specifically, we consider two sets of nodes: $\{x_i\}_{i=1}^{n}$representing the roots of $\pi_n(x)$ and $\{x_i^*\}_{i=1}^{n-1}$corresponding to the roots of $\pi_n'(x)$. Our study focuses on establishing the existence and uniqueness of the interpolatory polynomial, deriving its explicit representation, and analysing its order of convergence.

Keywords*: Lagrange Polynomial; Explicit Form; Fundamental Polynomials;*

1. Introduction

In 1975, L. G. Pál [7] introduced a modification of the Hermite-Fejér interpolation, where function values and first derivatives are prescribed at two interlaced sets of nodal points, denoted as $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$, respectively. These nodes are structured such that:

$$-\infty < x_1 < x_1^* < x_2 < \cdot \quad \cdot \quad \cdot < x_{n-1} < x_{n-1}^* < x_n < +\infty.$$

Here, the polynomials defining these nodes are given by:

$$\omega_n(x) = \prod_{i=1}^{n-1}(x - x_i)$$

Pál proved that for any given set of real numbers $\{b_i\}_{i=1}^{n}$ and $\{b_i^*\}_{i=1}^{n-1}$, there exist polynomial

$$R_{2n-1}(x_i) = b_i \qquad \text{i = 1, 2, 3, ..., n,}$$

,

$$R_{2n-1}(x_i^*) = b_i^* \qquad \text{i = 1, 2, 3, ..., n − 1,}$$

with additional condition $R_n(x_0) = 0$ where $x_0 \neq x_i$ for i=1,2,3,........,n and$\{b_i\}_{i=1}^{n}$ and $\{b_i^*\}_{i=1}^{n-1}$i=1 are arbitrary real numbers, whose convergence for $R_n(x)$ has been proved by S.A. Eneduanya on the roots of $\pi_n(x)$. Pál [13], Mathur P. and Datta S. [11] and many other authors [4][12][14][15][9] have discussed about various kind of interpolation problems. Pál [7] proved that when the values are fixed on one set of n points and derivative values on other set of n-1 points, then there exists no unique polynomial ≤ 2n − 2, but fixed function value at one more point not belonging to above set of n points there exists a unique polynomial of degree ≤ 2n − 1. In Eneduanya [15] investigated special case when

$$\pi_n(x) = n(n-1)\int_1^x P_{n-1}(x)dx = (1- x^2)P_{n-1}'(x) \qquad (1)$$

where P_{n-1} is the (n −1) th the Legendre polynomial with the usual normalization max $\{|P_{n-1}(x)| : x \in [-1, 1]\}$ = 1. For the uniqueness Eneduanya used also the additional condition nodal points $x_n^* = -1$. Szili [9] investigated the Pál -type interpolation on the roots of the Hermite-polynomials with the additional conditional point x_0 = 0. Both Szili and Eneduanya gave explicit formula and proved approximation theorems. Joo and Szabo [3] gave a common generalization of the classical Fejer interpolation and Pál interpolation. Szili [8] studied the inverse Pál interpolation problem on the roots of integrated Legendre polynomials. Later, Yamini Singh and R. Srivastava [14] studied an interpolation process on the roots of ultraspherical polynomials. In this paper, we have studied about an interpolation on the roots of polynomials $\pi_n(x)$. Pál type polynomials are a generalization of classical orthogonal polynomials and have been studied for their desirable approximation properties. By leveraging the structure of these polynomials, we analyse the behaviour of the associated interpolatory functions, their convergence characteristics, and their numerical stability. In this study, we investigate (0,1;0) interpolation constructed using the roots of a Pál-type interpolatory polynomial.

Preliminaries:

$Let\ P_{n-1}$(x) and $\pi_n(x)$ satisfy the differential equation are

$$(1\text{-}x^2)\ P'_{n-1}(x)\text{-}2xP'_{n-1}(x) + n(n-1)P_n(x) = 0. \quad (2)$$

$$(1\text{-}x^2)\ \pi''_{n-1}(x) + n(n-1)\pi_n(x) = 0 \quad (3)$$

Respectively.

$$l^*_{i,n}(x) = \frac{\pi'_n(x).}{\pi''_n(x^*_{i,n})(x-x^*_{i,n}).} \qquad (i = 1, 2, 3, \ldots.. n-1) \quad (4)$$

$$(1 - x_i^{*2}) \sim (i/n)^2 \qquad \text{(from ([2], (6.3.7))} \quad (5)$$

$$\left|P'_{n-1}(x^*_{i,n})\right| \sim i^{-3/2} n^2 \qquad (from\ ([2];\ (8{:}\,9{:}\,2)]), \quad (6)$$

2. Problem:

Let $x_{0,n},\ x_{1,n},\ x_{2,n}, \ldots\ldots,\ x_{n,n}$ be the roots of the polynomial $\pi_n(x)$ and $x^*_{1,n-1}\ x^*_{2,n-1}\ x^*_{3,n-1}\ \ldots,\ x^*_{n-1,n-1}$ be the roots of polynomial $\pi'_n(x)$. Let

$$-1 < x_{1,n} < x^*_{1,n} < x_{2,n} < \ldots\ldots\ldots < x_{n-1,n} < x^*_{n-1,n} < x_{n,n} < 1. \quad (7)$$

Further, we investigate the following problem: We determine a polynomial

$R_n(x)$ of lowest possible degree satisfying the conditions

$$R_n(f; x_{i,n}) = f(x_{i,n}) \qquad (i = 0,1, 2, 3, \ldots\ldots, n), \quad (8)$$

$$R'_n(f; x_{i,n}) = f'(x_{i,n}) \qquad (i = 1, 2, 3, \ldots\ldots,n), \quad (9)$$

$$R_n(f; x^*_{i,n}) = f\left(x^*_{i,n}\right) \qquad (i = 1, 2, 3, \ldots.., n-1). \quad (10)$$

where, $f(x_{i,n}),\ f'(x_{i,n})$ and $f(x^*_{i,n})$ are arbitrary given real numbers. Morever,

we have to prove the existence, uniqueness, explicit representation and order of

convergence of interpolatory polynomials.

3. Explicit Representation of Interpolatory polynomials:

Now let $f : [-1, 1] \to \mathbb{R}$ be a differentiable function. If n is even, then we get that

$$R_n(f; x) = \sum_{i=0}^{n} f(x_{i,n})A_{i,n}(x) + \sum_{i=1}^{n} f'\left(x_{i,n}\right)B_{i,n}(x) + \sum_{i=1}^{n-1} f\left(x^*_{i,n}\right)C_{i,n}(x) \quad (11)$$

is the uniquely determined polynomial of degree ≤ 3n−1 using the well known

relations for the Legendre polynomials we obtain that the polynomials $A_{i,n}(x)$,

$B_{i,n}(x)$ and $C_{i,n}(x)$satisfy the following requirement:

$$A_{i,n}(x_{j,n}) = \delta_{i,j} \quad (i = 0,1,2,3,\ldots,n;\ j = 0,1,2,3,\ldots,n)$$

$$A'_{i,n}(x_{j,n}) = 0, \quad (i = 1,2,3,\ldots\ldots,n;\ j = 1,2,3,\ldots\ldots,n) \quad (12)$$

$$A_{i,n}(x^*_{j,n}) = 0, \quad (i = 1,2,3,\ldots,n;\ j = 1,2,3,\ldots.,n-1)$$

$$B_{i,n}(x_{j,n}) = 0 \quad (i = 1,2,3,\ldots,n;\ j = 0,1,2,3,\ldots,n)$$

$$B'_{i,n}(x_{j,n}) = \delta_{i,j}, \quad (i = 1,2,3,\ldots\ldots,n;\ j = 1,2,3,\ldots\ldots,n) \quad (13)$$

$$B_{i,n}(x^*_{j,n}) = 0, \quad (i = 1,2,3,\ldots,n;\ j = 1,2,3,\ldots.,n-1)$$

$$C_{i,n}(x_{j,n}) = 0 \quad (i = 1,2,3,\ldots,n;\ j = 0,1,2,3,\ldots,n)$$

$$C'_{i,n}(x_{j,n}) = 0, \quad (i = 1,2,3,\ldots\ldots,n;\ j = 1,2,3,\ldots\ldots,n) \quad (14)$$

$$C_{i,n}(x^*_{j,n}) = \delta_{i,j}, \quad (i = 1,2,3,\ldots,n;\ j = 1,2,3,\ldots.,n-1)$$

where $\delta_{i,j}$ is the Kronecker symbol.

Lemma: The fundamental polynomial $A_{i,n}(x) for\ i = 0,1,2,\ldots\ldots,n$ satisfies the interpolatory conditions (12) is given by**:**

$$A_{i,n}(x_0) = \frac{\pi'_n(x_0).}{\pi'_n(x_{i,n})(1+x_{i,n}).}(l_{i,n}(x_0))^2(1+x_0) \quad (15)$$

$$A_{i,n}(x) = \frac{\pi'_n(x).}{\pi'_n(x_{i,n})(1+x_{i,n}).}(l_{i,n}(x))^2(1+x) - \frac{\pi_n(x)\pi'_n(x)l_{i,n}(x)}{\pi'_n(x_{i,n})}\left[l'_{i,n}(x_{i,n}) + \frac{1}{(1+x_{i,n})}\right] \quad (16)$$

where,

$$l_{i,n}(x) = \frac{\pi_n(x)}{\pi'_n(x_{i,n})(x-x_{i,n})} \quad (i = 1,2,3,\ldots\ldots n), \quad (17)$$

is the Lagrange fundamental polynomials corresponding to nodal points $x_{i,n}$

$(i = 1,2,\ldots\ldots,n)$.

Lemma: The fundamental polynomial $B_{i,n}(x)$for i = 1, 2,...,n that satisfies

The interpolatory conditions (13) is given by

$$B_{i,n}(x) = \frac{\pi_n(x)\pi'_n(x)l_{i,n}(x)}{\pi'_n(x_{i,n})}\left[\frac{x_{i,n}}{n(n-1)} + \frac{1}{2\pi'_n(x_{i,n})}\right] \quad (18)$$

Lemma: The fundamental polynomial $C_{i,n}(x)$ for $i = 1,2,\ldots,n-1$ satisfying the interpolatory conditions (14) is given by

$$C_{i,n}(x) = \frac{\pi_n^2(x)l^*_{i,n}(x)}{\pi_n^2(x^*_{i,n})} \quad (19)$$

4. Order of Convergence of the fundamental polynomials

Lemma: For the Lebesgue function of the fundamental polynomials

$$\sum_{i=0}^{n} |A_{i,n}(x)| = O(n^{-1/2})$$

$$(x \in [-1,1];\ n = 2,4,\dots), where\ O\ does\ not\ depends\ on\ x.$$

Proof: We have,

$A_{i,n}(x) = \frac{\pi_n'(x).}{\pi_n'(x_{i,n})(1+x_{i,n}).}(l_{i,n}(x))^2(1+x)$-$\frac{\pi_n(x)\pi_n'(x)l_{i,n}(x)}{\pi_n'(x_{i,n})}[l_{i,n}'(x_{i,n}) + \frac{1}{(1+x_{i,n})}]$

$$\sum_{i=0}^{n} |A_{i,n}(x)| =$$

$$\sum_{i=0}^{n} \left| \frac{\pi_n'(x).}{\pi_n'(x_{i,n})(1+x_{i,n}).}(l_{i,n}(x))^2(1+x) \right|$$

$$- \sum_{i=0}^{n} \left| \frac{\pi_n(x)\pi_n'(x)l_{i,n}(x)}{\pi_n'(x_{i,n})} \left[l_{i,n}'(x_{i,n}) + \frac{1}{(1+x_{i,n})} \right] \right|$$

$$= \sum_{i=0}^{n} \left| \frac{\pi_n'(x).}{\pi_n'(x_{i,n})(1+x_{i,n}).} \right| \left|(l_{i,n}(x))^2(1+x)\right| - \sum_{i=0}^{n} \left| \frac{\pi_n(x)\pi_n'(x)l_{i,n}(x)}{\pi_n'(x_{i,n})} \right|$$

$$\times \left| \left[l_{i,n}'(x_{i,n}) + \frac{1}{(1+x_{i,n})} \right] \right|$$

Since $|\pi_n(x)\ | = O(n^{1/2})$ and $|P_{n-1}(x)| \le 1,\ x \in [-1,1]\ \ (from\ [5;\ 2:3:4]\ ).$

$$= \sum_{i=0}^{n} \left| \frac{n(n-1)P_{n-1}(x)}{n(n-1)P_{n-1}(x_{i,n})(1+x_{i,n})} \right| \left|(l_{i,n}(x))^2(1+x)\right| -$$

$$\sum_{i=0}^{n} \left| \frac{O(n^{1/2})n(n-1)P_{n-1}(x)l_{i,n}(x)}{n(n-1)P_{n-1}(x_{i,n})} \right| \left| \left[l_{i,n}'(x_{i,n}) + \frac{1}{(1+x_{i,n})} \right] \right|$$

$$|P_{n-1}(x_{i,n})| = (8\pi i)^{-1/2}$$

(from([5],Lemma 2.1).

$$= \sum_{i=0}^{n} \left| \frac{1}{P_{n-1}(x_i)} \right| \left| \left(l_{i,n}(x)\right)^2 \right| - \sum_{i=0}^{n} \left| \frac{O(n^{1/2})l_{i,n}(x)}{P_{n-1}(x_{i,n})} \right| \left| \left[l_{i,n}'(x_{i,n}) + \frac{1}{(1+x_{i,n})} \right] \right|$$

$= O(n^{-1/2})$.

Hence, lemma 1 proved.

Lemma 2: For the Lebesgue function of the fundamental polynomials

$$\sum_{i=1}^{n} |B_{i,n}(x)| = O(n^{-5/2})$$

$$(x \in [-1,1];\ n = 2,4,\dots), where\ O\ does\ not\ depends\ on\ x.$$

Proof: We have,

$$B_{i,n}(x) = \frac{\pi_n(x)\pi'_n(x)l_{i,n}(x)}{\pi'^2_n(x_{i,n})}\left[\frac{x_{i,n}}{n(n-1)} + \frac{1}{2\pi'_n(x_{i,n})}\right]$$

$$\sum_{i=1}^{n}|B_{i,n}(x)|$$

$$= \sum_{i=1}^{n}\left|\frac{\pi_n(x)\pi'_n(x)l_{i,n}(x)}{\pi'^2_n(x_{i,n})}\left[\frac{x_{i,n}}{n(n-1)} + \frac{1}{2\pi'_n(x_{i,n})}\right]\right|$$

$$= \sum_{i=1}^{n}\frac{|\pi_n(x)|\,|\pi'_n(x)|\,|l_{i,n}(x)|}{|\pi'^2_n(x_{i,n})|}\left|\left[\frac{x_{i,n}}{n(n-1)} + \frac{1}{2\pi'_n(x_{i,n})}\right]\right|$$

$$= \sum_{i=1}^{n}\frac{|\pi_n(x)|\left|\pi'_n(x)\right|\,|l_{i,n}(x)|}{\left|n^2(n-1)^2P^2_{n-1}(x_{i,n})\right|}\left|\left[\frac{x_{i,n}}{n(n-1)} + \frac{1}{2\pi'_n(x_{i,n})}\right]\right|$$

$$= O(n^{1/2})\sum_{i=1}^{n}\frac{\left|\pi'_n(x)\right|\,\left|l_{i,n}(x)\right|}{\left|n^2(n-1)^2P^2_{n-1}(x_{i,n})\right|}\left|\frac{x_{i,n}}{n(n-1)} + \frac{1}{2n(n-1)P_{n-1}(x_{i,n})}\right|$$

$$= O(n^{-5/2}).$$

Hence, lemma 2 proved.

Lemma 3: For the Lebesgue function of the fundamental polynomials

$$\sum_{i=1}^{n-1}|C_{i,n}(x)| = O(n^{-3/2})$$

$$(x \in [-1,1];\ n = 2,4,\dots), where\ O\ does\ not\ depends\ on\ x.$$

Proof: We have,

$$C_{i,n}(x) = \frac{\pi^2_n(x)l^*_{i,n}(x)}{\pi^2_n(x^*_{i,n})}$$

$$\sum_{i=1}^{n-1}|C_{i,n}(x)| = \sum_{i=1}^{n-1}\left|\frac{\pi^2_n(x)l^*_{i,n}(x)}{\pi^2_n(x^*_{i,n})}\right|$$

$$= \sum_{i=1}^{n-1}\left|\frac{\pi^2_n(x)l^*_{i,n}(x)}{(1-x^{*2}_{i,n})^2P'^2_{n-1}(x^*_{i,n})}\right|$$

$$= O(n^1)\sum_{i=1}^{n-1}\left|\frac{l^*_{i,n}(x)}{\left(1-x^{*2}_{i,n}\right)^2P'^2_{n-1}(x^*_{i,n})}\right| = O(n^{-3/2})$$

Hence, lemma 3 proved.

Theorem: Let $f : [-1,1] \rightarrow \mathbb{R}$ be continuously differentiable function, then the sequence of the interpolation polynomials $R_n(f;\ x)$ $(n = 2,4,6,\ldots\ldots)$ given by (11) satisfy the following:

$$|R_n(f;\ x) - f(x)| = O\left(n^{-1}\omega\left(f';\frac{1}{n}\right)\right) \tag{20}$$

Where, $\omega(f';\delta)$ is the modulus of continuity of f'and O does not depend on x.

Proof: If $Q_n(x)$ is an arbitrary polynomial of degree ≤ 3n − 1 then by

uniqueness of the polynomial $R_n(f;\ x)$ we have

$$R_n(f;x) = \sum_{i=0}^{n} Q_n(x_{i,n})\, A_{i,n}(x) + \sum_{i=1}^{n} Q_n{}'(x_{i,n}) B_{i,n}(x) + \sum_{i=1}^{n-1} Q_n(x^*_{i,n})\, C_{i,n}(x) \tag{21}$$

Let $f : [-1,1] \to \mathbb{R}$ be a continuously differentiable function. It is well known (from, e.g. [2, Theorem1.3.3]) that there exists a polynomial $Q_n(x)$ of degree at most (3n-1) such that

$$|\, f(x) - Q_n(x)| = O\left(n^{-1}\omega\left(f';\frac{1}{n}\right)\right)$$

and

$$|f'(x) - Q_n'(x)| = O\left(\omega\left(f';\frac{1}{n}\right)\right).$$

Now,

$|f(x)-R_n(f;\ x)| \leq |\, f(x) - Q_n(x)| +$

$$\sum_{i=0}^{n}\left(Q_n(x_{i,n}) - f(x_{i,n})\right) A_{i,n}(x) + \sum_{i=1}^{n}(Q_n{}'(x_{i,n}) - f'(x_{i,n}))B_{i,n}(x) + \sum_{i=1}^{n-1}\left(Q_n(x^*_{i,n}) - f(x^*_{i,n})\right) C_{i,n}(x)$$

$$|f(x)-R_n(f;\ x)| = O\left(n^{-1}\omega\left(f';\frac{1}{n}\right)\right) + O\left(n^{-3/2}\omega\left(f';\frac{1}{n}\right)\right) + O\left(n^{-5/2}\omega\left(f';\frac{1}{n}\right)\right)$$

$$+O\left(n^{-5\backslash 2}\left(f';\frac{1}{n}\right)\right).$$

which completes the proof of theorem.

5. Conclusion:

In this work, we have established the existence, uniqueness, explicit formulation and convergence order of the given interpolatory problem under the condition that the nodes $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ are the roots of polynomials $\pi_n(x)$and $\pi_n'(x)$ respectively, along with an additional conditional point. Furthermore, if $f : [-1,1] \to \mathbb{R}$ be continuously differentiable function, then the sequence of the interpolation polynomials $R_n(f;x)$ and $R_n'(f;\ x)$ converge uniformly to $f(x)$ and $f'(x)$ respectively on [-1,1] as n → ∞.

Conflict of interest: There is no conflict of interest.

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